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OSCILLATION OF NONLINEAR IMPULSIVE PARABOLIC EQUATIONS OF NEUTRAL TYPE

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ABSTRACT. In this paper, oscillatory properties of solutions for certain nonlinear impulsive parabolic equations of neutral type with several delays are investigated and a series of new sufficient conditions and a necessary and sufficient condition for oscillation of the solutions are established.

1. Introduction. The theory of delay partial differential equations can be applied to many fields, such as to biology, population growth, engineering, generic repression, control theory and climate model. In the last few years, the fundamental theory of partial differential equations with deviating argument has undergone intensive development. The qualitative theory of this class of equations, however, is still in an initial stage of development. A few papers have been published on oscillation theory of partial differential equations with delay. Many have been done under the assumption that the state variables and system parameters change continuously. However, one may easily visualize situations in nature where abrupt change such as shock and disasters may occur. These phenomena are short-time perturbations whose duration is negligible in comparison with the duration of the whole evolution process. Consequently, it is natural to assume, in modeling these problems, that these perturbations act instantaneously, that is, in the form of impulses. In 1991, the first paper [**9**] on this class of equations was published. But, for instance, on oscillation theory of impulsive partial differential equations, only a few of papers have been published. Recently, Bainov, Minchev, Fu, Deng and Luo [**3–5, 10, 11, 14, 21**] investigated the oscillation of solutions of impulsive partial differential equations with or without deviating argument. But there is a scarcity

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in the study of oscillation theory of nonlinear impulsive parabolic equations with several delays.

In this paper, we shall discuss the oscillatory properties of solutions for a class of nonlinear impulsive parabolic equations with several delays (1), under the boundary conditions (3), (4). It should be noted that the equation we discuss here is nonlinear and boundary condition (3) also is nonlinear. Up to now, we do not find the work for oscillations of these kinds of problem.

$$\begin{aligned}(1)\quad &\frac{\partial}{\partial t}\left(u(t,x)+q(t)\,u(t-\mu,x)\right)\\ &= a(t)h(u)\Delta u+\sum_{i=1}^{m} a_i(t)h_i(u(t-\tau_i,x))\Delta u(t-\tau_i,x)\\ &\quad -\sum_{j=0}^{n} q_j(t,x)f_j(u(t-\sigma_j,x)),\quad t\neq t_k,\quad (t,x)\in R_+\times\Omega=G.\end{aligned}$$

$$u(t_k^+,x)-u(t_k^-,x)=bu(t_k,x), \tag{2}$$

with the boundary conditions

$$\frac{\partial u}{\partial n}=g(t,x,u),\quad (t,x)\in R_+\times\partial\Omega, \tag{3}$$

$$\frac{\partial u}{\partial n}+c(x)u=0,\quad (t,x)\in R_+\times\partial\Omega \tag{4}$$

and the initial condition $u(t,x)=\Phi(t,x)$, $(t,x)\in[-\delta,0]\times\Omega$. Here $\Omega\subset R^N$ is a bounded domain with boundary $\partial\Omega$ smooth enough and n is a unit exterior normal vector of $\partial\Omega$, $\delta=\max\{\mu,\tau_i,\sigma_j\}$, $\Phi(t,x)\in C^2([-\delta,0]\times\Omega,R)$, $c(x)\in C(\partial\Omega,(0,\infty))$.

This article is organized as follows. Section 2 studies the oscillatory properties of solutions for problems (1), (3). Section 3 discusses oscillatory properties of solutions for problems (1), (4). In Section 4, we, for the linear case, obtain a necessary and sufficient condition for oscillation of solutions.

Assume that the following conditions are fulfilled:

H$_1$) $a(t), a_i(t) \in PC(R_+, R_+)$, $q(t) \in C(R_+, (-1, 0])$; $\mu, \tau_i, \sigma_j =$ const > 0, $\sigma_0 = 0$, $q_j(t,x) \in C(R_+ \times \overline{\Omega}, (0, \infty))$, $i = 1, 2, \dots, m$, $j = 0, 1, 2, \dots n$; where PC denote the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t = t_k$ and left continuous at $t = t_k$, $k = 1, 2, \dots$, $t_k - t_{k-1} = \mu$.

H$_2$) $h'(u), h_i'(u), f_j(u) \in C(R, R)$; $f_j(u)/u \geq C_j =$ const > 0, for $u \neq 0$; $uh'(u) \geq 0$, $uh_i'(u) \geq 0$, $uh(u)g(t,x,u) < 0$, $uh_i(u)g(t,x,u) < 0$, $g(t,x,u)$ is continuous, $d_k =$ const > -1, $b =$ const > 0, $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $\lim_{t\to\infty} t_k = \infty$.

H$_3$) $u(t,x)$ is piecewise continuous in t with discontinuities of first kind only at $t = t_k$ and left continuous at $t = t_k$, $u(t_k, x) = u(t_k^-, x)$, $k = 1, 2, \dots$.

Let us construct the sequence $\{\bar{t}_k\} = \{t_k\} \cup \{t_{k\mu}\} \cup \{t_{ki}\} \cup \{t_{kj}\}$, where $t_{k\mu} = t_k + \mu, t_{ki} = t_k + \tau_i, t_{kj} = t_k + \sigma_j$ and $\bar{t}_k < \bar{t}_{k+1}$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, $k = 1, 2, \dots$.

Definition 1.1. By a solution of problems (1), (3) ((4)) with initial condition, we mean that any function $u(t,x)$ for which the following conditions are valid:

1. If $-\delta \leq t \leq 0$, then $u(t,x) = \Phi(t,x)$.

2. If $0 \leq t \leq \bar{t}_1 = t_1$, then $u(t,x)$ coincides with the solution of the problems (1), (2) and (3) ((4)) with initial condition.

3. If $\bar{t}_k < t \leq \bar{t}_{k+1}$, $\bar{t}_k \in \{t_k\} \setminus (\{t_{k\mu}\} \cup \{t_{ki}\} \cup \{t_{kj}\})$, then $u(t,x)$ coincides with the solution of the problems (1), (2) and (3) ((4)).

4. If $\bar{t}_k < t \leq \bar{t}_{k+1}$, $\bar{t}_k \in \{t_{k\mu}\} \bigcup \{t_{ki}\} \bigcup \{t_{kj}\}$, then $u(t,x)$ satisfies (3)((4)) and coincides with the solution of the following problem

$$
\begin{aligned}
\frac{\partial}{\partial t}&\left(u(t^+,x) + q(t)\,u((t-\mu)^+,x)\right)\\
&= a(t)h(u(t^+,x))\Delta u(t^+,x)\\
&\quad + \sum_{i=1}^{m} a_i(t)h_i(u((t-\tau_i)^+,x))\Delta u((t-\tau_i)^+,x)\\
&\quad - \sum_{j=0}^{n} q_j(t,x)f_j(u((t-\sigma_j)^+,x)), \quad (t,x) \in R_+ \times \Omega = G,
\end{aligned}
$$

$$u(\bar{t}_k^+, x) = u(\bar{t}_k, x), \quad \text{for} \quad \bar{t}_k \in \Big(\{t_{k\mu}\} \bigcup \{t_{ki}\} \bigcup \{t_{kj}\}\Big) \setminus \{t_k\},$$

or

$$u(\bar{t}_k^+, x) = (1+b)\, u(\bar{t}_k, x),$$
$$\text{for} \quad \bar{t}_k \in \Big(\{t_{k\mu}\} \bigcup \{t_{ki}\} \bigcup \{t_{kj}\}\Big) \bigcap \{t_k\}.$$

We introduce the notations: $\Gamma_k = \{(t,x) : t \in (t_k, t_{k+1}), x \in \Omega\}$, $\Gamma = \cup_{k=0}^{\infty}\Gamma_k$, $\bar{\Gamma}_k = \{(t,x) : t \in (t_k, t_{k+1}), x \in \bar{\Omega}\}$, $\bar{\Gamma} = \cup_{k=0}^{\infty}\bar{\Gamma}_k$, $v(t) = \int_\Omega u(t,x)\,dx$ and $p_j(t) = \min q_j(t,x), x \in \bar{\Omega}$.

Definition 1.2. The solution $u \in C^2(\Gamma) \cap C^1(\bar{\Gamma})$ of problems (1), (3) ((4)) is called nonoscillatory in the domain G if it is either eventually positive or eventually negative. Otherwise, it is called oscillatory.

2. Oscillation properties of the problems (1), (3). The following is the main theorem of this paper. The proof of the theorem needs the following lemma [**23**].

Lemma 2.1. *Let $\rho = \text{const} > 0, a_0(t)$, $p(t) \in ([0,+\infty), R)$ be locally summable functions and $p(t) > 0$; $y(t_k) = y(t_k^-)$, $k = 1, 2, \ldots$. If the following condition is satisfied*

$$\liminf_{t\to\infty} \int_{t-\rho}^{t} p(s) \exp\Big(\int_{s-\rho}^{s} a_0(r)\,dr\Big) \prod_{s-\rho<t_k<s} (1+d_k)^{-1}\,ds > \frac{1}{e},$$

then the following differential inequality has no eventually positive solution.

$$y'(t) + a_0(t)\,y(t) + p(t)\,y(t-\rho) \le 0, \quad t \ge 0, \quad t \ne t_k,$$
$$y(t_k^+) - y(t_k^-) = d_k\, y(t_k), \quad k = 1, 2, \ldots.$$

Theorem 2.2. *Suppose that the conditions* H_1), H_2) *and the following condition* (5) *hold for some* $j \in \{1, \ldots, n\}$,

(5)

$$\liminf_{t\to\infty} \int_{t-\sigma_j}^{t} C_j p_j(s) \exp\Big(\int_{s-\sigma_j}^{s} C_0 p_0(r)\,dr\Big) \prod_{s-\sigma_j<t_k<s} (1+b)^{-1}\,ds > \frac{1}{e}.$$

Then every solution of problems (1), (3) *oscillates in* G.

Proof. Suppose that the assertion is not true and $u(t,x)$ is a nonoscillatory solution of problems (1), (3). Without loss of generality, we may assume that there exists a $t_0 \geq T$ such that $u(t,x) > 0, u(t-\mu,x) > 0$, $u(t-\tau_i,x) > 0$, $i = 1,2,\dots,m$ and $u(t-\sigma_j,x) > 0$, $j = 1,2,\dots,n$ for any $(t,x) \in [t_0,\infty) \times \Omega$.

For $t \geq t_0$, $t \neq t_k$, $k = 1,2,\dots$, integrating (1) with respect to x over Ω yields

$$\begin{aligned}\frac{d}{dt}\Big[\int_\Omega (u(t,x) + q(t)\,u(t-\mu,x))\,dx\Big]\\ &= a(t)\int_\Omega h(u)\Delta u\,dx\\ &\quad + \sum_{i=1}^m a_i(t)\int_\Omega h_i(u(t-\tau_i,x))\Delta u(t-\tau_i,x)\,dx\\ &\quad - \sum_{j=0}^n \int_\Omega q_j(t,x) f_j(u(t-\sigma_j,x))\,dx, \quad t \geq t_0, \quad t \neq t_k.\end{aligned}$$

By Green's formula and the boundary condition we have

$$\begin{aligned}\int_\Omega h(u)\Delta u\,dx &= \int_{\partial\Omega} h(u)\,\frac{\partial u}{\partial n}\,ds - \int_\Omega h'(u)\,|\mathrm{gradu}|^2\,dx\\ &\leq -\int_\Omega h'(u)\,|\mathrm{gradu}|^2\,dx \leq 0,\\ \int_\Omega h_i(u(t-\tau_i,x))\Delta u(t-\tau_i,x)\,dx &\leq 0.\end{aligned}$$

From condition H_2), we can easily obtain

$$\int_\Omega q_j(t,x) f_j(u(t-\sigma_j,x))\,dx \geq C_j p_j(t)\int_\Omega u(t-\sigma_j,x)\,dx.$$

Then $v(t) > 0$, it follows that

$$\frac{d}{dt}\,[v(t) + q(t)\,v(t-\mu)] + \sum_{j=0}^n C_j p_j(t)\,v(t-\sigma_j) \leq 0,$$
$$t \geq t_0, \quad t \neq t_k.$$

Hence, we obtain

$$\frac{d}{dt}\,[v(t)+q(t)\,v(t-\mu)]+C_0p_0(t)\,v(t)+C_jp_j(t)\,v(t-\sigma_j)\le 0. \tag{6}$$

Now in inequality (6), set

$$y(t)=v(t)+q(t)\,v(t-\mu). \tag{7}$$

Then we have

$$y'(t)+C_jp_j(t)\,v(t-\sigma_j)\le 0,\quad t\ge t_0,\quad t\ne t_k. \tag{8}$$

From inequality (8) it is easy to see that $y(t)$ is nonincreasing on intervals $[t_k,t_{k+1})$ and, together with condition $b>0$, $u(t_k^+,x)-u(t_k^-,x)=bu(t_k,x)$, we can easily obtain that $y(t)$ is either eventually positive or eventually negative.

(1) If we suppose that $y(t)$ is eventually negative, then it is easy to see that $\lim_{t\to\infty}y(t)=-\infty$. From equality (7), we can get that $v(t)$ is unbounded, consequently there exists $\{s_k:k\to\infty,s_k\to\infty\}$, such that $y(s_k)<0$, $v(s_k)=\max v(r)$, $r\in[t_0,s_k]$. Therefore $y(s_k)=v(s_k)+q(s_k)v(s_k-\mu)\ge v(s_k)[1+q(s_k)]\ge 0$. This contradicts $y(s_k)<0$.

(2) If we suppose that $y(t)$ is eventually positive, then from equality (7) we get $y(t)<v(t)$ and, from inequality (6), we obtain the following differential inequality

$$y'(t)+C_0p_0(t)\,y(t)+C_jp_j(t)\,y(t-\sigma_j)\le 0,\quad t\ge t_0,\quad t\ne t_k, \tag{9}$$

For $t>t_0$, $t=t_k$, $k=1,2,\dots$, since $q(t)$ is continuous on $[t_0,+\infty)$, it is easy to verify that

(10)

$$\begin{aligned} y(t_k^+)-y(t_k^-)&=v(t_k^+)-v(t_k^-)+q(t_k^+)\,v(t_k^+-\mu)-q(t_k^-)\,v(t_k^--\mu)\\ &=bv(t_k)+q(t_k)\,bv(t_k-\mu)=by(t_k). \end{aligned}$$

Hence we obtain that $y(t)>0$ is an eventually positive solution of differential inequality (9), (10). But, according to Lemma 2.1 and condition (5), the differential inequality (9), (10) has no eventually positive solution. This is a contradiction. This ends the proof of the theorem. □

Similarly, we can obtain the following theorems.

Theorem 2.3. *Suppose that the conditions* H_1), H_2) *and the following condition hold for some* $j \in \{1, \dots, n\}$,

$$\limsup_{t\to\infty} \int_{t-\sigma_j}^{t} C_j p_j(s) \exp\Big(\int_{s-\sigma_j}^{s} C_0 p_0(r)\,dr\Big) \prod_{s-\sigma_j<t_k<s} (1+b)^{-1}\,ds > 1.$$

Then every solution of the problems (1), (3) *oscillates in* G.

If we assume that $m_1\,\mu \ge \sigma_j$ for some integer m_1, then we have the following Theorems 2.4, 2.5.

Theorem 2.4. *Suppose that the conditions* H_1), H_2) *and the following condition hold for some* $j \in \{1, \dots, n\}$,

$$\liminf_{t\to\infty} \int_{t-\sigma_j}^{t} C_j p_j(s) \exp\Big(\int_{s-\sigma_j}^{s} C_0 p_0(r)\,dr\Big)\,ds > \frac{(1+b)^{m_1}}{e}.$$

Then every solution of problems (1), (3) *oscillates in* G.

Theorem 2.5. *Suppose that the conditions* H_1), H_2) *and the following condition hold for some* $j \in \{1, \dots, n\}$,

$$\limsup_{t\to\infty} \int_{t-\sigma_j}^{t} C_j p_j(s) \exp\Big(\int_{s-\sigma_j}^{s} C_0 p_0(r)\,dr\Big)\,ds > (1+b)^{m_1}.$$

Then every solution of problems (1), (3) *oscillates in* G.

More generally, we have the following Theorem 2.6.

Theorem 2.6. *Suppose that the conditions of Theorem* 2.2 *still hold and the condition* (5) *is replaced by the differential inequality* (9), (10) *which has no eventually positive solution. Then every solution of problems* (1), (3) *oscillates in* G.

The proofs are easy so we just omit them.

It should be noted that obviously all solutions of problems (1), (3) are oscillatory if $b < -1$.

3. Oscillation properties of problems (1), (4). Making use of the following lemma of eigenvalue, we can obtain many results for problem (1), (4). We suppose that $h(u), h_i(u)$ are constants (suppose them all to be 1).

Lemma 3.1. *Suppose that* λ_0 *is the smallest eigenvalue of the problem*

$$\begin{aligned} \Delta\varphi(x) + \lambda\varphi(x) = 0, &\quad x \in \Omega \\ \frac{\partial\varphi(x)}{\partial n} + c(x)\varphi(x) = 0, &\quad x \in \partial\Omega. \end{aligned}$$

and $\varphi(x)$ *is the corresponding eigenfunction of* λ_0. *Then* $\lambda_0 > 0$, $\varphi(x) > 0$, $x \in \Omega$.

Theorem 3.2. *Suppose that the conditions* H_1), H_2) *and the following condition hold for some* $j \in \{1, \dots, n\}$.

$$\liminf_{t\to\infty} \int_{t-\sigma_j}^{t} C_j p_j(s) \exp\Big(\int_{s-\sigma_j}^{s} (\lambda_0 a(r) + C_0 p_0(r))\, dr\Big) \times \prod_{s-\sigma_j < t_k < s} (1+b)^{-1}\, ds > \frac{1}{e}. \tag{11}$$

Then every solution of problems (1), (4) *oscillates in* G.

Proof. Suppose that the assertion is not true and $u(t,x)$ is a nonoscillatory solution of problems (1), (4). Without loss of generality, we may assume that there exists a $t_0 \geq T$ such that $u(t,x) > 0,\ u(t-\mu,x) > 0$, $u(t-\tau_i,x) > 0,\ i = 1,2,\dots,m$ and $u(t-\sigma_j,x) > 0,\ j = 1,2,\dots,n$, for any $(t,x) \in [t_0,\infty) \times \Omega$.

For $t \geq t_0,\ t \neq t_k,\ k = 1,2,\dots$, multiplying equation (1) with $\varphi(x)$, which is the same as that in Lemma 3.1 and then integrating (1) with respect to x over Ω we have

$$\begin{aligned}\frac{d}{dt}\Big[\int_\Omega (u(t,x) + q(t)\,u(t-\mu,x))\varphi(x)\,dx\Big] \\ = a(t)\int_\Omega \Delta u\varphi(x)dx + \sum_{i=1}^m a_i(t)\int_\Omega \Delta u(t-\tau_i,x)\varphi(x)\,dx \\ - \sum_{j=0}^n \int_\Omega q_j(t,x) f_j(u(t-\sigma_j,x))\varphi(x)\,dx.\end{aligned}$$

By Green's formula and the boundary condition, we obtain

$$\int_\Omega u\Delta\varphi\,dx - \int_\Omega \varphi\Delta u\,dx = \int_{\partial\Omega} \frac{\partial\varphi}{\partial n}\,u\,ds - \int_{\partial\Omega}\frac{\partial u}{\partial n}\,\varphi\,ds = 0.$$

It follows that

$$\begin{aligned}\int_\Omega \Delta u(t,x)\varphi(x)\,dx &= \int_\Omega \Delta\varphi(x)\,u(t,x)\,dx \\ &= -\lambda_0\int_\Omega \varphi(x)\,u(t,x)\,dx, \\ \int_\Omega \Delta u(t-\tau_j,x)\varphi(x)\,dx &= \int_\Omega \Delta\varphi(x)\,u(t-\tau_i,x)\,dx \\ &= -\lambda_0\int_\Omega \varphi(x)\,u(t-\tau_i,x)\,dx.\end{aligned}$$

From the condition H_2), we can easily obtain

$$\int_\Omega q_j(t,x) f_j(u(t-\sigma_j,x))\varphi(x)\,dx \geq C_j p_j(t)\int_\Omega u(t-\sigma_j,x)\varphi(x)\,dx. \tag{12}$$

Denote $v(t) = \int_\Omega u(t,x)\varphi(x)\,dx$. Then $v(t) > 0$; it follows that we have

$$\frac{d}{dt}\left[v(t) + q(t)\,v(t-\mu)\right] + \lambda_0 a(t)\,v(t) + \lambda_0 \sum_{i=1}^{m} a_i(t)\,v(t-\tau_i) + \sum_{j=0}^{n} C_j p_j(t)\,v(t-\sigma_j) \le 0, \quad t \ge t_0, \quad t \ne t_k. \tag{13}$$

Hence we obtain the similar differential inequality as (5).

$$\frac{d}{dt}\left[v(t)+q(t)\,v(t-\mu)\right]+(\lambda_0 a(t)+C_0 p_0)\,v(t)+C_j p_j(t)\,v(t-\sigma_j) \le 0. \tag{14}$$

The following proof is similar to that used in Theorem 2.2. We omit it. This ends the proof of the theorem. □

Theorem 3.3. *Suppose that the conditions* H$_1$), H$_2$) *and the following condition hold for some* $j \in \{1,\dots,n\}$,

$$\limsup_{t\to\infty} \int_{t-\sigma_j}^{t} C_j p_j(s) \exp\Big(\int_{s-\sigma_j}^{s} (\lambda_0 a(r) + C_0 p_0(r))\,dr\Big) \times \prod_{s-\sigma_j<t_k<s} (1+b)^{-1}\,ds > 1. \tag{15}$$

Then every solution of problems (1), (4) *oscillates in* G.

If we assume that $m_1\,\mu \ge \sigma_j$ for some integer m_1, then we have the following Theorems 3.4, 3.5.

Theorem 3.4. *Suppose that the conditions* H$_1$), H$_2$) *and the following condition hold for some* $j \in \{1,\dots,n\}$,

$$\liminf_{t\to\infty} \int_{t-\sigma_j}^{t} C_j p_j(s) \exp\Big(\int_{s-\sigma_j}^{s} (\lambda_0 a(r)+C_0 p_0(r))\,dr\Big)\,ds > \frac{(1+b)^{m_1}}{e}.$$

Then every solution of problems (1), (4) *oscillates in* G.

Theorem 3.5. *Suppose that the conditions* H_1), H_2) *and the following condition hold for some* $j \in \{1, \dots, n\}$,

$$\limsup_{t\to\infty} \int_{t-\sigma_j}^{t} C_j p_j(s) \exp\Big(\int_{s-\sigma_j}^{s} (\lambda_0 a(r) + C_0 p_0(r))\, dr\Big)\, ds > (1+b)^{m_1}.$$

Then every solution of problems (1), (4) *oscillates in* G.

The proofs are easy, we just omit them.

Theorem 3.6. *Suppose that the conditions* H_1), H_2) *and the following condition hold for some* $a_i(t)$,

$$\liminf_{t\to\infty} \int_{t-\tau_i}^{t} \lambda_0 a_i(s) \exp\Big(\int_{s-\tau_i}^{s} \lambda_0 a(r)\, dr\Big) \prod_{s-\tau_i<t_k<s} (1+b)^{-1}\, ds > \frac{1}{e}.$$

Then every solution of problems (1), (4) *oscillates in* G.

Proof. From differential inequality (13) we can obtain

$$\frac{d}{dt}[v(t) + q(t)v(t-\mu)] + \lambda_0 a(t) v(t) + \lambda_0 a_i(t) v(t-\tau_i) \le 0, \qquad t \ge t_0, \quad t \ne t_k. \tag{17}$$

The following proof is the same as that used in Theorem 3.2. We just omit it. This ends the proof of Theorem 3.6. □

It should be noted that the criterion in this theorem only depends on diffusion coefficient $a_i(t)$.

Theorem 3.7. *Suppose that the conditions* H_1), H_2) *and the following condition hold for some* $a_i(t)$,

$$\limsup_{t\to\infty} \int_{t-\tau_i}^{t} \lambda_0 a_i(s) \exp\Big(\int_{s-\tau_i}^{s} \lambda_0 a(r)\, dr\Big) \prod_{s-\tau_i<t_k<s} (1+b)^{-1}\, ds > 1. \tag{18}$$

Then every solution of problems (1), (4) *oscillates in* G.

4. Necessary and sufficient condition. In this section, we will establish a necessary and sufficient condition for oscillation of impulsive parabolic equations with several delays. We consider the following linear problem.

$$\frac{\partial}{\partial t}\left(u(t,x)+q(t)\,u(t-\mu,x)\right) = a(t)\Delta u+\sum_{i=1}^{m} a_i(t)\Delta u(t-\tau_i,x)-\sum_{j=0}^{n} p_j(t)\,u(t-\sigma_j,x), \quad t\neq t_k, \quad (t,x)\in R_+\times\Omega=G \tag{19}$$

$$u(t_k^+,x)-u(t_k^-,x)=bu(t_k,x), \quad k=1,2,\dots \tag{20}$$

with boundary condition (4).

Theorem 4.1. *Every solution of the problem* (19), (20), (4) *is oscillatory in domain* G *if and only if every solution of the following impulsive delay differential equation* (21), (22) *is oscillatory.*

$$\frac{d}{dt}\left[v(t)+q(t)\,v(t-\mu)\right]+a(t)\lambda_0 v(t) + \lambda_0\sum_{i=1}^{m} a_i(t)\,v(t-\tau_i)+\sum_{j=0}^{n} p_j(t)\,v(t-\sigma_j)=0, \tag{21}$$

$$v(t_k^+)-v(t_k^-)=bv(t_k), \quad k=1,2,\dots . \tag{22}$$

Proof. Sufficiency. Using reduction to absurdity. Let $u(t,x)$ be a nonoscillatory solution of the problem (19), (20), (4). Without loss of generality, we may assume that there exists a $t_0\ge T$ such that $u(t,x)>0$, $u(t-\mu,x)>0$, $u(t-\tau_i,x)>0$ and $u(t-\sigma_j,x)>0$, $i=1,\dots,m$; $j=1,\dots,n$ for any $(t,x)\in[t_0,+\infty)\times\Omega$.

For $t\ge t_0$, $t\neq t_k$, $k=1,2,\dots$, multiplying equation (19) with $\varphi(x)$, which is the same as that in Lemma 3.1, then integrating (19) with respect to x over Ω we have

$$\begin{aligned}(23)\quad & \frac{d}{dt}\int_\Omega [u(t,x)+q(t)u(t-\mu)]\varphi(x)\,dx \\ &= a(t)\int_\Omega \Delta u(t,x)\varphi(x)\,dx + \sum_{i=1}^m a_i(t)\int_\Omega \Delta u(t-\tau_i,x)\varphi(x)\,dx \\ &\qquad - \sum_{j=0}^n \int_\Omega p_j(t)u(t-\sigma_j,x)\varphi(x)\,dx.\end{aligned}$$

By Green's formula and boundary condition, we have

$$\int_\Omega u\Delta\varphi(x)\,dx - \int_\Omega \varphi(x)\Delta u\,dx = \int_{\partial\Omega} u\,\frac{\partial\varphi(x)}{\partial n}\,ds - \int_{\partial\Omega}\varphi(x)\,\frac{\partial u}{\partial n}\,ds = 0.$$

It follows that

$$\int_\Omega \varphi(x)\Delta u\,dx = \int_\Omega u\Delta\varphi(x)\,dx = -\lambda_0\int_\Omega \varphi(x)u(t,x)\,dx$$

$$\begin{aligned}\int_\Omega \varphi(x)\Delta u(t-\tau_i,x)\,dx &= \int_\Omega u(t-\tau_i,x)\Delta\varphi(x)\,dx \\ &= -\lambda_0\int_\Omega \varphi(x)u(t-\tau_i,x)\,dx.\end{aligned}$$

Denote $v(t)=\int_\Omega \varphi(x)u(t,x)dx$, then $v(t)>0$. It follows from that we can easily obtain

$$\begin{aligned}(24)\quad & \frac{d}{dt}\,[v(t)+q(t)\,v(t-\mu)] + a(t)\lambda_0 v(t) \\ &\qquad + \lambda_0\sum_{i=1}^m a_i(t)\,v(t-\tau_i) + \sum_{j=0}^n p_j(t)\,v(t-\sigma_j) = 0.\end{aligned}$$

For $t>t_0$, $t=t_k$, $k=1,2,\dots$, we have

$$\int_\Omega u(t_k^+,x)\,dx - \int_\Omega u(t_k^-,x)\,dx = b\int_\Omega u(t_k,x)\,dx.$$

This implies

$$(25)\qquad v(t_k^+)-v(t_k^-)=bv(t_k).$$

Hence we obtain that $v(t) > 0$ satisfies equation (21), (22). This means that impulsive delay differential equation (21), (22) has a nonoscillatory solution. A contradiction. This ends the proof of sufficient condition.

Necessity (still using reduction to absurdity). Let $v(t)$ be a nonoscillatory solution of the equation (21), (22). Without loss of generality, we may assume that there exists a t_1 large enough such that $v(t) > 0$, $v(t-\mu) > 0$, $v(t-\tau_i) > 0$ and $v(t-\sigma_j) > 0$, $i = 1, \dots, m$; $j = 1, \dots, n$ for any $t \in [t_1, +\infty)$.

For $t \geq t_1$, $t \neq t_k$, $k = 1, 2, \dots$, set $u(t,x) = v(t)\varphi(x)$; we have $u(t,x) > 0$ and we can easily obtain

$$\Delta u(t,x) = \Delta[v(t)\varphi(x)] = v(t)\Delta\varphi(x) = -\lambda_0 v(t)\varphi(x)$$
$$\Delta u(t-\tau_i, x) = \Delta[v(t-\tau_i)\varphi(x)] = v(t-\tau_i)\Delta\varphi(x) = -\lambda_0 v(t-\tau_i)\varphi(x).$$

Making use of these results, from equation (21). We obtain

$$\frac{d}{dt}\left[(v(t) + q(t)\, v(t-\mu))\varphi(x)\right] + a(t)\lambda_0 v(t)\varphi(x) + \lambda_0 \sum_{i=1}^{m} a_i(t)\, v(t-\tau_i)\varphi(x) + \sum_{j=0}^{n} p_j(t)\, v(t-\sigma_j)\varphi(x) = 0. \tag{26}$$

This means that $u(t,x) = v(t)\varphi(x)$ satisfies equation (19).

For $t \geq t_1$, $t = t_k$, $k = 1, 2, \dots$, from the conditions (22), it is easy to see that function $u(t,x) = v(t)\varphi(x)$ satisfies (20). And because $(\partial\varphi(x)/\partial n) + c(x)\varphi(x) = 0$, $x \in \partial\Omega$. That is, $u(t,x) = v(t)\varphi(x)$ also satisfies boundary condition (4). This indicates that problems (19), (20) and (4) have a nonoscillatory solution. This is a contradiction. This ends the proof of Theorem 4.1. □

Example.

$$\frac{\partial}{\partial t}\Big(u(t,x) - \frac{1}{2}\, u(t-\pi, x)\Big) = 2u^2\Delta u - u(t-\pi, x)\exp[u(t-\pi, x)]^2, \quad t \neq k\,\pi, \quad (t,x) \in R_+ \times \Omega = G. \tag{27}$$

$$u(t_k^+, x) - u(t_k^-, x) = 2u(t_k, x), \quad t = k\,\pi \tag{28}$$

with the boundary condition

$$\frac{\partial u}{\partial n} = -cu, \quad (t,x) \in R_+ \times \partial\Omega, \quad c > 0. \tag{29}$$

It is easy to verify all hypotheses of Theorem 2.2 for this case, and thus all solutions of (27)–(29) are oscillatory.

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