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LOCAL CONNECTEDNESS IN HYPERSPACES

JANUSZ J. CHARATONIK AND ALEJANDRO ILLANES

ABSTRACT. Variants of local connectedness as local connectedness, local arcwise connectedness, strong local connectedness and strong local arcwise connectedness at a point are studied for the following hyperspaces of a compact Hausdorff space X: $C_n(X)$, $C_\infty(X)$, $F_n(X)$, $F_\infty(X)$ and 2^X.

1. Introduction. In [**20**] it is proved that if X is a Hausdorff compact space, then local connectedness and local arcwise connectedness of the hyperspace $C(X)$ of all subcontinua of X and of the hyperspace 2^X of all nonempty closed subsets of X are equivalent at any point. In [**21**] it is shown that for metric spaces the above properties are equivalent to another one, namely to local k-connectedness. In the present paper the above equivalences are studied for further hyperspaces: $C_n(X)$ of all members of 2^X that have no more than n components, $C_\infty(X)$ of all members of 2^X that have finitely many components, $F_n(X)$ of all members of 2^X that have no more than n points, and $F_\infty(X)$ of all members of 2^X that consist of finitely many points. The obtained results complete not only the above mentioned papers [**20**] and [**21**], but also a number of other ones related to the same topic of local connectivity properties of hyperspaces as, e.g., [**4**–**7, 9**–**12, 18, 19**] and others.

The paper consists of six sections. In the first one we collect, for reader information and completeness of this paper, some known results about local connectedness at a point of the hyperspace 2^X, i.e., at a nonempty closed subset of X. The second and the third sections are devoted to variants of local connectedness at a point of the hyperspace $C_n(X)$. We study local connectedness, local arcwise connectedness, strong local connectedness and strong local arcwise connectedness of

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these hyperspaces at a point. Section 4 deals with local k-connectedness and LC^∞ of $C_n(X)$ at a point, for compact metric spaces X. In Section 5 we discuss the same variants of local connectedness of the hyperspace $C_\infty(X)$ at a point. The last, sixth, section is devoted to investigation of the same kind of problems for the hyperspaces of finite subsets of X, i.e., for $F_n(X)$ and $F_\infty(X)$.

The symbol $\mathbf{N}$ stands for the set of all positive integers. All considered spaces are assumed to be Hausdorff, and all mappings are continuous. A *continuum* means a compact connected space.

A space X is said to be *locally connected at a point* $p \in X$ provided that for each open subset U of X such that $p \in U$ the point p is an interior point of a component of U, see [**16**, Section 49, p. 227]. Note that some authors use the term *connected im kleinen at a point* in the sense of "locally connected" as defined above, see e.g. [**13**, p. 113] or [**23**, p. 132]. A space X is said to be *strongly locally connected at a point* $p \in X$ provided that for each open subset U of X with $p \in U$ there exists a connected open subset V of X such that $p \in V \subset U$ (some authors use the term *locally connected at a point* in the same sense, see e.g., [**23**, p. 132]).

Observe that

(0.1) *if X is compact and Hausdorff, then X is locally connected at a point p if and only if X has a basis of neighborhoods at p composed of continua*, see e.g., [**20**, p. 120].

An *arc* means a continuum having exactly two noncut points, as defined, e.g., in [**28**, p. 36] (and named a generalized arc in [**14**, p. 114]). A space is said to be *arcwise connected* provided that every two of its points can be joined by an arc contained in the space. A space X is said to be *locally arcwise connected at a point* $p \in X$ provided that given an open neighborhood U of p in X there exists an open neighborhood V of p with $V \subset U$ such that if $x \in V$, then there is an arc $A \subset U$ joining p and x; equivalently, given an open subset U of X such that $p \in U$, there exists an arcwise connected set W such that $p \in \operatorname{int}_X(W) \subset W \subset U$. A space is said to be *locally arcwise connected* provided that it is locally arcwise connected at each of its points. A space X is said to be *strongly locally arcwise connected at a point* p provided that for each open neighborhood U of p in X there is an open arcwise connected neighborhood V of p such that $V \subset U$

(note that the term "locally arcwise connected at a point" is used in the same sense in [**11**, p. 41]).

For a given point x in the Euclidean k-dimensional space $\mathbf{R}^k$ the symbol $\|x\|$ denotes the distance from x to the origin 0. We will use the symbol $\mathbf{S}_k$ for the k-dimensional sphere, i.e., $\mathbf{S}_k = \{x \in \mathbf{R}^{k+1} : \|x\| = 1\}$; the symbol $\mathbf{B}_k$ for the k-dimensional ball, i.e., $\mathbf{B}_k = \{x \in \mathbf{R}^k : \|x\| \leq 1\}$; and $N(x, \varphi)$, where $\varphi > 0$, for an φ-neighborhood of the point x.

Let $k \in \{0\} \cup \mathbf{N}$. A space X is said to be *locally connected in dimension k at a point* $p \in X$ (briefly, we will write *locally k-connected at p*) provided that for each open subset U of X such that $p \in U$ there exists an open set V with $p \in V \subset U$ and such that each mapping $f : \mathbf{S}_k \to X$ for which $f(\mathbf{S}_k) \subset V$ has a continuous extension $f^* : \mathbf{B}_{k+1} \to X$ such that $f^*(0) = p$ and $f^*(\mathbf{B}_{k+1}) \subset U$, see [**16**, Section 53, p. 346]. A space X is said to be *locally connected in dimension k* provided that it has the above property at each of its points. Note that

(0.2) *local* 0-*connectedness* (*at a point*) *is equivalent to local arcwise connectedness* (*at the point*), compare [**16**, Section 53, p. 351] and [**1**, Chapter 1, Section 17, p. 30].

It should be observed that local connectedness in dimension k at a point p is sometimes understood in a slightly different way; the difference is that the condition $f^*(0) = p$ is not required, compare, e.g., [**1**, p. 30]; see [**21**, p. 30].

A space X is said to be an LC^k-*space* (at a point $p \in X$), where $k \in \{0\} \cup \mathbf{N}$, if X is locally i-connected (at p) for each $i \in \{0, 1, \dots, k\}$; and an LC^∞-*space* (at a point p) means an LC^k-space (at p) for every $k \in \{0\} \cup \mathbf{N}$.

Given a space X, we let 2^X denote the hyperspace of all nonempty compact subsets of X equipped with the *Vietoris topology*, see [**23**, p. 10] and [**14**, Definition 1.1, p. 3] defined as follows. Given a finite collection, $U_1, U_2, \dots, U_m$, of open sets of X, we define

$$\langle U_1, \dots, U_m \rangle = \left\{ A \in 2^X : A \subset \bigcup_{k=1}^{m} U_k \text{ and } A \cap U_k \neq \varnothing \text{ for each } k \in \{1, \dots, m\} \right\}.$$

Then the family of all subsets of 2^X of the form $\langle U_1, \dots, U_m \rangle$ forms a basis for the Vietoris topology for 2^X.

We denote by $C(X)$ the hyperspace of all subcontinua of X, i.e., of all connected elements of 2^X; further, for a given $n \in \mathbf{N}$, we let $C_n(X)$ denote the hyperspace of all elements of 2^X having at most n components. Thus $C(X) = C_1(X)$ and $C_n(X) \subset C_{n+1}(X)$ for each $n \in \mathbf{N}$, and we define $C_\infty(X) = \{A \in 2^X : A \text{ has finitely many components}\}$. Similarly, we denote by $F_n(X)$ the hyperspace of all elements of 2^X consisting of at most n points and define $F_\infty(X) = \{A \in 2^X : A \text{ is finite}\}$. Thus $C_\infty(X) = \cup\{C_n(X) : n \in \mathbf{N}\}$ and $F_\infty(X) = \cup\{F_n(X) : n \in \mathbf{N}\}$. All the hyperspaces mentioned above are considered as subspaces of the hyperspace 2^X (thus they are equipped with the inherited topology induced by the Vietoris topology).

Let us note the following statements.

Statement 0.3. *For each compact Hausdorff space X the hyperspace 2^X is compact and normal.*

Proof. Indeed, since X compact and Hausdorff, the hyperspace 2^X is also compact and Hausdorff, see [**8**, p. 244], and thus it is normal, see [**8**, Theorem 3.1.9, p. 125]. □

Statement 0.4. *For each compact Hausdorff space X and for each $n \in \mathbf{N}$ the hyperspace $C_n(X)$ is compact and normal.*

Proof. By the previous statement, the hyperspace 2^X is compact and normal. Since normality is a hereditary property with respect to closed subspaces, to complete the proof it is enough to show that $C_n(X)$ is a closed subspace of 2^X.

Let $A \in 2^X \setminus C_n(X)$. Then A has at least $n+1$ components. Thus there exist nonempty, closed and pairwise disjoint subsets $A_1, \dots, A_{n+1}$

of A such that $A = A_1 \cup \cdots \cup A_{n+1}$. Choose pairwise disjoint open subsets $U_1, \dots, U_{n+1}$ such that $A_i \subset U_i$ for each $i \in \{1, \dots, n+1\}$. Then $A \in \langle U_1, \dots, U_{n+1} \rangle \subset 2^X \setminus C_n(X)$. The proof is complete. $\square$

For each compact Hausdorff space X and for each $n \in \mathbf{N}$ we have $F_n(X) \subset C_n(X)$, whence $F_\infty(X) \subset C_\infty(X)$. Further, since each element of 2^X, i.e., each nonempty closed subset of X, can be approximated by finite sets, it follows that

(0.5) *for each compact Hausdorff space X the hyperspaces $F_\infty(X)$ and, consequently, $C_\infty(X)$, are dense subspaces of 2^X.*

To simplify notation, we put

$$\langle U_1, \dots, U_m \rangle_n = C_n(X) \cap \langle U_1, \dots, U_m \rangle$$
$$\langle U_1, \dots, U_m \rangle_\infty = C_\infty(X) \cap \langle U_1, \dots, U_m \rangle.$$

For each nonempty closed subset $\mathcal{A}$ of 2^X denote by $\cup\mathcal{A}$ the union of all elements of $\mathcal{A}$. Then $\cup : 2^{2^X} \to 2^X$ is a surjective mapping, see [**23**, Lemma 1.48, p. 100]; compare [**14**, Exercise 11.5, p. 91].

An *order arc* in a hyperspace $\mathcal{H}(X)$ is an arc $\alpha \subset \mathcal{H}(X)$ such that the partial ordering of containment for $\mathcal{H}(X)$ agrees on α with the total ordering on α, see [**14**, p. 110].

The reader is referred to monographs [**14**] and [**23**] as well as to papers [**17, 18**] and [**19**] for needed information about the above defined hyperspaces.

The property of Kelley has been originally introduced for metric continua, see [**15**, p. 26]; for more information see [**23**, p. 538 and Chapter 16] or [**14**, p. 167], where references for further results in this area are given. A pointed version of this property has been defined by Wardle in [**27**, p. 291]. The concept has been extended to compact Hausdorff spaces in [**2**, p. 210] and in [**20**, p. 124] as follows. A compact Hausdorff space X is said to have the *property of Kelley at a point* $p \in X$ provided that for each subcontinuum K of X containing the point p and for each open neighborhood $\mathcal{U}$ of K in $C(X)$ there is an open neighborhood V of p in X such that if $q \in V$, then there is a continuum L in X containing the point q and belonging to $\mathcal{U}$, i.e., such that $q \in L \in \mathcal{U}$. The space is said to have the *property of Kelley* provided that it has the property of Kelley at each of its points.

1. Local connectedness at a point in 2^X – summary of known results. In this section we collect, for a compact Hausdorff space X, some known equivalences that are related to the considered four variants of local connectivity, as local connectedness, strong local connectedness, local arcwise connectedness and strong local arcwise connectedness of the hyperspace 2^X at a point $A \in 2^X$.

The following three equivalences were proved by Goodykoontz in [**9**, Theorem 1, p. 388, Theorem 2, p. 390] and [**11**, Theorem 1, p. 42] originally for metric continua and extended to compact Hausdorff spaces in [**20**, Theorem 3, p. 122].

Theorem 1.1. *Let X be a compact Hausdorff space, and let $A \in C(X)$.*

(1.1.1) *The hyperspace 2^X is locally connected at A if and only if for each open set $U \subset X$ containing A there exists a connected set V such that $A \subset \mathrm{int}_X(V) \subset V \subset U$.*

(1.1.2) *The hyperspace 2^X is strongly locally connected at A if and only if for each open set $U \subset X$ containing A there exists an open connected set V such that $A \subset V \subset U$.*

(1.1.3) *The hyperspace 2^X is strongly locally arcwise connected at A if and only if for each open set $U \subset X$ containing A there exists an open set V such that $A \subset V \subset U$ and such that whenever B is a closed subset of V, there exists a continuum K such that $B \subset K \subset V$.*

Further, the following equivalences were proved by Goodykoontz in [**9**, Theorems 4 and 5, pp. 393, 395], respectively, and [**11**, Theorem 2, p. 43] again for metric continua, and, as it is observed in [**20**, p. 121], they remain true for compact Hausdorff spaces with almost the same proofs, see [**20**, Theorem 4, p. 122] (for local arcwise connectedness see [**20**, Theorem 8, p. 123]).

Theorem 1.2. *Let X be a compact Hausdorff space, and let $A \in 2^X$. The hyperspace 2^X is locally connected (strongly locally connected, locally arcwise connected, strongly locally arcwise connected) at a point A if and only if 2^X is locally connected (strongly locally connected, strongly locally arcwise connected, respectively) at each component of A.*

The following two equivalences were proved by Makuchowski in [**20**, Lemmas 5 and 6, pp. 122 and 123, respectively].

Theorem 1.3. *Let X be a compact Hausdorff space, and let $A \in C(X)$.*

(1.3.1) *The hyperspace 2^X is locally arcwise connected at A if and only if for each open set $U \subset X$ containing A there exists an open set V such that $A \subset V \subset U$ and such that whenever B is a closed set and $B \subset V$, there exists a continuum K such that $B \subset K \subset U$.*

(1.3.2) *For each open set U of X containing A there exists a connected set V such that $A \subset \mathrm{int}_X(V) \subset V \subset U$ if and only if for each open set U of X containing A there exists an open set V such that $A \subset V \subset U$ and such that whenever B is a closed set contained in V, there exists a continuum K such that $B \subset K \subset U$.*

Recall also the following result shown in [**20**, Theorem 9, p. 124].

Theorem 1.4. *Let X be a compact Hausdorff space and $A \in 2^X$. The hyperspace 2^X is locally connected at A if and only if it is locally arcwise connected at A.*

Concerning the above result let us mention that for the case of $C(X)$ there is an equivalence for local connectedness of $C(X)$ at an element $A \in C(X)$, see [**10**, Theorem 2, p. 358]. However, the following question remains open.

Question 1.5 (Goodykoontz, Jr. [**23**, Question 1.144, p. 156]). What are necessary and sufficient conditions for $C(X)$ to be strongly locally connected at $A \in C(X)$?

2. Local connectedness at a point in $C_n(X)$ – The general case. The present section contains results concerning the four variants, mentioned in Section 1, of local connectivity at a point for the hyperspace $C_n(X)$ of a compact Hausdorff space X in the general case when the space X is arbitrary, i.e., when no additional assumptions are needed on X. First, it is shown in Theorem 2.4 that local connectedness and local arcwise connectedness, both at a given point of $C_n(X)$, are equivalent. Second, it is proved that, for $n \geq 2$, local connectedness of C_n at $A \in C_{n-1}(X)$, i.e., when A has less than n components, is equivalent to this property of 2^X at each component of A, and that for $n = 1$ only one implication holds, Theorem 2.5 and Example 2.7. The four properties, now considered globally, are considered for the Cartesian products of a finite number of factors, Proposition 2.10 and Corollary 2.11; the results are applied to characterize the properties at a point $A \in C_n(X) \setminus C_{n-1}(X)$, i.e., when A has exactly n components, in Theorem 2.12. Further, Theorem 2.18 ties local connectedness of 2^X at a point with other variants of local connectedness of $C_n(X)$ at this point. Finally, in Theorem 2.21 and Corollary 2.22 it is shown that local connectedness and strong local connectedness at a point A of the hyperspace $C_n(X)$ are equivalent if A has less than n components. The same equivalence in case when A has exactly n components is discussed in the next section (it needs an additional assumption that the space X has the property of Kelley).

We start with three lemmas.

Lemma 2.1. *Let X be a compact Hausdorff space, $n \in \mathbf{N}$, and let a connected subset $\mathcal{B}$ of 2^X be such that $\mathcal{B} \cap C_n(X) \neq \varnothing$. Then $\cup\mathcal{B}$ has at most n components.*

Proof. Suppose on the contrary that $\cup\mathcal{B}$ has more than n components. Then there are $n+1$ nonempty and pairwise separated subsets $A_1, \dots, A_{n+1}$ of X such that $\cup\mathcal{B} = A_1 \cup \dots \cup A_{n+1}$. Fix an element $B \in \mathcal{B} \cap C_n(X)$. Then B has at most n components, and $B \subset A_1 \cup \dots \cup A_{n+1}$. Assume that $A_1, \dots, A_m$ are such that $A_i \cap B \neq \varnothing$ for each $i \in \{1, \dots, m\}$ and $A_i \cap B = \varnothing$ for each $i \in \{m+1, \dots, n+1\}$. Thus $m \leq n$. Let

$$\mathcal{K} = \{C \in \mathcal{B} : C \subset A_1 \cup \dots \cup A_m\}$$

and

$$\mathcal{L} = \{C \in \mathcal{B} : C \cap (A_{m+1} \cup \dots \cup A_{n+1}) \neq \varnothing\}.$$

Since $B \in \mathcal{K}$, the set $\mathcal{K}$ is nonempty. To see that also $\mathcal{L}$ is nonempty, fix a point $p \in A_{n+1} \subset \cup\mathcal{B}$. Then there is an element C of $\mathcal{B}$ such that $p \in C$. Thus $C \cap (A_{m+1} \cup \dots \cup A_{n+1}) \neq \varnothing$, so $C \in \mathcal{L}$, and thus $\mathcal{L} \neq \varnothing$.

Since $\cup\mathcal{B} = (A_1 \cup \cdots \cup A_m) \cup (A_{m+1} \cup \cdots \cup A_{n+1})$, it follows that $\mathcal{B} = \mathcal{K} \cup \mathcal{L}$.

We will show that the sets $\mathcal{K}$ and $\mathcal{L}$ are separated.

First, suppose on the contrary that there exists a $D \in \mathrm{cl}_{2^X}(\mathcal{K}) \cap \mathcal{L}$. Define $\mathcal{Z} = \{E \in 2^X : E \subset \mathrm{cl}_X(A_1 \cup \cdots \cup A_m)\}$ and notice that $\mathcal{Z}$ is closed in 2^X (by the definition of the Vietoris topology) and that $\mathcal{K} \subset \mathcal{Z}$. Therefore $\mathrm{cl}_{2^X}(\mathcal{K}) \subset \mathcal{Z}$, whence $D \in \mathcal{Z}$. On the other hand $D \in \mathcal{L}$, which implies that $D \cap (A_{m+1} \cup \cdots \cup A_{n+1}) \neq \varnothing$. It follows from these two conditions that $\mathrm{cl}_X(A_1 \cup \cdots \cup A_m) \cap (A_{m+1} \cup \cdots \cup A_{n+1}) \neq \varnothing$ which is a contradiction because the unions $A_1 \cup \cdots \cup A_m$ and $A_{m+1} \cup \cdots \cup A_{n+1}$ are separated.

Second, suppose on the contrary that there exists a $D \in \mathcal{K} \cap \mathrm{cl}_{2^X}(\mathcal{L})$. Define now $\mathcal{Z} = \{E \in 2^X : E \cap \mathrm{cl}_X(A_{m+1} \cup \cdots \cup A_{n+1}) \neq \varnothing\}$, and note that $\mathcal{Z}$ is closed in 2^X (again by the definition of the Vietoris topology) and that $\mathcal{L} \subset \mathcal{Z}$. Thus $\mathrm{cl}_{2^X}(\mathcal{L}) \subset \mathcal{Z}$, whence $D \in \mathcal{Z}$, i.e., $D \cap \mathrm{cl}_X(A_{m+1} \cup \cdots \cup A_{n+1}) \neq \varnothing$. On the other hand, $D \in \mathcal{K}$ means $D \subset A_1 \cup \cdots \cup A_m$, whence it follows that $\mathrm{cl}_X(A_{m+1} \cup \cdots \cup A_{n+1}) \cap (A_1 \cup \cdots \cup A_m) \neq \varnothing$, a contradiction again. Therefore $\mathcal{K}$ and $\mathcal{L}$ are separated, contrary to the connectedness of $\mathcal{B}$. The proof is complete. □

Lemma 2.2. *Let X be a compact Hausdorff space, $n \in \mathbf{N}$, and let a subcontinuum $\mathcal{B}$ of 2^X be such that $\mathcal{B} \cap C_n(X) \neq \varnothing$. Then $\cup\mathcal{B}$ is a closed subset of X with at most n components, i.e., $\cup\mathcal{B} \in C_n(X)$.*

Proof. Put $B = \cup\mathcal{B}$, and let $p \in \mathrm{cl}_X(B)$. For each closed neighborhood M of p in X, let $\mathcal{A}_M = \{A' \in \mathcal{B} : A' \cap M \neq \varnothing\}$. Then $\mathcal{A}_M$ is a closed, therefore a compact, subset of $\mathcal{B}$. If M and N are closed neighborhoods of p in X with $M \subset N$, then $\mathcal{A}_M \subset \mathcal{A}_N$. Further, each $\mathcal{A}_M$ is nonempty, because for a given M, the condition $M \cap B \neq \varnothing$ implies $M \cap A' \neq \varnothing$ for some $A' \in \mathcal{B}$, whence $A' \in \mathcal{A}_M$. Thus $\{\mathcal{A}_M : M \text{ is a closed neighborhood of } p\}$ is a family of compact subsets of $\mathcal{B}$ with the finite intersection property. Hence there exists an element $A_0 \in \mathcal{B}$ such that $A_0 \in \mathcal{A}_M$ for each M. Thus $p \in \mathrm{cl}_X(A_0) = A_0$. Therefore $p \in B$, and so B is closed in X. Finally, B has at most n components by Lemma 2.1. Thus $B \in C_n(X)$, as needed. The proof is complete. □

Lemma 2.3. *Let X be a compact Hausdorff space, $n \in \mathbf{N}$, and let α be an order arc in 2^X such that $\cap \alpha \in C_n(X)$. Then $\alpha \subset C_n(X)$.*

Proof. Take $A \in \alpha$, and define $\beta = \{B \in \alpha : B \subset A\}$. Then β is a subarc of α and $A = \cup \beta$. Then $A \in C_n(X)$ by Lemma 2.2, and so $\alpha \subset C_n(X)$. □

The following result extends [**20**, Theorem 10, p. 124]. Its proof is modeled on the corresponding proof in [**20**].

Theorem 2.4. *Let X be a compact Hausdorff space, $n \in \mathbf{N}$ and $A \in C_n(X)$. Then the following conditions are equivalent*:

(2.4.1) $C_n(X)$ *is locally connected at A*;

(2.4.2) $C_n(X)$ *is locally arcwise connected at A.*

Proof. The implication (2.4.2) $\Longrightarrow$ (2.4.1) is known, see [**20**, Observation 1, p. 120]. To show the opposite implication, assume that $C_n(X)$ is locally connected at A. Let $\mathcal{U} = \langle U_1, \dots , U_m\rangle_n$ be a neighborhood of A in $C_n(X)$ such that it is a member of a base in the Vietoris topology. By the local connectedness of $C_n(X)$ at A there is a continuum $\mathcal{K} \subset C_n(X)$ contained in $\mathcal{U}$ and such that $A \in \operatorname{int}_{C_n(X)}(\mathcal{K})$, compare (0.1). The set $\operatorname{int}_{C_n(X)}(\mathcal{K})$ is the open set required in the definition of local arcwise connectedness. Indeed, let $K = \cup \mathcal{K}$. Then by Lemma 2.2 it follows that K is a compact subset of X that has at most n components.

Observe that $K \subset U_1 \cup \dots \cup U_m$, and that for each $i \in \{1, \dots , m\}$ we have $\varnothing \neq A \cap U_i \subset K \cap U_i$. Hence $K \in \mathcal{U}$. Take $B \in \operatorname{int}_{C_n(X)}(\mathcal{K}) \subset \mathcal{K}$, whence $B \subset K$. Recall that $A \in \mathcal{K}$. We will show the following.

Claim. Given $B \in \mathcal{K}$, each component of K intersects B.

Suppose on the contrary that there exists a component C of K such that $C \cap B = \varnothing$. By the cut wire theorem, see [**24**, p. 72] for the metric formulation, and a remark following its proof, referring to compact Hausdorff spaces in [**24**, p. 82], there exist disjoint closed in X subsets L and M such that $K = L \cup M$, $C \subset L$ and $B \subset M$. Let

$\mathcal{M} = \{D \in \mathcal{K} : D \subset M\}$ and $\mathcal{L} = \{D \in \mathcal{K} : D \cap L \neq \varnothing\}$. Then $\mathcal{M}$ and $\mathcal{L}$ are closed, disjoint and nonempty subsets of $\mathcal{K}$ such that $\mathcal{K} = \mathcal{M} \cup \mathcal{L}$. This contradicts the connectedness of $\mathcal{K}$ and proves the claim.

Thus by the Claim each component of K intersects both A and B, and therefore, see [**14**, Theorem 15.3, p. 120], there are order arc α_1 from B to K and order arc α_2 from A to K. Then an arc joining A and B can easily be constructed in $\alpha_1 \cup \alpha_2 \subset \mathcal{U}$. The proof is complete. □

For another proof of the implication from (2.4.1) to (2.4.2) see Remark 2.23.

In the next two theorems we study several local connectivity conditions of the hyperspace $C_n(X)$ at its element A. We distinguish separately two cases, depending on whether the number of components of A is less than n (Theorem 2.5) or it is exactly n (Theorem 2.12).

Since the case of $n = 1$ has already been investigated in the past in a sequence of papers, see, e.g., [**9–12, 20, 21**], we may assume $n \geq 2$. However, it should be underlined that this assumption is indispensable in some results, so that a conclusion obtained for any $n \geq 2$ need not be true for $n = 1$. Each such case will be indicated and/or discussed separately.

Theorem 2.5. *Let X be a compact Hausdorff space, and let $n \in \mathbf{N}$ be fixed.*

(2.5.1) *If $n = 1$, let $A \in C(X)$. If 2^X is locally connected at A, then $C(X)$ is strongly locally arcwise connected at A.*

(2.5.2) *If $n \geq 2$, let $A \in C_{n-1}(X) \subset C_n(X)$. Then $C_n(X)$ is locally connected at A if and only if 2^X is locally connected at each component of A.*

Proof. The statement (2.5.1) was proved in [**10**, Theorem 1, p. 358] for the metric case; the proof for the general case is exactly the same.

To show (2.5.2) assume that $n \geq 2$, and let $A_1, \ldots, A_m$ be the different components of A, where $m \leq n - 1$.

Necessity. Assume that $C_n(X)$ is locally connected at A. Fix $j \in \{1, \dots, m\}$. In order to show that 2^X is locally connected at A_j, it is enough to see that if U is an open subset of X with $A_j \subset U$, then by (1.1.1) of Theorem 1.1 there exists a connected subset V of X such that $A_j \subset \operatorname{int}_X(V) \subset V \subset U$. Let $V_1, \dots, V_m$ be mutually disjoint open subsets of X such that $A_i \subset V_i$ for each $i \in \{1, \dots, m\}$ and $V_j \subset U$. Let $\mathcal{V} = \langle V_1, \dots, V_m \rangle_n$. Then $A \in \mathcal{V}$. Since $C_n(X)$ is locally connected at A, there exists a connected subset $\mathcal{B}$ of $C_n(X)$ such that $A \in \operatorname{int}_{C_n(X)}(\mathcal{B}) \subset \mathcal{B} \subset \mathcal{V}$.

Let $B = \operatorname{cl}_X(\cup \mathcal{B})$. Since $A \in \mathcal{B}$, the union $\cup \mathcal{B}$ has at most m components. Thus B has at most m components. For a given $C \in \mathcal{B}$ we have $C \in \langle V_1, \dots, V_m \rangle_n$, so $C \subset V_1 \cup \cdots \cup V_m$. Thus $A \subset B \subset V_1 \cup \cdots \cup V_m$. Hence B has at least m components. Therefore, B has exactly m components, and they are $B \cap V_1, \cdots, B \cap V_m$. Let $V = B \cap V_j$. Then $A_j \subset V \subset V_j \subset U$. So, we only need to prove that $A_j \subset \operatorname{int}_X(V)$.

Since $A \in \operatorname{int}_{C_n(X)}(\mathcal{B})$, there exists a basic open set $\mathcal{W} = \langle W_1, \dots, W_k \rangle_n$ in $C_n(X)$ such that $A \in \mathcal{W} \subset \mathcal{B}$. Let $W = W_1 \cup \cdots \cup W_k$. Then $A_j \subset V_j \cap W$. We will show that $V_j \cap W \subset V$. Let $p \in V_j \cap W$. Since $m < n$, it follows that $A \cup \{p\} \in C_n(X)$ and $A \cup \{p\} \in \langle W_1, \dots, W_k \rangle_n \subset \mathcal{B}$. Thus $A \cup \{p\} \subset B$ and $p \in B \cap V_j = V$. We have shown that $V_j \cap W \subset V$. Therefore $A \subset \operatorname{int}_X(V) \subset V \subset U$. The proof of the necessity is complete.

Sufficiency. Assume that 2^X is locally connected at each component $A_1, \dots, A_m$ of A. To show that $C_n(X)$ is locally connected at A take an open subset $\mathcal{U}$ of $C_n(X)$ such that $A \in \mathcal{U}$, and let $\mathcal{W} = \langle W_1, \dots, W_k \rangle_n$ be a basic open set in $C_n(X)$ such that $A \in \mathcal{W} \subset \mathcal{U}$. Put $W = W_1 \cup \cdots \cup W_k$. Let $V_1, \dots, V_m$ be open subsets of X such that $A_i \subset V_i \subset \operatorname{cl}_X(V_i) \subset W$ for each $i \in \{1, \dots, m\}$ and such that $\operatorname{cl}_X(V_1), \dots, \operatorname{cl}_X(V_m)$ are pairwise disjoint. By (1.1.1) of Theorem 1.1 for each $i \in \{1, \dots, m\}$, there exists a connected set C_i such that $A_i \subset \operatorname{int}_X(C_i) \subset C_i \subset V_i$. Let $\mathcal{Z} = \langle \operatorname{cl}_X(C_1), \dots, \operatorname{cl}_X(C_m) \rangle_n \cap \langle W_1, \dots, W_k \rangle_n$. Then $A \in \operatorname{int}_{C_n(X)}(\mathcal{Z}) \subset \mathcal{Z} \subset \mathcal{U}$. Each $B \in \mathcal{Z}$ intersects each of the components of $T = \operatorname{cl}_X(C_1) \cup \cdots \cup \operatorname{cl}_X(C_m)$ and $B \subset T$. Then, by [**14**, Theorem 15.3, p. 120], there exists an order arc α from B to T. It is easy to show that $\alpha \subset \mathcal{Z}$. Thus $\mathcal{Z}$ is a connected neighborhood of A, in $C_n(X)$, and therefore $C_n(X)$, is locally connected at A. The proof of the sufficiency is complete, and thus (2.5.2) is proved.

The proof is complete. □

Remark 2.6. (a) The implication in (2.5.1) has been shown for metric continua X in [**9**, Theorem 3, p. 390].

(b) The implication in (2.5.1) cannot be replaced by an equivalence, i.e., the converse implication is not true in general. The next example shows this.

Example 2.7. There is a metric continuum X and a subcontinuum A of X such that $C(X)$ is strongly locally arcwise connected at A while 2^X is not locally connected at A.

Proof. In the Cartesian coordinates in the plane put $v^- = (0, -1)$, $v^+ = (0, 1)$, and for each $n \in \mathbf{N}$, let $e_n^- = (-(1/n), 0)$ and $e_n^+ = ((1/n), 0)$. For any two points p and q let $\overline{pq}$ stand for the straight line segment from p to q. Define

$$X = \left(\bigcup\{\overline{v^- e_n^-} \cup \overline{v^+ e_n^+} : n \in \mathbf{N}\}\right) \bigcup \{\overline{v^- v^+}\}.$$

Thus X is the one-point union of two harmonic fans with the (only) accumulation points of their sets of end points identified. The reader can verify that X has the need properties for $A = [-1/2, 1/2] \times \{0\} \subset \overline{v^- v^+}$. □

Let X be a compact Hausdorff space and $n \in \mathbf{N}$. Accept the following notation. Let $(C(X))^n$ be the Cartesian product of n copies of the space $C(X)$. Define a function

$$\text{(2.8)} \quad \varphi : (C(X))^n \longrightarrow C_n(X) \quad \text{by} \quad \varphi(A_1, \dots, A_n) = A_1 \cup \dots \cup A_n.$$

To prove the next result we need a lemma on properties of the function φ and a proposition on various concepts of local connectedness in the Cartesian products.

Lemma 2.9. *The above defined function φ has the following properties.*

(2.9.1) φ *is continuous.*

(2.9.2) *For each* $n \in \mathbf{N}$ *with* $n \geq 2$ *and for each* $A_0 \in \varphi^{-1}(C_n(X) \setminus C_{n-1}(X))$, *there exists an open subset* $\mathcal{Y}$ *of* $(C(X))^n$ *such that* $A_0 \in \mathcal{Y}$, *the set* $\varphi(\mathcal{Y})$ *is open in* $C_n(X)$, *and the restriction* $\varphi|\mathcal{Y} : \mathcal{Y} \to \varphi(\mathcal{Y})$ *is a homeomorphism.*

Proof. To prove (2.9.1) let $B_0 = (B_1, \dots, B_n) \in (C(X))^n$, and let $\mathcal{U} = \langle U_1, \dots, U_k \rangle_n$ be a basic open set of $C_n(X)$ such that $\varphi(B_0) \in \mathcal{U}$. Put $B = \varphi(B_0) = B_1 \cup \dots \cup B_n$. For each $i \in \{1, \dots, n\}$, define $J_i = \{j \in \{1, \dots, k\} : B_i \cap U_j \neq \varnothing\}$. Let

$$\mathcal{U}_i = \Big\{D \in C(X) : D \subset \bigcup\{U_j : j \in J_i\} \text{ and } D \cap U_j \neq \varnothing \text{ for each } j \in J_i\Big\}.$$

Finally put $\mathcal{U}_0 = \mathcal{U}_1 \times \dots \times \mathcal{U}_n$. Since the sets $U_1, \dots, U_k$ are open by the definition, it follows that each $\mathcal{U}_i$, for $i \in \{1, \dots, n\}$, is open by the definition of the Vietoris topology, and thus $\mathcal{U}_0$ is open in $(C(X))^n$. Observe that $B_0 \in \mathcal{U}_0$. It is easy to show that $\varphi(\mathcal{U}_0) \subset \mathcal{U}$. Therefore φ is continuous.

To show (2.9.2) let $A_0 = (A_1, \dots, A_n) \in \varphi^{-1}(C_n(X) \setminus C_{n-1}(X)) \subset (C(X))^n$, and put $A = \varphi(A_0) = A_1 \cup \dots \cup A_n$. Then $A_1, \dots, A_n$ are different components of A. Choose open subsets $U_1, \dots, U_n$ of X such that $A_i \subset U_i$ for each $i \in \{1, \dots, n\}$ and that $\mathrm{cl}_X(U_1), \dots, \mathrm{cl}_X(U_n)$ are pairwise disjoint. Let $\mathcal{W} = \mathrm{cl}_{C(X)}\langle U_1 \rangle_1 \times \dots \times \mathrm{cl}_{C(X)}\langle U_n \rangle_1$. Then $\mathcal{W}$ is a compact neighborhood of A_0 in $(C(X))^n$. Let $\mathcal{V} = \langle U_1, \dots, U_n \rangle_n$. Clearly, $A \in \mathcal{V} \subset \varphi(\mathcal{W})$. Thus $A \in \mathrm{int}_{C_n(X)}(\varphi(\mathcal{W}))$. It is easy to show that $\varphi \mid \mathcal{W} : \mathcal{W} \to \varphi(\mathcal{W})$ is one-to-one. Therefore $\varphi \mid \mathcal{W}$ is a homeomorphism. Put $\mathcal{Y} = \varphi^{-1}(\mathrm{int}_{C_n(X)}(\varphi(\mathcal{W})))$, and observe that it satisfies the required conditions. □

Proposition 2.10. *Let* X *and* Y *be compact Hausdorff spaces. Then the product* $X \times Y$ *is locally connected* (*strongly locally connected, locally arcwise connected, strongly locally arcwise connected*) *if and only if the factors* X *and* Y *are locally connected* (*strongly locally connected, locally arcwise connected, strongly locally arcwise connected, respectively*).

Proof. We will argue for the case of strongly locally arcwise connected spaces; the proof for other cases is quite similar.

Necessity. Assume that $X \times Y$ is strongly locally arcwise connected. It ought to be proved that X is such, too (the argument for Y is the same). To this aim take a point $x \in X$ and an open subset U of X with $x \in U$. Let $\pi_1 : X \times Y \to X$ be the natural projection. Fix $y \in Y$. Then $(x, y) \in \pi_1^{-1}(U)$, and the set $\pi_1^{-1}(U)$ is open in $X \times Y$. Since $X \times Y$ is strongly locally arcwise connected, there exists an arcwise connected and open in $X \times Y$ subset W such that $(x, y) \in W \subset \pi_1^{-1}(U)$. Projecting under π_1 we get $x \in \pi_1(W) \subset \pi_1(\pi_1^{-1}(U)) = U$. Since the projection π_1 is an open mapping, $\pi_1(W)$ is open in X. We have to show that $\pi_1(W)$ is arcwise connected. So, take points $\pi_1(u), \pi_1(v) \in \pi(W)$, where $u, v \in W$. Since W is arcwise connected, there is an arc α in W joining u and v. Thus $\pi_1(u), \pi_1(v) \in \pi_1(\alpha)$. Since the continuum $\pi_1(\alpha)$ is a continuous image of the arc α, it follows from [**26**, Theorem 9, p. 201] that it is arcwise connected. Thus there exists an arc $\beta \subset \pi_1(\alpha) \subset \pi_1(W)$ joining $\pi_1(u)$ and $\pi_1(v)$. Therefore $\pi_1(W)$ is arcwise connected, and the proof of the necessity is finished.

Sufficiency. Assume that X and Y are strongly locally arcwise connected. Take $(p, q) \in X \times Y$ and a (basic) open set of the form $P \times Q$ in $X \times Y$ such that $(p, q) \in P \times Q$, P is open in X and Q is open in Y. By the assumption there are subsets U of X and V of Y, both arcwise connected, such that U is open in X, V is open in Y, with $p \in U \subset P$ and $q \in V \subset Q$. Then their product $U \times V$ is open in $X \times Y$, arcwise connected and satisfies $(p, q) \in U \times V \subset P \times Q$, as needed.

The proof is complete. □

Applying finite induction, Proposition 2.10 leads to the following corollary.

Corollary 2.11. *Let $n \in \mathbf{N}$ be fixed and, for each $i \in \{1, \ldots, n\}$, let the space X_i be compact and Hausdorff. Then the product $X_1 \times \cdots \times X_n$ is locally connected* (*strongly locally connected, locally arcwise connected, strongly locally arcwise connected*) *if and only if each of the factors X_i is locally connected* (*strongly locally connected, locally arcwise connected, strongly locally arcwise connected, respectively*).

Now we are ready to prove the previously mentioned result.

Theorem 2.12. *Let X be a compact Hausdorff space, $n \in \mathbf{N}$ fixed, and let $A \in C_n(X) \setminus C_{n-1}(X)$ have $A_1, \ldots, A_n$ as its different components. Put $A_0 = (A_1, \ldots, A_n) \in (C(X))^n$. Then the following are equivalent*:

(2.12.1) *$C_n(X)$ is locally connected (strongly locally connected, locally arcwise connected, strongly locally arcwise connected, respectively) at A,*

(2.12.2) *$(C(X))^n$ is locally connected (strongly locally connected, locally arcwise connected, strongly locally arcwise connected, respectively) at A_0,*

(2.12.3) *$C(X)$ is locally connected (strongly locally connected, locally arcwise connected, strongly locally arcwise connected, respectively) at each A_i for $i \in \{1, \ldots, n\}$.*

Proof. Note that if $n = 1$ then the symbol $C_{n-1}(X)$ is not defined and it can be taken as the empty set. In this case the conditions (2.12.1), (2.12.2) and (2.12.3) coincide, so it is nothing to prove. Therefore we can assume $n \geq 2$.

Let φ be as in (2.8). Then $A = \varphi(A_0) = A_1 \cup \cdots \cup A_n$, so the equivalence between (2.12.1) and (2.12.2) is immediate from Lemma 2.9. The equivalence between (2.12.2) and (2.12.3) is an obvious consequence of Corollary 2.11. The proof is finished. □

Note that in Theorem 2.5 local connectedness of $C_n(X)$ at A is characterized by local connectedness of 2^X at each component of A. In Theorem 2.12 however we cannot substitute 2^X for $C(X)$, as Example 2.7 shows.

To prove the next result we introduce a notation and prove a lemma.

Let X be a compact Hausdorff space. Fix an arbitrary $n \in \mathbf{N}$. For each open subset $\mathcal{U}$ of 2^X and for each $m \in \{1, \ldots, n\}$ let

$$\mathcal{U}[m, n] = \{A \in C_n(X) : \text{there exist nonempty pairwise disjoint closed and open subsets } A_1, \ldots, A_m \text{ of } A \text{ such that } A_1 \cup \cdots \cup A_m \in \mathcal{U}\}.$$

Lemma 2.13. *The set $\mathcal{U}[m,n]$ is open in $C_n(X)$.*

Proof. Let $A \in \mathcal{U}[m,n]$ and let $A_1, \dots, A_m$ be as in the definition of $\mathcal{U}[m,n]$. Put $A_0 = A_1 \cup \dots \cup A_m$. Since $\mathcal{U}$ is open in 2^X, there exist open subsets $U_1, \dots, U_k$ of X such that $A_0 \in \langle U_1, \dots, U_k\rangle \subset \mathcal{U}$. Put $U = U_1 \cup \dots \cup U_k$. Let $V_1, \dots, V_m, V_{m+1}$ be pairwise disjoint open subsets of X such that $A_1 \subset V_1, \dots, A_m \subset V_m$, $A \setminus A_0 \subset V_{m+1}$ and $V_1 \cup \dots \cup V_m \subset U$.

For each $i \in \{1, \dots, m\}$, let $J_i = \{j \in \{1, \dots, k\} : A_i \cap U_j \neq \varnothing\}$. Notice that $J_i \neq \varnothing$ for each i.

Let

$$\mathcal{V} = \{B \in C_n(X) : B \subset V_1 \cup \dots \cup V_{m+1},\ B \cap V_{m+1} \neq \varnothing \text{ and } B \cap V_i \cap U_j \neq \varnothing \text{ for all } i \in \{1, \dots, m\} \text{ and } j \in J_i\} \quad \text{if } A \setminus A_0 \neq \varnothing,$$

$$\mathcal{V} = \{B \in C_n(X) : B \subset V_1 \cup \dots \cup V_m \text{ and } B \cap V_i \cap U_j \neq \varnothing \text{ for all } i \in \{1, \dots, m\} \text{ and } j \in J_i\} \quad \text{if } A \setminus A_0 = \varnothing.$$

It can easily be proved that, in both cases, $\mathcal{V}$ is open in $C_n(X)$ and $A \in \mathcal{V} \subset \mathcal{U}[m,n]$. Thus $\mathcal{U}[m,n]$ is open. □

Let again X be a compact Hausdorff space; let $A \in C(X)$, and assume that the hyperspace 2^X is locally connected at A. Fix $n \in \mathbf{N}$, and let $\mathcal{U} = \langle U_1, \dots, U_k\rangle_n$ be a basic open set in $C_n(X)$ such that $A \in \mathcal{U}$. Define $U = U_1 \cup \dots \cup U_k$ and $\mathcal{U}' = \langle U_1, \dots, U_k\rangle \subset 2^X$. By (1.1.1) of Theorem 1.1, there exist closed connected subsets $Q_{-1}, Q_0, Q_1, \dots, Q_n, Q_{n+1}$ of X such that

$$A \subset \operatorname{int}_X(Q_{-1}) \subset Q_{-1} \subset \operatorname{int}_X(Q_0) \subset Q_0 \subset \operatorname{int}_X(Q_1) \subset Q_1 \subset \dots \subset \operatorname{int}_X(Q_n) \subset Q_n \subset \operatorname{int}_X(Q_{n+1}) \subset Q_{n+1} \subset U.$$

Further, define

$$\begin{aligned} V_i &= \operatorname{int}_X(Q_i) && \text{for each} \quad i \in \{-1, 0, 1, \dots, n, n+1\}, \\ W_i &= V_i \setminus Q_{i-1} && \text{for each} \quad i \in \{0, 1, \dots, n+1\}, \\ Z_i &= V_{i+1} \setminus Q_{i-2} && \text{for each} \quad i \in \{1, \dots, n\}, \end{aligned}$$

and, for each $i \in \{1, \dots, n-1\}$, define

$$\mathcal{U}_i = \langle Z_i, W_i\rangle[1, n] \quad \cap (\langle V_{-1}\rangle \cap \mathcal{U}')[i, n] \cap \langle V_{-1} \cup Z_i\rangle_n,$$
$$\mathcal{V}_i = \langle V_i, V_{i-1}\rangle[i+1, n] \cap (\langle V_{-1}\rangle \cap \mathcal{U}')[i, n] \cap \langle V_{i+1}\rangle_n.$$

Finally, put $\mathcal{U}_0 = \langle V_0, V_1\rangle_n \cap \mathcal{U}$, and let

$$\mathcal{V} = \mathcal{U}_0 \cup \mathcal{U}_1 \cup \dots \cup \mathcal{U}_{n-1} \cup \mathcal{V}_1 \cup \dots \cup \mathcal{V}_{n-1}. \tag{2.14}$$

In particular, if $n = 1$, then $\mathcal{V} = \mathcal{U}_0$.

Proposition 2.15. *Let X be a compact Hausdorff space, and let 2^X be locally connected at $A \in C(X)$. Under the above notation, let $\mathcal{V}$ be defined by* (2.14). *Then for each element $B \in \mathcal{V}$ there exists an arc α joining B and A in $\mathcal{V}$ with the property that for each $C \in \alpha$ the number of components of C is less than or equal to the number of components of B.*

Proof. Note that $A \in \mathcal{U}_0$, whence $A \in \mathcal{V}$ by (2.14). According to Lemma 2.13 the set $\mathcal{V}$ is open in $C_n(X)$.

Take an element $B \in \mathcal{U}_i \cup \mathcal{V}_i$ for some $i \in \{1, \dots, n-1\}$ and recall that $B \in (\langle V_{-1}\rangle \cap \mathcal{U}')[i, n]$. Therefore B contains a closed subset which belongs to $\mathcal{U}'$. Thus B intersects each U_j for $j \in \{1, \dots, k\}$. Moreover, $B \subset V_{i+1} \subset U$. Hence $B \in \mathcal{U}$. Since $\mathcal{U}_0 \subset \mathcal{U}$, it follows that $\mathcal{V} \subset \mathcal{U}$.

Accept the following notation. Given two subsets $\mathcal{R}$ and $\mathcal{S}$ of $\mathcal{V}$, we write $\mathcal{R} \rightrightarrows \mathcal{S}$ to indicate that for each element $R \in \mathcal{R}$ there exist an element $S \in \mathcal{S}$ and an arc $\alpha \subset \mathcal{V}$ that joins R and S such that for each $C \in \alpha$ the number of components of C is less than or equal to the number of components of R. Thus, to prove the proposition, it is enough to show the following:

$$\mathcal{U}_0 \rightrightarrows \mathcal{U}_1 \cup \{A\}, \quad \mathcal{U}_1 \rightrightarrows \mathcal{V}_1, \quad \mathcal{V}_1 \rightrightarrows \mathcal{U}_2 \cup \{A\}, \quad \mathcal{U}_2 \rightrightarrows \mathcal{V}_2, \quad \dots,$$
$$\mathcal{U}_{n-2} \rightrightarrows \mathcal{V}_{n-2}, \quad \mathcal{V}_{n-2} \rightrightarrows \mathcal{U}_{n-1} \cup \{A\}, \quad \mathcal{U}_{n-1} \rightrightarrows \mathcal{V}_{n-1}, \quad \mathcal{V}_{n-1} \rightrightarrows \{A\}.$$

First, we show that $\mathcal{U}_0 \rightrightarrows \mathcal{U}_1 \cup \{A\}$. Let $B \in \mathcal{U}_0$. Then $B \subset V_1$, $B \cap V_0 \neq \varnothing$ and $B \in \mathcal{U}$. Denote by C the union of all components of B that intersect Q_0, and put $D = B \setminus C$. Thus $B = C \cup D$. Let $C_1 = Q_0 \cup C$. Then $C_1 \in C(X)$. Let α be an order arc in $C_n(X)$ from C to C_1, and let β be an order arc in $C(X)$ from A to C_1. Put $\gamma = \{D \cup E : E \in \alpha \cup \beta\}$. Then γ is the union of two order arcs, one joining $B = C \cup D$ to $C_1 \cup D$, and the other joining $C_1 \cup D$ to $A \cup D$. Thus it is possible to find an arc $\delta \subset \gamma$ which joins B to $A \cup D$. By Lemma 2.3 the number of components of each element of $\alpha \cup \beta$ is less than or equal to the number of components of B. It is easy to show that $\gamma \subset \mathcal{U}_0 \subset \mathcal{V}$. If $D = \varnothing$, then δ joins B to A. Otherwise $A \cup D \in \mathcal{U}_1$. So, we have shown that $\mathcal{U}_0 \rightrightarrows \mathcal{U}_1 \cup \{A\}$.

Now we show that $\mathcal{U}_i \rightrightarrows \mathcal{V}_i$ for each $i \in \{1, \dots, n-1\}$. Let $B \in \mathcal{U}_i$. Since V_{-1} and Z_i are open and disjoint, B is of the form $B = C \cup D$, where C and D are closed, disjoint and nonempty, and

$$C \subset V_{-1}, \quad D \subset Z_i, \quad D \cap W_i \neq \varnothing,$$
$$C \text{ has at least } i \text{ components, and } C \in \mathcal{U}'.$$

Define D_1 (D_2 and D_3, respectively) as the union of all components of D that intersect Q_i and are not contained in V_i (are contained in V_i, are disjoint with Q_i, respectively). Then $D = D_1 \cup D_2 \cup D_3$ and $D \cap W_i \neq \varnothing$ implies that $D_1 \cup D_2 \neq \varnothing$. Consider two cases.

Case 1. $D_1 = \varnothing$. In this case D_2 is a nonempty subset of V_i. Thus $C \cup D_2 \in \langle V_i, V_{i-1} \rangle$ and $C \cup D_2$ has at least $i+1$ components. Since $B = C \cup D \subset V_{i+1}$, it follows that $B \in \mathcal{V}_i$.

Case 2. $D_1 \neq \varnothing$. Let $E_1, \dots, E_r$ be all the different components of D_1. Each E_k intersects Q_i and is not contained in V_i. Thus we may choose a point $p_k \in (E_k \cap Q_i) \setminus V_i$. Let Y_1 and Y_2 be open subsets of X such that

$$Q_{i-2} \subset Y_1 \subset \mathrm{cl}_X(Y_1) \subset V_{i-1} \subset Q_{i-1} \subset Y_2 \subset \mathrm{cl}_X(Y_2) \subset V_i.$$

For each $k \in \{1, \dots, r\}$, let F_k be the component of $Q_i \setminus Y_1$ that contains p_k. Then $F_k \in C(X)$ and, by the boundary bumping theorem, see [**23**, Theorem 20.2, p. 626], there exists a point $q_k \in F_k \cap \mathrm{cl}_{Q_i}(Y_1) = F_k \cap \mathrm{cl}_X(Y_1) \subset F_k \cap V_{i-1}$. Let G_k be the component of $F_k \cap \mathrm{cl}_X(Y_2)$ that contains q_k. Thus $G_k \in C(X)$. Notice that $p_k \in F_k \setminus \mathrm{cl}_X(Y_2)$. So, we can again apply the same boundary bumping theorem to conclude that $\varnothing \neq G_k \cap \mathrm{bd}_{F_k}(F_k \cap \mathrm{cl}_X(Y_2)) \subset G_k \cap \mathrm{bd}_X(\mathrm{cl}_X(Y_2)) \subset G_k \cap \mathrm{bd}_X(Y_2) \subset G_k \cap W_i$. Thus $G_k \cap W_i \neq \varnothing$. Note that $F_k \subset V_{i+1} \setminus Q_{i-2} \subset Z_i$. Put

$$F = \bigcup \{F_k : k \in \{1, \dots, r\}\} \quad \text{and} \quad G = \bigcup \{G_k : k \in \{1, \dots, r\}\},$$

and observe that $G \subset D_1 \cup F$.

Let α (β) be an order arc from D_1 to $D_1 \cup F$ (from G to $D_1 \cup F$, respectively), and put $\gamma = \{C \cup L \cup D_2 \cup D_3 \in C_n(X) : L \in \alpha \cup \beta\}$. It is easy to show that there is an arc $\delta \subset \gamma$ that joins $B = C \cup D_1 \cup D_2 \cup D_3$ to the set $B_1 = C \cup G \cup D_2 \cup D_3$. Notice that $G \cup D_2$ is a nonempty subset of $(\mathrm{cl}_X(Y_2) \setminus Y_1) \cup D_2 \subset V_i \setminus Q_{i-2}$. Thus $C \cup G \cup D_2$ has at least $i+1$ components and belongs to $\langle V_i, V_{i-1} \rangle$. Therefore $B_1 \in \mathcal{V}_i$. Notice that, for each $L \in \gamma$, we have

$$\begin{aligned} L &\subset C \cup D_1 \cup F \cup D_2 \cup D_3 \subset B \cup F \subset B \cup (Q_i \setminus Y_1) \\ &\subset B \cup Z_i \subset V_{-1} \cup Z_i, \end{aligned}$$

and L contains D_1 or $C \cup G \cup D_2 \cup D_3$, so $L \cap W_i \neq \varnothing$. It implies that $L \in \mathcal{U}_i$. Thus we have connected, by an arc in $\mathcal{V}$, the set B to $B_1 \in \mathcal{V}_i$. This completes the proof of $\mathcal{U}_i \rightrightarrows \mathcal{V}_i$.

Now we show that $\mathcal{V}_i \rightrightarrows \mathcal{U}_{i+1} \cup \{A\}$ for each $i \in \{1, \dots, n-2\}$. Let $B \in \mathcal{V}_i$. Let $\mathcal{C}$ ($\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, respectively) be the set of components of B that are contained in V_{-1} (that intersect V_{i-1} but are not contained in V_{-1}, intersect Q_{i-1} but do not intersect V_{i-1}, intersect Q_i but do not intersect Q_{i-1}, do not intersect Q_i, respectively). Some of the sets $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ may be empty. Since $B \in \mathcal{V}_i$, the set $\mathcal{C}$ has at least i elements, $\cup\mathcal{C} \in \mathcal{U}'$, $\mathcal{C} \cup \mathcal{D}$ has at least $i+1$ elements, and $(\cup\mathcal{F}) \cup (\cup\mathcal{G}) \subset Z_i$. For each $L \in \mathcal{D}$, choose a point $p_L \in (L \cap V_{i-1}) \setminus V_{-1}$, and for each $L \in \mathcal{E}$, choose a point $p_L \in L \cap Q_{i-1}$.

Let $B_1 = (\cup\mathcal{C}) \cup \{p_L : L \in \mathcal{D} \cup \mathcal{E}\} \cup (\cup\mathcal{F}) \cup (\cup\mathcal{G})$. Since $B_1 \subset B$ and each component of B intersects B_1, there exists an order arc α from B_1 to B. It is easy to show that each element of α belongs to $\mathcal{V}_i$. Therefore we have joined B to B_1 by an arc in $\mathcal{V}$.

Let Y_1 be an open subset of X such that $\cup\mathcal{C} \subset Y_1 \subset \mathrm{cl}_X(Y_1) \subset V_{-1}$. For each $L \in \mathcal{D} \cup \mathcal{E}$, let M_L be the component of $Q_{i-1} \setminus Y_1$ containing

p_L. By the boundary bumping theorem, see [**23**, Theorem 20.2, p. 626], there exists a point $q_L \in M_L \cap \mathrm{cl}_{Q_{i-1}}(Y_1) = M_L \cap \mathrm{cl}_X(Y_1) \subset V_{-1}$. Let

$$B_2 = (\cup\mathcal{C}) \cup \left(\bigcup\left\{M_L : L \in \mathcal{D} \cup \mathcal{E}\right\}\right) \cup (\cup\mathcal{F}) \cup (\cup\mathcal{G}),$$
$$B_3 = (\cup\mathcal{C}) \cup \big(\{q_L : L \in \mathcal{D} \cup \mathcal{E}\}\big) \cup (\cup\mathcal{F}) \cup (\cup\mathcal{G}).$$

Then $B_1 \cup B_3 \subset B_2$, and each component of B_2 intersects both B_1 and B_3. Thus there are order arcs β and γ from B_1 to B_2 and from B_3 to B_2, respectively. It is easy to check that $\beta \cup \gamma \subset \mathcal{V}_i$. Therefore we can join B_1 to B_3 by an arc in $\mathcal{V}$. Notice that $B_3 \in (\langle V_{-1}\rangle \cap \mathcal{U}')[i+1, n]$ and $B_3 \subset V_{-1} \cup (V_{i+1} \setminus Q_{i-1})$. Further, $B_3 = C_1 \cup D_1$, where C_1 and D_1 are closed subsets of X such that $C_1 \subset V_{-1}$, $C_1 \in \mathcal{U}'$, C_1 has at least $i+1$ components and $D_1 = (\cup\mathcal{F}) \cup (\cup\mathcal{G}) \subset V_{i+1} \setminus Q_{i-1}$.

Let Y_2 be an open subset of X such that $\mathrm{cl}_X(Y_1) \subset Y_2 \subset \mathrm{cl}_X(Y_2) \subset V_{-1}$. For each $L \in \mathcal{F}$ choose a point $x_L \in L \cap Q_i$. Let $B_4 = C_1 \cup \{x_L : L \in \mathcal{F}\} \cup (\cup\mathcal{G})$. Then $B_4 \subset B_3$ and each component of B_3 intersects B_4. Thus there exists an order arc δ from B_4 to B_3. Notice that $\delta \subset \mathcal{V}_i$. Therefore we have joined B_3 to B_4 by an arc in $\mathcal{V}$.

For each $L \in \mathcal{F}$, let N_L be the component of $Q_i \setminus Y_2$ that contains x_L. By the boundary bumping theorem, see [**23**, Theorem 20.2, p. 626], there exists a point $y_L \in N_L \cap \mathrm{cl}_{Q_i}(Y_2) = N_L \cap \mathrm{cl}_X(Y_2) \subset V_{-1}$. Let

$$B_5 = C_1 \cup \left(\bigcup\{N_L : L \in \mathcal{F}\}\right) \cup (\cup\mathcal{G}),$$
$$B_6 = C_1 \cup \big(\{y_L : L \in \mathcal{F}\}\big) \cup (\cup\mathcal{G}).$$

Then $B_4 \cup B_6 \subset B_5$ and each component of B_5 intersects both B_4 and B_6. Thus there exist order arcs ζ and η from B_4 to B_5 and from B_6 to B_5, respectively. Note that $\zeta \cup \eta \subset \mathcal{V}_i$. Thus we can join B_4 to B_6 by an arc in $\mathcal{V}$. Observe that $B_6 = C_3 \cup D_3$, where C_3 and D_3 are closed subsets of X such that $C_3 \subset V_{-1}$, C_3 has at least $i+1$ components, $C_3 \in \mathcal{U}'$ and $D_3 \subset V_{i+1} \setminus Q_i$.

Consider two cases.

Case 1. $D_3 = \varnothing$. In this case $B_6 = C_3 \in \langle V_0, V_1\rangle_n \cap \mathcal{U}$ and $B_6 \subset V_{-1}$. Let κ and λ be order arcs from B_6 to Q_{-1} and from

A to Q_{-1}, respectively. Clearly, each element of $\kappa \cup \lambda$ belongs to $\langle V_0, V_1\rangle_n \cap \mathcal{U} \subset \mathcal{V}$. Thus it is possible to join B_6 to A by an arc in $\mathcal{V}$.

Case 2. $D_3 \neq \varnothing$. In this case $B_6 \in \mathcal{U}_{i+1}$.

This finishes the proof of $\mathcal{V}_i \rightrightarrows U_{i+1} \cup \{A\}$.

Finally, we intend to show that $V_{n-1} \rightrightarrows \{A\}$.

Let $B \in \mathcal{V}_{n-1} = \langle V_{n-1}, V_{n-2}\rangle[n,n] \cap (\langle V_{-1}\rangle \cap \mathcal{U}')[n-1,n] \cap \langle V_n\rangle_n$. Define C_4 as the union of all components of B that are contained in V_{-1}, and let $D_4 = B \setminus C_4$. Then C_4 has at least $n-1$ components. Consider three cases.

Case 1. $D_4 = \varnothing$. In this case $B = C_4 \in \langle V_0, V_1\rangle_n \cap \mathcal{U}$ and $B \subset V_{-1}$. Let ξ, σ be order arcs from B to Q_{-1} and from A to Q_{-1}, respectively. Clearly, each element of $\xi \cup \sigma$ belongs to $\langle V_0, V_1\rangle_n \cap \mathcal{U} \subset \mathcal{V}$. Thus it is possible to join B to A by an arc in $\mathcal{V}$.

Case 2. $D_4 \neq \varnothing$ and $D_4 \cap V_{-1} \neq \varnothing$. In this case D_4 is a subcontinuum of X and $B \subset V_{n-1}$. Fix a point $p \in D_4 \cap V_{-1}$. Then it is possible to find an order arc from $C_4 \cup \{p\}$ to B. Observe that each element of such an arc is in $\mathcal{V}_{n-1}$. Since $C_4 \cup \{p\} \in \mathcal{V}_{n-1}$ and $C_4 \cup \{p\} \subset V_{-1}$, we can proceed as in Case 1 to join $C_4 \cup \{p\}$ to A by an arc in $\mathcal{V}$.

Case 3. $D_4 \neq \varnothing$ and $D_4 \cap V_{-1} = \varnothing$. In this case D_4 is a subcontinuum of X, $B \subset V_{n-1}$ and $D_4 \cap V_{n-2} \neq \varnothing$. Fix a point $q \in (D_4 \cap V_{n-2}) \setminus V_{-1}$. Let Y_3 be an open subset of X such that $C_4 \subset Y_3 \subset \mathrm{cl}_X(Y_3) \subset V_{-1}$. Let M be the component of $Q_{n-2} \setminus Y_3$ that contains q. By the boundary bumping theorem, see [**23**, Theorem 20.2, p. 626], there exists a point $x \in M \cap \mathrm{cl}_{Q_{n-2}}(Y_3) = M \cap \mathrm{cl}_X(Y_3)$. Using respective order arcs from $C_4 \cup \{q\}$ to B, from $C_4 \cup \{q\}$ to $C_4 \cup M$ and from $C_4 \cup \{x\}$ to $C_4 \cup M$, it is possible to join B to $C_4 \cup \{x\}$ by an arc in $\mathcal{V}_{n-1}$. Since $C_4 \cup \{x\} \in \mathcal{V}_{n-1}$ and $C_4 \cup \{x\} \subset V_{-1}$, we can proceed as in Case 1 to join $C_4 \cup \{x\}$ to A by an arc in $\mathcal{V}$.

With this we finish Case 3 and the proof of $\mathcal{V}_{n-1} \rightrightarrows \{A\}$.

Therefore the proof of the proposition is complete. $\square$

The following corollary is a consequence of Proposition 2.15.

Corollary 2.16. *Let X be a compact Hausdorff space, 2^X locally connected at $A \in C(X)$ and, for a fixed $n \in \mathbf{N}$, let $\mathcal{U} = \langle U_1, \dots, U_k \rangle_n$ be a basic open set in $C_n(X)$ such that $A \in \mathcal{U}$. If $\mathcal{V}$ is defined by* (2.14), *then $A \in \mathcal{V} \subset \mathcal{U}$ and $\mathcal{V}$ is an arcwise connected open subset of $C_n(X)$.*

Corollary 2.16 leads to the following result, that generalizes [**10**, Theorem 2, p. 358] as well as implication (2.5.1) of Theorem 2.5.

Theorem 2.17. *Let X be a compact Hausdorff space, and let 2^X be locally connected at $A \in C(X)$. Then, for each $n \in \mathbf{N}$, the hyperspace $C_n(X)$ is strongly locally arcwise connected at A.*

The converse implication to that of Theorem 2.17 is also true. Moreover, we have the following result.

Theorem 2.18. *Let X be a compact Hausdorff space, and let $A \in C(X)$. Then the following conditions are equivalent*:

(2.18.1) *the hyperspace 2^X is locally connected at A*;

(2.18.2) *for each $n \in \mathbf{N}$ the hyperspace $C_n(X)$ is strongly locally arcwise connected at A*;

(2.18.3) *there exists an integer $n \geq 2$ such that $C_n(X)$ is strongly locally arcwise connected at A*;

(2.18.4) *for each $n \in \mathbf{N}$ the hyperspace $C_n(X)$ is locally connected at A*;

(2.18.5) *there exists an integer $n \geq 2$ such that $C_n(X)$ is locally connected at A.*

Proof. (2.18.1) $\Longrightarrow$ (2.18.2) is Theorem 2.17; the implications (2.18.2) $\Longrightarrow$ (2.18.3) $\Longrightarrow$ (2.18.5) and (2.18.2) $\Longrightarrow$ (2.18.4) $\Longrightarrow$

(2.18.5) are immediate; and finally (2.18.5) $\Longrightarrow$ (2.18.1) by Theorems 2.5 and 1.2. □

Recall the following well-known example.

Example 2.19. There exist a metric continuum X and a point $p \in X$ such that:

(2.19.1) X is locally connected at p;

(2.19.2) X is not strongly locally connected at p;

(2.19.3) 2^X is locally connected at $\{p\}$;

(2.19.4) 2^X is not strongly locally connected at $\{p\}$.

Proof. In the Cartesian coordinates in the plane put $p = (0, 0)$ and, for each $k \in \mathbf{N}$, let H_k be the cone with the vertex $v_k = ((1/k), 0)$ over the set $E_k = \{v_{k+1}\} \cup \cup\{(1/(k+1), (1/i)) : i \in \{k+1, k+2, \dots, \}\}$. Then each H_k is homeomorphic to the harmonic fan. The union

$$X = \{p\} \cup \bigcup\{H_k : k \in \mathbf{N}\}$$

is the needed continuum. It is pictured in [**13**, Figures 3–9, p. 113] and in [**24**, Figure 5.22, p. 84], where assertions (2.19.1) and (2.19.2) are shown. Assertions (2.19.3) and (2.19.4) are consequences of the previous two by [**9**, Corollaries 1 and 2, pp. 389 and 390], respectively. □

Remarks 2.20. (a) In (2.18.1) the local connectedness of 2^X cannot be replaced by the strong local connectedness because of Example 2.19.

(b) Example 2.7 shows that we cannot put $n = 1$ in (2.18.3) and (2.18.5).

In Theorem 2.17 the strong local arcwise connectedness of the hyperspace $C_n(X)$ at an element A was established if $A \in C(X) \subset C_n(X)$. Now we are going to prove a similar result for an arbitrary $A \in C_n(X)$. However, the situation differs depending on either $A \in C_{n-1}(X) \subset C_n(X)$ or $A \in C_n(X) \setminus C_{n-1}(X)$. The latter case needs an additional assumption (that X has the property of Kelley), and it will be discussed in the next section.

Theorem 2.21. *Let X be a compact Hausdorff space, and let $n \in \mathbf{N}$ with $n \geq 2$ fixed. If the hyperspace $C_n(X)$ is locally connected at $A \in C_{n-1}(X) \subset C_n(X)$, then $C_n(X)$ is strongly locally arcwise connected at A.*

Proof. For some $m \leq n-1$, let $A_1, \dots, A_m$ be the different components of A. By Theorem 2.5 the hyperspace 2^X is locally connected at each A_i, and by Theorem 2.17 the hyperspace $C_n(X)$ is strongly locally arcwise connected also at each A_i. Let $V_1, \dots, V_m$ be open mutually disjoint subsets of X such that $A_i \subset V_i$ for each $i \in \{1, \dots, m\}$, and $\mathcal{U} = \langle U_1, \dots, U_k \rangle_n$ be a basic open set in $C_n(X)$ such that $A \in \mathcal{U}$. For each $i \in \{1, \dots, m\}$ let $F_i = \{j \in \{1, \dots, m\} : A_i \cap U_j \neq \varnothing\}$ and

$$\mathcal{U}_i = \Big\{ B \in C_n(X) : B \cap U_j \neq \varnothing \text{ for each } j \in F_i \text{ and } B \subset \bigcup \{U_j : j \in F_i\} \Big\} \cap \langle V_i \rangle_n.$$

Then $\mathcal{U}_i$ is an open subset of $C_n(X)$, $A_i \in \mathcal{U}_i$, and $\{1, \dots, m\} = F_1 \cup \dots \cup F_m$. By Corollary 2.16 there exists an open subset $\mathcal{V}_i$ of $C_n(X)$ such that $A_i \in \mathcal{V}_i \subset \mathcal{U}_i$ and, if $B \in \mathcal{V}_i$, then there exists an arc α joining B and A_i in $\mathcal{V}_i$ having the property that for each element $C \in \alpha$ the number of components of C is less than or equal to the number of components of B, see Proposition 2.15.

Let $\mathcal{V} = \{B \in \langle V_1, \dots, V_m \rangle_n : B \cap V_i \in \mathcal{V}_i \text{ for each } i \in \{1, \dots, m\}\}$. We are going to prove that $\mathcal{V}$ is an open arcwise connected subset of $C_n(X)$ such that $A \in \mathcal{V} \subset \mathcal{U}$. Clearly, $A \in \mathcal{V}$.

Given $B \in \mathcal{V}$ and $i \in \{1, \dots, m\}$, let $B_i = B \cap V_i$. Since $B_i \in \mathcal{V}_i$ and $\mathcal{V}_i$ is open in $C_n(X)$, there exists a basic open subset $\mathcal{W}_i = \langle W_1^{(i)}, \dots, W_{k_i}^{(i)} \rangle_n$ of $C_n(X)$ such that $B_i \in \mathcal{W}_i \subset \mathcal{V}_i$. For each $i \in \{1, \dots, m\}$ and each $j \in \{1, \dots, k_i\}$ let $Z_j^{(i)} = W_j^{(i)} \cap V_i$.

Let $\mathcal{W} = \langle Z_1^{(1)}, \dots, Z_{k_1}^{(1)}; Z_1^{(2)}, \dots, Z_{k_2}^{(2)}; \dots; Z_1^{(m)}, \dots, Z_{k_m}^{(m)} \rangle_n$. Therefore $\mathcal{W}$ is open in $C_n(X)$ and $B \in \mathcal{W}$. For $C \in \mathcal{W}$ and $i \in \{1, \dots, m\}$ let $C_i = C \cap V_i$. Since $\varnothing \neq C \cap Z_1^{(i)} \subset C \cap V_i$, it follows that

$C_i \neq \varnothing$. Note that $C \subset V_1 \cup \cdots \cup V_m$. So each C_i is closed, and $C \in \langle V_1, \ldots, V_m \rangle_n$. For each $i \in \{1, \ldots, m\}$ and each $j \in \{1, \ldots, k_i\}$ we have $\varnothing \neq C \cap Z_j^{(i)} \subset C_i \cap W_j^{(i)}$. Given a point $p \in C_i \subset V_i$, there exist $l \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, k_l\}$ such that $p \in Z_j^{(l)} \subset V_l$. Since $V_1, \ldots, V_m$ are pairwise disjoint, we get $l = i$. Thus $C_i \subset Z_1^{(i)} \cup \cdots \cup Z_{k_i}^{(i)}$. Hence $C_i \in \mathcal{W}_i \subset \mathcal{V}_i$. This proves that $C \in \mathcal{V}$. So, we have shown that $\mathcal{W} \subset \mathcal{V}$. Since $\mathcal{W}$ is open in $C_n(X)$ and $B \in \mathcal{W} \subset \mathcal{V}$, it follows that $\mathcal{V}$ is open in $C_n(X)$.

Now we show that $\mathcal{V}$ is arcwise connected. Let $B \in \mathcal{V}$. For each $i \in \{1, \ldots, m\}$ let $B_i = B \cap V_i$. Then $B_i \in \mathcal{V}_i$. By the definition of $\mathcal{V}_i$ there exists an arc α_i joining B_i and A_i in $\mathcal{V}_i$ having the property that for each element C of α_i the number of components of C is less than or equal to the number of components of B_i. Let $\sigma_1 = \{(B \setminus V_1) \cup C : C \in \alpha_1\}$. Then σ_1 is an arc in $C_n(X)$ (since the number of components of each element of σ_1 is less than or equal to the number of components of B) that joins B to $(B \setminus V_1) \cup A_1$. Note that each element of σ_1 belongs to $\mathcal{V}$. Using a similar argument it is possible to join $(B \setminus V_1) \cup A_1$ to $(B \setminus (V_1 \cup V_2)) \cup A_1 \cup A_2$ by an arc in $\mathcal{V}$. Repeating this procedure it is possible to join B to A in $\mathcal{V}$. Therefore $\mathcal{V}$ is arcwise connected.

Given $B \in \mathcal{V}$ and $i \in \{1, \ldots, m\}$, let (as previously) $B_i = B \cap V_i$. Since $B_i \in \mathcal{V}_i \subset \mathcal{U}_i$, it follows that $B \subset U_1 \cup \cdots \cup U_k$. For each $j \in \{1, \ldots, k\}$ there exists $i \in \{1, \ldots, m\}$ such that $j \in F_i$, whence $B_i \cap U_j \neq \varnothing$. Therefore $B \in \mathcal{U}$. So, we have shown that $\mathcal{V} \subset \mathcal{U}$. This completes the proof. □

Corollary 2.22. *Let X be a compact Hausdorff space, $n \in \mathbf{N}$ with $n \geq 2$, and let $A \in C_{n-1}(X) \subset C_n(X)$. Then the hyperspace $C_n(X)$ is locally connected at A if and only if $C_n(X)$ is strongly locally arcwise connected at A.*

Remark 2.23. As it was mentioned earlier, the implication from (2.4.1) to (2.4.2) in the proof of Theorem 2.4 can be obtained as a consequence of Theorem 2.21 (for the case that $A \in C_{n-1}(X)$), Theorem 2.12 and [**20**, Theorem 10, p. 124].

3. Local connectedness at a point in $C_n(X)$ – the property of Kelley. The next results concerning local connectedness at a point in $C_n(X)$ need an additional assumption on X, namely the property of Kelley. We start with a result that is a continuation of our investigation from the previous section. Namely, in Theorem 2.21, the implication was shown from local connectedness to strong local arcwise connectedness of the hyperspace $C_n(X)$ at its element A provided that $A \in C_{n-1}(X)$. Now we are going to prove a similar result if $A \in C_n(X) \setminus C_{n-1}(X)$. But this will be done under an additional assumption that the considered space X has the property of Kelley. As we will see in Example 3.4, this assumption is essential for the result. Finally, a characterization of the property of Kelley in terms of hyperspace is obtained in Theorem 3.5.

Theorem 3.1. *Let X be a compact Hausdorff space having the property of Kelley, and let $n \in \mathbf{N}$. If the hyperspace $C_n(X)$ is locally connected at $A \in C_n(X) \setminus C_{n-1}(X)$ (for $n = 1$ we assume $A \in C(X)$), then $C_n(X)$ is strongly locally arcwise connected at A.*

Proof. Note again that if $n = 1$ the symbol $C_{n-1}(X)$ can simply be omitted, and that, in this case, the result is just Theorem 12 of [**20**, p. 125]. Thus we can consider the case $n \geq 2$ only.

Let $A_1, \dots, A_n$ be the different components of A. Since $C_n(X)$ is locally connected at A, it follows from Theorem 2.12 that $C(X)$ is locally connected at each A_i. Further, since X has the property of Kelley, $C(X)$ is strongly locally arcwise connected at each A_i by [**20**, Theorem 12, p. 125]. Applying again Theorem 2.12 we conclude that $C_n(X)$ is strongly locally arcwise connected at A, as needed. The proof is complete. □

Corollary 3.2. *Let X be a compact Hausdorff space having the property of Kelley, and let $n \in \mathbf{N}$. Then the hyperspace $C_n(X)$ is locally connected at $A \in C_n(X) \setminus C_{n-1}(X)$ (for $n = 1$ we assume $A \in C(X)$) if and only if $C_n(X)$ is strongly locally arcwise connected at A.*

Corollaries 2.22 and 3.2 imply the next one.

Corollary 3.3. *Let X be a compact Hausdorff space having the property of Kelley, and let $n \in \mathbf{N}$ be fixed. Then the hyperspace $C_n(X)$ is locally connected at $A \in C_n(X)$ if and only if $C_n(X)$ is strongly locally arcwise connected at A.*

As it was said before, the property of Kelley for X is an indispensable assumption in Theorem 3.1. This can be seen by the following example.

Example 3.4. There exists a continuum X which does not have the property of Kelley and which contains, for each integer $n \geq 2$, a subset $A \in C_n(X) \setminus C_{n-1}(X)$ such that $C_n(X)$ is locally connected, while not strongly locally arcwise connected, at A.

Proof. In [**10**, Example 3, p. 361] a metric continuum X and its subcontinuum M are constructed such that

(a) X does not have the property of Kelley,

(b) $C(X)$ is locally connected at M,

(c) $C(X)$ is not strongly locally connected at M.

In the set $X \setminus M$ choose $n-1$ distinct points $p_2, \dots, p_n$ such that

(d) X is locally connected at each p_i for $i \in \{2, \dots, n\}$,

and define $A = M \cup \{p_2, \dots, p_n\}$. Putting $A_1 = M$ and $A_i = \{p_i\}$ for each $i \in \{2, \dots, n\}$, we see that A_i (for $i \in \{1, 2, \dots, n\}$) are the components of A. Thus $A \in C_n(X) \setminus C_{n-1}(X)$. It follows from (b) and (d) that $C(X)$ is locally connected at each component of A. Hence the equivalence of conditions (2.12.1) and (2.12.3), for the local connectedness, in Theorem 2.12 implies that $C_n(X)$ is locally connected at A. Further, according to the same equivalence (now for the strong local arcwise connectedness) condition (c) implies that $C_n(X)$ is not strongly locally arcwise connected at A. The proof is complete. □

In the next theorem we present a characterization of the property of Kelley in terms of hyperspaces. The result is an extension of [**20**, Theorem 11, p. 125].

Theorem 3.5. *Let X be a compact Hausdorff space, and let $n \in \mathbf{N}$. Then the following conditions are equivalent*:

(3.5.1) *X has the property of Kelley*;

(3.5.2) *for each $n \in \mathbf{N}$ the union of any open subset of $C_n(X)$ is an open subset of X*;

(3.5.3) *there is an $n \in \mathbf{N}$ such that the union of any open subset of $C_n(X)$ is an open subset of X*;

(3.5.4) *the union of any open subset of $C(X)$ is an open subset of X.*

Proof. We will show the following circle of implications:

$$(3.5.1) \implies (3.5.2) \implies (3.5.3) \implies (3.5.4) \implies (3.5.1).$$

The first implication is easy to check. Let $\mathcal{U}$ be an open subset of $C_n(X)$, and let $x \in \cup\mathcal{U}$. Thus there is an $A \in \mathcal{U}$ such that $x \in A$. Since X has the property of Kelley at x, there exists an open subset V of X such that $x \in V$, and for each point $q \in V$ there is a subcontinuum L of X with $q \in L \in C(X) \cap \mathcal{U} \subset \mathcal{U}$. Therefore $V \subset \cup\mathcal{U}$. Hence $\cup\mathcal{U}$ is open. Thus (3.5.2) follows.

The implication (3.5.2) $\implies$ (3.5.3) is obvious.

To prove that (3.5.3) $\implies$ (3.5.4) we may assume that $n \geq 2$. Let $\mathcal{U}$ be an open subset of $C(X)$, and let $U = \cup\mathcal{U}$. Take a point $p \in U$. Then there exists $A \in \mathcal{U}$ such that $p \in A$. If $A = X$, then $X = A \subset U \subset X$, whence $U = X$. Therefore there is a neighborhood of p contained in U, and we are done. So, we may assume that $A \neq X$. Fix points $q_2, \dots, q_n \in X \setminus A$, and choose pairwise disjoint open subsets $V_1, V_2, \dots, V_n$ in X such that $A \subset V_1$ and $q_i \in V_i$ for each $i \in \{2, \dots, n\}$. Since $\mathcal{U}$ is open in $C(X)$ and $A \in \mathcal{U}$, there exists a basic neighborhood $\mathcal{U}_0 = \langle U_1, \dots, U_m\rangle_1$ of A in $C(X)$ such that $A \in \mathcal{U}_0 \subset \mathcal{U} \cap \langle V_1\rangle_1$. Let $\mathcal{U}_1 = \langle U_1 \cap V_1, \dots, U_m \cap V_1, V_2, \dots, V_n\rangle_n$, and put $W = \cup\mathcal{U}_1$. By hypothesis W is open in X. Further, since $A \cup \{q_2, \dots, q_n\} \in \mathcal{U}_1$ and $p \in A \subset A \cup \{q_2, \dots, q_n\}$, we infer that $p \in W \cap V_1$. Let $x \in W \cap V_1$. Then there exists $B \in \mathcal{U}_1$ such that $x \in B$. Thus $x \in B \cap V_1$. Since $B \in \mathcal{U}_1$, we have $B \subset V_1 \cup V_2 \cup \dots \cup V_n$ and B intersects each one of the sets $V_1, V_2, \dots, V_n$. Consequently, B has exactly n components, namely $B \cap V_1, B \cap V_2, \dots, B \cap V_n$. Since $B \in \mathcal{U}_1$, it follows that $B \cap V_1 \in \langle U_1 \cap V_1, \dots, U_m \cap V_1\rangle_1 \subset \mathcal{U}_0 \subset \mathcal{U}$.

Thus $x \in B \cap V_1 \subset U$. So, we have shown that $W \cap V_1 \subset U$. Therefore $p \in W \cap V_1 \subset U$. This completes the proof that U is open in X. Hence (3.5.4) follows.

Finally, the implication (3.5.4) $\Longrightarrow$ (3.5.1) is shown in [**20**, Theorem 11, p. 125]. The proof is complete. $\square$

4. Local k-connectedness at a point in $C_n(X)$. In Theorem 2.4 it was proved that, for compact Hausdorff spaces and for any $n \in \mathbf{N}$, local connectedness and local arcwise connectedness of the hyperspace $C_n(X)$ at a point are equivalent. In the present section we show that for compact *metric* spaces the above properties are equivalent to another one, viz. to LC^k-connectedness for each non-negative integer k, i.e., to LC^∞ at the considered point, Corollary 4.2.

The following result is an extension of [**21**, Theorem 2, p. 30].

Theorem 4.1. *Let X be a compact metric space, $n \in \mathbf{N}$ and $A \in C_n(X)$. Then the hyperspace $C_n(X)$ is locally connected at the point A if and only if $C_n(X)$ is LC^∞ at A.*

Proof. We have to prove that for $C_n(X)$ local connectedness at A is equivalent to being LC^k at A for each $k \in \{0\} \cup \mathbf{N}$. If $k = 0$, then local connectedness of $C_n(X)$ at A is equivalent to local arcwise connectedness at A by Theorem 2.1 above, thus to local connectedness of $C_n(X)$ at A at dimension 0, see (0.2). Therefore we may assume that $k \in \mathbf{N}$. It is shown in [**21**, Theorem 1, p. 30] that local k-connectedness of a space at a point implies local connectedness of the space at this point. Therefore it remains to prove the opposite implication only.

Let the hyperspace $C_n(X)$ be locally connected at $A \in C_n(X)$, and let k be a positive integer. We show that $C_n(X)$ is locally k-connected at A. For some positive integer $r \leq n$, let $A_1, \dots, A_r$ be the different components of A. Consider two cases.

Case 1. $r < n$, i.e., $A \in C_{n-1}(X) \subset C_n(X)$. Let $\mathcal{U} = \langle U_1, \dots, U_m \rangle_n$ be a basic element in the Vietoris topology in $C_n(X)$ such that $A \in \mathcal{U}$. Let $V_1, \dots, V_r$ be pairwise disjoint open subsets of X such that $A_i \subset V_i$ for each $i \in \{1, \dots, r\}$. Put $U = U_1 \cup \dots \cup U_m$. Since $C_n(X)$ is locally connected at A, by Theorem 2.5 and [**20**, Theorem 3, p. 122] for each $i \in \{1, \dots, r\}$, there exists a closed connected subset K_i of X such that $A_i \subset \text{int}_X(K_i) \subset K_i \subset V_i \cap U$. We claim that the set $\mathcal{V} = \langle \text{int}_X(K_1), \dots, \text{int}_X(K_r)\rangle_n \cap \mathcal{U}$ is the open set required in the definition of local k-connectedness. Let $K = K_1 \cup \dots \cup K_r$. Notice that $A \subset K$ and $K \in \mathcal{U}$.

Take a mapping f from the k-sphere $\mathbf{S}_k$ to $\mathcal{V}$ and observe that $S = \cup f(\mathbf{S}_k)$ is compact and it has at most n components according to Lemma 2.2. Notice that $S \in \mathcal{V}$ and S intersects each set K_i. Thus there exists an order arc from S to K in $C_n(X)$. Since $A \subset K$ and A intersects each K_i, there exists, again in $C_n(X)$, an order arc from A to K. Hence there exist mappings $\alpha : [0, (1/3)] \to C_n(X)$ and $\beta : [(1/3), (2/3)] \to C_n(X)$ such that $\alpha(0) = A$, $\alpha(1/3) = K$, $\beta(1/3) = K$, $\beta(2/3) = S$; and if $0 \le s \le t \le 1/3$, then $\alpha(s) \subset \alpha(t)$ and, if $1/3 \le s \le t \le 2/3$, then $\beta(t) \subset \beta(s)$.

Now, recalling that $\mathbf{B}_{k+1}$ stands for the $(k+1)$-ball having $\mathbf{S}_k$ as its boundary, we construct an extension $f^* : \mathbf{B}_{k+1} \to \mathcal{U}$ of the mapping f. Namely, for each $x \in \mathbf{B}_{k+1}$ we define

$$f^*(x) = \begin{cases} \alpha(\|x\|) & \text{if } 0 \le \|x\| \le 1/3, \\ \beta(\|x\|) & \text{if } 1/3 \le \|x\| \le 2/3, \\ \cup f(\mathbf{S}_k \cap N((x/\|x\|), 6(1-\|x\|))) & \text{if } 2/3 \le \|x\| \le 1. \end{cases}$$

We see that f^* is continuous, $f^*(0) = A$ and $f^*(x) = f(x)$ for each $x \in \mathbf{S}_k$.

Case 2. $r = n$, i.e., $A \in C_n(X) \setminus C_{n-1}(X)$. It is easy to show that a finite product is locally k-connected at a point if and only if each one of the factors is locally k-connected at the projections of the point. So, this case follows from Lemma 2.9, Theorem 2.12 and [**21**, Theorem 2, p. 30]. The proof is complete. □

Theorems 2.4 and 4.1 can be summarized as follows.

Corollary 4.2. *Let X be a compact metric space, $n \in \mathbf{N}$ and $A \in C_n(X)$. Then the following conditions are equivalent*:

(2.4.1) *$C_n(X)$ is locally connected at A*;

(2.4.2) $C_n(X)$ *is locally arcwise connected at* A;

(4.2.1) $C_n(X)$ *is* LC^∞ *at* A.

Remark 4.3. In connection with the equivalences of Corollary 4.2, recall that each of local arcwise connectedness as well as LC^∞ of a space at a point implies local connectedness of the space at the point (the former implication is obvious; the latter one is shown in [**21**, Theorem 1, p. 30]). The opposite implications are not true in general, see [**21**, Remark b), p. 33].

It is tempting to have the equivalence of Theorem 4.1 (and ones in Corollary 2.2, too) not only for metric spaces, but in general, if X is a compact *Hausdorff* space. However, it is not the case, as can be seen from the example below.

Example 4.4. If X denotes the long segment, and if $n \in \mathbf{N}$, then

(4.4.1) $C_n(X)$ is locally connected at X, while

(4.4.2) $C_n(X)$ is not LC^0 at X.

Proof. Let ω_1 stand for the least uncountable ordinal, and let Ω denote the (ordered) set of all ordinal numbers $\alpha \leq \omega_1$. For each ordinal $\xi \leq \omega_1$ we consider the open (closed) ordinal space defined respectively by

$$\Gamma(\xi) = \{\alpha \in \Omega : \alpha < \xi\} \quad \text{and} \quad \Gamma(\xi] = \{\alpha \in \Omega : \alpha \leq \xi\}$$

equipped with the order topology. Then the *long segment* X, see [**8**, p. 237]; it is called the *extended long line* in [**25**, Example 46, p. 71] is defined as the space obtained from the closed ordinal space $\Gamma(\omega_1] = \Omega$ by placing between each ordinal α and its successor $\alpha + 1$ a copy of the open unit interval $(0, 1)$. Then X is linearly ordered, and we give it the order topology. Further, for each $y \in X$ we consider open and closed initial segments:

$$[0, y) = \{x \in X : x < y\} \quad \text{and} \quad [0, y] = \{x \in X : x \leq y\}.$$

Thus, in particular, we have $[0, \omega_1] = X$.

To show (4.4.1) note that by (1.1.1) of Theorem 1.1 the hyperspace 2^X is locally connected at its point X, whence it follows by Theorem 2.17 that $C_n(X)$ is strongly locally arcwise connected at X. So (4.4.1) is satisfied.

To prove (4.4.2) suppose on the contrary that $C_n(X)$ is LC^0 at X. Let $\mathcal{U} = \langle X \rangle_n$. Then $\mathcal{U}$ is open in $C_n(X)$ and $X \in \mathcal{U}$. According to the definition of LC^0 at X there exists an open subset $\mathcal{V}$ of $C_n(X)$ such that $X \in \mathcal{V} \subset \mathcal{U}$ and that each mapping $f : S^0 = \{-1, 1\} \to \mathcal{V}$ has a continuous extension $f^* : [-1, 1] \to \mathcal{U}$ of f with the property $f^*(0) = X$.

Since $\mathcal{V}$ is open, there is an ordinal number $\gamma < \omega_1$ such that $[0, \gamma] \in \mathcal{V}$. Let $f : S^0 = \{-1, 1\} \to \mathcal{V}$ be given by $f(-1) = [0, \gamma] = f(1)$. By the choice of $\mathcal{U}$ there exists a mapping $f^* : [-1, 1] \to \mathcal{U}$ such that $f^*(-1) = [0, \gamma] = f^*(1)$ and $f^*(0) = X$. For each $t \in [0, 1]$, define $G_t = \cup\{f^*(s) : s \in [t, 1]\}$, and let $r = \sup\{t \in [0, 1] : G_t = X\}$. Then $X = f^*(0) \subset G_0$, whence $G_0 = X$. Thus r is well defined.

Further, $G_1 = f^*(1) = [0, \gamma] \in \langle [0, \gamma + 1) \rangle_n$, which is an open set. So, by the continuity of f^*, there exists $t_0 \in [0, 1)$ such that $f^*(s) \in \langle [0, \gamma + 1) \rangle_n$ for each $s \in [t_0, 1]$. Therefore $G_s \subset [0, \gamma + 1)$, whence $G_s \neq X$ for each $s \in [t_0, 1]$. Thus $r \leq t_0 < 1$.

So, we can choose a strictly decreasing sequence $\{r_m\}$ such that $r < \cdots < r_2 < r_1 < 1$ with $\lim r_m = r$. Then, by Lemma 2.2, for each $m \in \mathbf{N}$ the set G_{r_m} is a proper subcontinuum of X containing 0 (because $0 \in [0, \gamma] \subset G_{r_m}$). Therefore there is an ordinal number $\gamma_m < \omega_1$ such that $G_{r_m} \subset [0, \gamma_m]$. Consequently, there is $\lambda < \omega_1$ such that $\gamma_m < \lambda$, whence $G_{r_m} \subset [0, \lambda]$ for each $m \in \mathbf{N}$.

Note that for any $t \in (r, 1]$ there is an index $m(t) \in \mathbf{N}$ such that $r_m < t$ for each $m \in \mathbf{N}$ with $m > m(t)$. This implies that $f^*(t) \subset G_{r_m}$ for each $m > m(t)$. Then, by the continuity of f^* we conclude that $f^*(t) \subset [0, \lambda]$ for each $t \in [r, 1]$.

In particular, we get $f^*(r) \subset [0, \lambda] \subset [0, \lambda + 1)$, which in an open subset of X. Thus $f^*(r) \in \langle [0, \lambda + 1) \rangle_n$. Since $f^*(0) = X$, we conclude that $0 < r$. Therefore there is a number $q \in (0, r)$ such that $f^*(s) \in \langle [0, \lambda + 1) \rangle_n$ for each $s \in [q, r]$. Then $f^*(s) \subset [0, \lambda + 1)$ for each $s \in [q, 1)$, which implies that $G_q \neq X$.

On the other hand, by the definition of supremum, there is $t \in (q, r]$ such that $G_t = X$. But $G_t \subset G_q$ implies $G_q = X$. This contradiction completes the proof. □

5. Local connectedness at a point in $C_\infty(X)$. In this section it is shown that, for each compact Hausdorff space X, local connectedness, local arcwise connectedness, and strong local connectedness of $C_\infty(X)$ at a point are equivalent to the corresponding properties of 2^X, Theorems 5.2 and 5.3, while for strong local arcwise connectedness only one implication is true: from 2^X to $C_\infty(X)$, Theorem 5.5 and Example 5.6. Further, a characterization of strong local arcwise connectedness of $C_\infty(X)$ at a point is obtained in Theorem 5.4.

We start with a lemma that concerns local connectedness of a regular, i.e., a T_3, space at a point of a dense subspace. The lemma is not related to the hyperspace theory. Its proof is quite standard, and it is attached here for the sake of completeness only.

Lemma 5.1. *Let D be a dense subspace of a regular space X, and let $p \in D$. Then the following implications hold.*

(5.1.1) *If D is locally connected at p, then X is locally connected at p, too.*

(5.1.2) *If D is strongly locally connected at p, then X is strongly locally connected at p, too.*

Proof. To show (5.1.1) assume that D is locally connected at p. Let U and V be open subsets of X such that $p \in V \subset \mathrm{cl}_X(V) \subset U$. Since D is locally connected at p, there exists a connected subset K of D such that $p \in \mathrm{int}_D(K) \subset K \subset V \cap D$. Put $W = \mathrm{int}_D(K)$. Then there exists an open subset Z of X such that $W = Z \cap D$. Since D is dense in X, it follows that $W \subset Z \subset \mathrm{cl}_X(W) \subset \mathrm{cl}_X(K) \subset \mathrm{cl}_X(V) \subset U$. So, $\mathrm{cl}_X(K)$ is a connected neighborhood of p in X.

To show (5.1.2) assume that D is strongly locally connected at p. Let U and V be open subsets of X such that $p \in V \subset \mathrm{cl}_X(V) \subset U$. Since D is strongly locally connected at p, there exists an open connected subset W of D such that $p \in W \subset V \cap D$. Let Z be an open subset of X such that $W = Z \cap D$. Since D is dense in X, it follows that $Z \subset \mathrm{cl}_X(W)$. So $W \subset Z \subset \mathrm{cl}_X(W)$. Thus Z is an open connected subset of X such that $p \in Z \subset \mathrm{cl}_X(W) \subset \mathrm{cl}_X(V) \subset U$. Therefore X is strongly locally connected at p, as needed. □

Theorem 5.2. *Let X be a compact Hausdorff space and $A \in C_\infty(X)$. Then the following conditions are equivalent*:

(5.2.1) 2^X *is locally connected at* A;

(5.2.2) 2^X *is locally arcwise connected at* A;

(5.2.3) $C_\infty(X)$ *is locally connected at* A;

(5.2.4) $C_\infty(X)$ *is locally arcwise connected at* A.

Proof. Since the equivalence between (5.2.1) and (5.2.2) has been shown in [**20**, Theorem 9, p. 124], see Theorem 1.4 above, it is enough to show the following circle of implications:

$$(5.2.4) \implies (5.2.3) \implies (5.2.1) \implies (5.2.4).$$

The implication (5.2.4) $\implies$ (5.2.3) is clear. Since 2^X is normal, thus regular, by Statement 0.3 and since $C_\infty(X)$ is its dense subspace according to (0.5), the implication (5.2.3) $\implies$ (5.2.1) is a consequence of implication (5.1.1) of Lemma 5.1. To close the circle of implications it remains to prove that (5.2.1) implies (5.2.4).

So, assume (5.2.1). To show (5.2.4) let $\mathcal{U} = \langle U_1, \dots, U_m \rangle_\infty$ be a basic open set in $C_\infty(X)$ such that $A \in \mathcal{U}$. Let $A_1, \dots, A_n$ be the components of A. Choose pairwise disjoint open subsets $V_1, \dots, V_n$ such that $A_i \subset V_i \subset U_1 \cup \dots \cup U_m$ for each $i \in \{1, \dots, n\}$. By Theorem 1.2 the hyperspace 2^X is locally connected at each component A_i. Applying normality of 2^X we infer that for each $i \in \{1, \dots, n\}$, there is an open set W_i such that $A_i \subset W_i \subset \mathrm{cl}_X(W_i) \subset V_i$. Further, by (1.1.1) of Theorem 1.1, there is a connected set D_i such that $A_i \subset \mathrm{int}_X(D_i) \subset D_i \subset W_i \subset V_i$. Let $B_i = \mathrm{cl}_X(D_i)$ is a closed connected subset of X such that $A_i \subset \mathrm{int}_X(B_i) \subset B_i \subset V_i$. Let $B = B_1 \cup \dots \cup B_n$, and observe that $B_1, \dots, B_n$ are the components of B. Define $\mathcal{V} = \mathcal{U} \cap \langle \mathrm{int}_X(B_1), \dots, \mathrm{int}_X(B_n) \rangle_\infty$, and note that $\mathcal{V}$ is open in $C_\infty(X)$ and $A \in \mathcal{V} \subset \mathcal{U}$.

Take $C \in \mathcal{V}$. Then $C \subset \mathrm{int}_X(B_1) \cup \dots \cup \mathrm{int}_X(B_n)$ and $C \cap \mathrm{int}_X(B_i) \neq \varnothing$ for each $i \in \{1, \dots, n\}$. Then $C \cap \mathrm{int}_X(B_i)$ is closed, nonempty, and it has a finite number of components for each $i \in \{1, \dots, n\}$. It follows that $C \subset B$ and each component of B intersects C. Thus there is an order arc $\alpha(C)$ in $C_\infty(X)$ joining C to B. Let $\mathcal{D} = \cup\{\alpha(C) : C \in \mathcal{V}\}$ So, $\mathcal{D}$ is arcwise connected and $\mathcal{V} \subset \mathcal{D}$. Thus $A \in \operatorname{int}_{C_\infty(X)}(\mathcal{D})$, and therefore $\mathcal{D}$ is an arcwise connected neighborhood of A in $C_\infty(X)$.

It remains to prove that $\mathcal{D} \subset \mathcal{U}$. To this aim it is enough to show that $\alpha(C) \subset \mathcal{U}$ for each $C \in \mathcal{V}$. So, take $C \in \mathcal{V}$ and $E \in \alpha(C)$. Then $C \subset E \subset B$. Since $C \in \mathcal{V} \subset \mathcal{U} = \langle U_1, \dots, U_m\rangle_\infty$, we infer that $C \cap U_j \neq \varnothing$ for each $j \in \{1, \dots, m\}$. Since $C \subset E$ it follows that $E \cap U_j \neq \varnothing$ for each $j \in \{1, \dots, m\}$. Further, we have $E \subset B \subset U_1 \cup \cdots \cup U_m$. Therefore $E \in \mathcal{U}$, and thus (5.2.4) is shown.

The proof is complete. □

Theorem 5.3. *Let X be a compact Hausdorff space and $A \in C_\infty(X)$. Then $C_\infty(X)$ is strongly locally connected at A if and only if 2^X is strongly locally connected at A.*

Proof. Necessity. Assume that $C_\infty(X)$ is strongly locally connected at A. Since 2^X is normal, thus regular, by Statement 0.3 and, since $C_\infty(X)$ is its dense subspace according to (0.5), the needed implication is a consequence of implication (5.1.2) of Lemma 5.1.

Sufficiency. Assume that 2^X is strongly locally connected at A. To show that also $C_\infty(X)$ is, take a basic open set $\mathcal{U} = \langle U_1, \dots, U_m\rangle_\infty$ such that $A \in \mathcal{U}$. Let $A_1, \dots, A_n$ be the components of A. Choose pairwise disjoint open sets $V_1, \dots, V_n$ such that for each $i \in \{1, \dots, n\}$ we have $A_i \subset V_i \subset U_1 \cup \cdots \cup U_m$. By (1.1.2) of Theorem 1.1 for each $i \in \{1, \dots, n\}$, there exists an open connected set W_i such that $A_i \subset W_i \subset V_i$. Let $\mathcal{V} = \mathcal{U} \cap \langle W_1, \dots, W_n\rangle_\infty$. Then $A \in \mathcal{V} \subset \mathcal{U}$ and $\mathcal{V}$ is open in $C_\infty(X)$. It remains to show that $\mathcal{V}$ is connected.

To this aim take $B \in \mathcal{V}$. Then $B \subset W_1 \cup \cdots \cup W_n$ and $B \cap W_i \neq \varnothing$ for each $i \in \{1, \dots, n\}$. Fix a point $a_1 \in A_1$. Consider the set $\mathcal{A} = \{B \cup \{p\} : p \in W_1\}$. Since the function $g : W_1 \to \mathcal{A}$ defined by $g(p) = B \cup \{p\}$ is continuous and onto, it follows that $\mathcal{A}$ is connected. Observe that $\mathcal{A} \subset \mathcal{V}$. Thus B and $B \cup \{a_1\}$ can be joined by a connected subset of $\mathcal{V}$. By [**23**, Theorem 1.8, p. 59] there is an order arc α in 2^X from $B \cup \{a_1\}$ to $B \cup A_1$. Clearly, $\alpha \subset \mathcal{V}$. Therefore B and $B \cup A_1$ can be joined by a connected subset of $\mathcal{V}$. Similarly, $B \cup A_1$ and $B \cup A_1 \cup A_2$

can be joined by a connected subset of $\mathcal{V}$. Proceeding in this way we conclude that B and $B \cup A$ can be joined by a connected subset of $\mathcal{V}$. Similarly, A and $B \cup A$ can be joined by a connected subset of $\mathcal{V}$. Hence $\mathcal{V}$ is connected.

Therefore $C_\infty(X)$ is strongly locally connected at A, and the proof is complete. $\square$

Theorem 5.4. *Let X be a compact Hausdorff space. For some $n \in \mathbf{N}$, let $A = A_1 \cup \cdots \cup A_n \in C_n(X) \subset C_\infty(X)$, where $A_1, \ldots, A_n$ are the components of A. Then $C_\infty(X)$ is strongly locally arcwise connected at A if and only if for each open subset U of X with $A \subset U$ there exist pairwise disjoint open subsets $V_1, \ldots, V_n$ of X such that*

(a) *$A_i \subset V_i \subset U$ for each $i \in \{1, \ldots, n\}$, and*

(b) *if $B \in C_\infty(X)$ and $B \subset V_i$ for some $i \in \{1, \ldots, n\}$, then there exists a subcontinuum K of X such that $B \subset K \subset V_i$.*

Proof. *Necessity.* Assume that $C_\infty(X)$ is strongly locally arcwise connected at A. Let U be an open subset U of X with $A \subset U$, and let $U_1, \ldots, U_n$ be pairwise disjoint open subsets of X such that $A_i \subset U_i \subset U$ for each $i \in \{1, \ldots, n\}$. Put $\mathcal{U} = \langle U_1, \ldots, U_n \rangle_\infty$. Then $A \in \mathcal{U}$. By assumption there exists an open arcwise connected subset $\mathcal{V}$ of $C_\infty(X)$ such that $A \in \mathcal{V} \subset \mathcal{U}$.

Let $V = \cup \mathcal{V}$. We will prove that V is open. To this aim take $p \in V$. Then there exists an element $B \in \mathcal{V}$ with $p \in B$. Since $\mathcal{V}$ is an open subset of $C_\infty(X)$, there exists a basic open set $\langle W_1, \ldots, W_k \rangle_\infty$ such that $B \in \langle W_1, \ldots, W_k \rangle_\infty \subset \mathcal{V}$. We may assume that $p \in W_1$. Given $q \in W_1$, we have $B \cup \{q\} \in \langle W_1, \ldots, W_k \rangle_\infty$. Thus $q \in V$. So, we have shown that $p \in W_1 \subset V$. Therefore V is open.

For each $i \in \{1, \ldots, n\}$, put $V_i = V \cap U_i$. Then V_i are pairwise disjoint open subsets of X. Since $A \subset V$, we have $A_i \subset V_i$ for each $i \in \{1, \ldots, n\}$.

Now let $i \in \{1, \ldots, n\}$ and $B \in C_\infty(X)$ be such that $B \subset V_i$. Let $B_1, \ldots, B_s$ be the different components of B. Fix points $p_1 \in B_1, \ldots, p_s \in B_s$. Let $j \in \{1, \ldots, s\}$. Since $p_j \in V_i$, there exists $K_j \in \mathcal{V}$ such that $p_j \in K_j$. Since $\mathcal{V}$ is arcwise connected, there exists an arc α_j in $\mathcal{V}$ that joins K_j and A. Put $D_j = \cup \alpha_j \subset V$. By Lemma 2.2

the set D_j has at most n components. Since $\alpha_j \subset \mathcal{U}$, we conclude that $D_j \subset U_1 \cup \cdots \cup U_n$. Since $A \subset D_j$, it follows that D_j intersects each one of the sets $U_1, \dots, U_n$. Therefore D_j has n components, and they are $D_j \cap U_1, \dots, D_j \cap U_n$. Hence $D_j \cap U_i$ is a subcontinuum of V_i, and it contains the point p_j. Thus $K = A_i \cup B \cup ((D_1 \cup \cdots \cup D_s) \cap U_i)$ is a subcontinuum of V_i. This completes the proof of the necessity.

Sufficiency. Let $\mathcal{U} = \langle W_1, \dots, W_k \rangle$ be a basic open set in 2^X such that $A \in \mathcal{U} \cap C_\infty(X)$. Put $U = W_1 \cup \cdots \cup W_k$. By assumption there exist pairwise disjoint open subsets $V_1, \dots, V_n$ of X such that conditions (a) and (b) are satisfied.

Let $\mathcal{V} = \mathcal{U} \cap \langle V_1, \dots, V_n \rangle_\infty$. Then $A \in \mathcal{V} \subset \mathcal{U}$. We will show that $\mathcal{V}$ is arcwise connected. To this aim take $D \in \mathcal{V}$. For each $i \in \{1, \dots, n\}$ the set $D \cap V_i$ belongs to $C_\infty(X)$, so there exists a subcontinuum K_i of X such that $A_i \cup (D \cap V_i) \subset K_i \subset V_i$. Put $K = K_1 \cup \cdots \cup K_n$. Then $A \subset K$, $D \subset K$, and each component of K intersects both A and D. Thus there exist order arcs α and β from A to K and from D to K, respectively. Clearly, $\alpha \cup \beta \subset \mathcal{V}$. Thus $\mathcal{V}$ is arcwise connected, and thereby $C_\infty(X)$ is strongly locally arcwise connected at A. The proof is complete. □

Theorem 5.5. *Let X be a compact Hausdorff space, and let $A \in C_\infty(X)$. If 2^X is strongly locally arcwise connected at A, then also $C_\infty(X)$ is strongly locally arcwise connected at A.*

Proof. Assume that 2^X is strongly locally arcwise connected at $A \in C_\infty(X)$. Let $A_1, \dots, A_n$ be the components of A. Then 2^X is strongly locally arcwise connected at each component A_i of A by Theorem 1.2.

To prove the conclusion, take an open subset U of X with $A \subset U$. Choose pairwise disjoint open subsets $U_1, \dots, U_n$ such that $A_i \subset U_i \subset U$ for each $i \in \{1, \dots, n\}$. Since the hyperspace 2^X is strongly locally arcwise connected at each $A_i \in C(X)$, the characterization (1.1.3) of Theorem 1.1 (applied to each A_i and the open set U_i separately) implies that there is an open set V_i such that $A_i \subset V_i \subset U_i$ having the property that

$(*)$ if $B \in 2^X$ and $B \subset V_i$, then there is a subcontinuum K_i of X such that $B \subset K_i \subset V_i$.

Since $C_\infty(X) \subset 2^X$, each set V_i satisfies $(*)$ for $B \in C_\infty(X)$, whence (b) of Theorem 5.4 holds. Thus Theorem 5.4 can be applied and thereby $C_\infty(X)$ is strongly locally arcwise connected at A. □

The converse implication to that of Theorem 5.5 is not true in general, even for metric continua X. The next example shows this.

Example 5.6. There exist a metric continuum X and a point $p \in X$ such that $C_\infty(X)$ is strongly locally arcwise connected at $\{p\}$, while 2^X is not.

Proof. The example is constructed in the 3-space $\mathbf{R}^3$ equipped with the Cartesian coordinate system.

For each $n \in \mathbf{N}$, let $p_n = ((1/(n+1)^2), 0, (1/(n+1)))$ and $q_n = (0, 1-(1/(n+1)), 0)$. Consider a sequence of arcs α_n in $\mathbf{R}^3$ such that each α_n joins p_n and q_n, $\operatorname{Lim} \alpha_n$ is the segment $\alpha = \{0\}\times[0,1]\times\{0\}$, the arcs α_n are pairwise disjoint and, if $\pi_2 : \mathbf{R}^3 \to \mathbf{R}$ denotes the natural projection on the second coordinate, then the restriction $\pi_2|\alpha_n \to [0,1]$ is one-to-one for each $n \in \mathbf{N}$. Putting $D = \alpha \cup \cup\{\alpha_n : n \in \mathbf{N}\}$, we see that D is an arcwise connected subcontinuum of $\mathbf{R}^3$. Note that $D \setminus \{(0,1,0)\}$ is an open arcwise connected subset of D.

Given a subcontinuum A of $\mathbf{R}^3$, a point $b \in \mathbf{R}^3$ and a number $t \in \mathbf{R}$, let $b + tA = \{b + ta \in \mathbf{R}^3 : a \in A\}$.

Using this notation, for each $n \in \mathbf{N}$ let $D_n = (0, (1/2^n), 0) + (1/2^n)D$. Putting $p = (0,0,0)$ we finally define

$$X = \{p\} \cup \{D_n : n \in \mathbf{N}\}.$$

For each $n \in \mathbf{N}$, let

$$U_n = (\{p\} \cup D_n \cup D_{n+1} \cup \cdots) \setminus \{(0, \tfrac{1}{2^{n-1}}, 0)\} = (\pi_2|X)^{-1}([0, \tfrac{1}{2^{n-1}})).$$

Then U_n is an open arcwise connected subset of X, and the family $\{U_n : n \in \mathbf{N}\}$ is a local basis of neighborhoods of p in X.

To see that $C_\infty(X)$ is strongly locally arcwise connected at $\{p\}$ we apply Theorem 5.4. Let U be an open subset of X such that $p \in U$. Take $m \in \mathbf{N}$ such that $p \in U_m \subset U$ and $B \in C_\infty(X)$ with

$B \subset U_m$. Fix points $w_1, \dots, w_k \in B$ such that each component of B intersects $\{w_1, \dots, w_k\}$. Since U_m is arcwise connected, there exist arcs $\gamma_1, \dots, \gamma_k$ in U_m such that for each $i \in \{1, \dots, k\}$ the arc γ_i joins w_i with p. Hence the set $C = B \cup \gamma_1 \cup \dots \cup \gamma_k$ is a subcontinuum of U_m containing B. Therefore $C_\infty(X)$ is strongly locally arcwise connected at $\{p\}$.

To verify that 2^X is not strongly locally arcwise connected at $\{p\}$ it is enough to show that $\mathcal{U} = N_{2^X}(\{p\}, 1/2)$ (the 1/2-neighborhood of $\{p\}$ in 2^X) does not contain any open arcwise connected subset $\mathcal{V}$ with $\{p\} \in \mathcal{V}$. Suppose on the contrary that there exists such a set $\mathcal{V}$. Let $m = \min\{k \in \mathbf{N} : (0, (1/2^{k-1}), 0) \in \cup\mathcal{V}\}$. Since $\mathcal{V}$ is open, there exists $k \in \mathbf{N}$ such that $\{(0, (1/2^k), 0)\} \in \mathcal{V}$, so m is well defined. Since $\cup\mathcal{V} \subset \cup\mathcal{U} \subset N_X(p, (1/2))$, we infer that $m \geq 3$, and since $(0, (1/2^{m-1}), 0) \in \cup\mathcal{V}$, there exists $B \in \mathcal{V}$ such that $(0, (1/2^{m-1}), 0) \in B$.

For each $n \in \mathbf{N}$, let

$$B_n = B \cup \left((0, \tfrac{1}{2^{m-1}}, 0) + \tfrac{1}{2^{m-1}}\{p_n, p_{n+1}, \dots\}\right).$$

Then $\operatorname{Lim} B_n = B$. Since $\mathcal{V}$ is open, there exists $n \in \mathbf{N}$ such that $B_n \in \mathcal{V}$. Since $\mathcal{V}$ is arcwise connected, there exists an arc α in $\mathcal{V}$ joining B_n with $\{p\}$. Put $A = \cup\alpha$. Then A is a subcontinuum of X containing B_n. Clearly, $(0, (1/2^{m-2}), 0) \in A$. Thus there exists $E \in \alpha \subset \mathcal{V}$ such that $(0, (1/2^{m-2}), 0) \in E \subset \cup\mathcal{V}$. This contradicts the choice of m and completes the proof that 2^X is not strongly locally arcwise connected at $\{p\}$.

The proof of the properties of the example is finished. □

6. Local connectedness at a point in $F_n(X)$ and $F_\infty(X)$. In the present, last section of the paper the previously considered four variants of local connectivity at a point are studied for the hyperspaces of finite subsets of a compact Hausdorff space X. Characterizations are obtained for local connectedness and strong local connectedness at a point of $F_n(X)$ and $F_\infty(X)$, while for local arcwise connectedness and strong local arcwise connectedness of these hyperspaces at a point only sufficient conditions are shown, which appear to be also necessary if the space X is metric. But we do not have any examples showing that metrizability is essential to get the reverse implications.

We start with two lemmas.

Lemma 6.1. *Let X be a compact Hausdorff space, $n, r \in \mathbf{N}$ with $r \leq n$, and let $A = \{a_1, \ldots, a_r\} \in F_n(X)$, where $a_1, \ldots, a_r$ are different. Let $U_1, \ldots, U_r$ be pairwise disjoint open subsets of X such that $a_i \in U_i$ for each $i \in \{1, \ldots, r\}$ and $\mathcal{C} \subset \langle U_1, \ldots, U_r\rangle \cap F_n(X)$ such that $A \in \mathcal{C}$. Then the following implications hold.*

(6.1.1) *If $\mathcal{C}$ is open in $F_n(X)$, then $\cup\mathcal{C}$ is open in X.*

(6.1.2) *If $\mathcal{C}$ is connected, then $(\cup\mathcal{C}) \cap U_i$ is connected for each $i \in \{1, \ldots, r\}$.*

(6.1.3) *If $\mathcal{C}$ is arcwise connected and X is metric, then $(\cup\mathcal{C}) \cap U_i$ is arcwise connected for each $i \in \{1, \ldots, r\}$.*

Proof. To show (6.1.1) take a point $p \in \cup\mathcal{C}$. Then there exists $B \in \mathcal{C}$ such that $p \in B$. Since $\mathcal{C}$ is open in $F_n(X)$, there exists a basic open set $\mathcal{V} = \langle V_1, \ldots, V_m\rangle \cap F_n(X)$ in $F_n(X)$ such that $B \in \mathcal{V} \subset \mathcal{C}$. Suppose that $V_1, \ldots V_s$ are the sets V_j, each of which contains p. We claim that $p \in V_1 \cap \cdots \cap V_s \subset \cup\mathcal{C}$. Take $x \in V_1 \cap \cdots \cap V_s$. Then $\{x\} \cup (B \setminus \{p\}) \in \langle V_1 \cap \cdots \cap V_m\rangle \cap F_n(X) \subset \mathcal{C}$. Thus $x \in \cup\mathcal{C}$. We have shown that $p \in V_1 \cap \cdots \cap V_s \subset \cup\mathcal{C}$. Therefore $\cup\mathcal{C}$ is open.

In order to prove (6.1.2) and (6.1.3), put $\mathcal{D} = \langle U_1, \ldots, U_r\rangle \cap F_n(X)$, $\psi : \mathcal{D} \to F_n(X)$ by $\psi(D) = D \cap U_1$. First, we show that ψ is continuous. Let $D \in \mathcal{D}$ and $\mathcal{W} = \langle W_1, \ldots, W_m\rangle \cap F_n(X)$ be such that $\psi(D) \in \mathcal{W}$. Let $\mathcal{V} = \langle W_1 \cap U_1, \ldots, W_m \cap U_1, U_2, \ldots, U_r\rangle \cap F_n(X)$. It is easy to prove that $D \in \mathcal{V}$ and $\psi(\mathcal{V}) \subset \mathcal{W}$. Hence ψ is continuous.

Now we are ready to prove (6.1.2). Since $\mathcal{C}$ is connected and ψ is continuous, $\psi(\mathcal{C}) = \{C \cap U_1 : C \in \mathcal{C}\}$ is connected. Since $\{a_1\} = \psi(A) \in \psi(\mathcal{C})$, it follows that $\psi(\mathcal{C}) \cap C(X) \neq \varnothing$. By Lemma 2.1 the union $\cup\,\psi(\mathcal{C})$ is connected. But $\cup\,\psi(\mathcal{C}) = \cup\{C \cap U_1 : C \in \mathcal{C}\} = U_1 \cap (\cup\mathcal{C})$. Thus $U_1 \cap (\cup\mathcal{C})$ is connected. An argument for the connectedness of $U_i \cap (\cup\mathcal{C})$ for any $i \in \{1, \ldots, r\}$ is the same.

To show (6.1.3) take a point $p \in (\cup\mathcal{C}) \cap U_1 = \cup\,\psi(\mathcal{C})$. Then there exists $C \in \mathcal{C}$ such that $p \in \psi(C)$. Since $\psi(\mathcal{C})$ is arcwise connected, there exists a mapping $\alpha : [0,1] \to \psi(\mathcal{C})$ with $\alpha(0) = \psi(C)$ and $\alpha(1) = \psi(A) = \{a_1\}$. Thus $\alpha([0,1])$ is a locally connected metric subcontinuum of X, see [**3**, Lemma 2.2, p. 252]. Since $p, a_1 \in \cup\,\alpha([0,1])$, there exists a mapping $\beta : [0,1] \to \cup\alpha([0,1])$ such that $\beta(0) = p$ and $\beta(1) = a_1$. Note that $\beta([0,1]) \subset \cup\alpha([0,1]) \subset \cup\psi(\mathcal{C}) = (\cup\mathcal{C}) \cap U_1$. Therefore $(\cup\mathcal{C}) \cap U_1$ is arcwise connected. The proof is complete. □

Lemma 6.2. *Let X be a compact Hausdorff space, $n, r \in \mathbf{N}$ with $r \leq n$, and let $A = \{a_1, \dots, a_r\} \in F_n(X)$, where $a_1, \dots, a_r$ are different. For each $i \in \{1, \dots, r\}$, let U_i and C_i be such that $a_i \in C_i \subset U_i$ and that $U_1, \dots, U_r$ are open and pairwise disjoint subsets of X. Further, let*

$$\mathcal{C} = \big\{B \in F_n(X) : B \subset C_1 \cup \dots \cup C_r \text{ and } B \cap C_i \neq \varnothing \\ \text{for each } i \in \{1, \dots, r\}\big\}.$$

Then the following implications hold.

(6.2.1) *If each C_i is open in X, then $\mathcal{C}$ is open in $F_n(X)$.*

(6.2.2) *If each C_i is connected, then $\mathcal{C}$ is connected.*

(6.2.3) *If each C_i is arcwise connected, then $\mathcal{C}$ is arcwise connected.*

Proof. Implication (6.2.1) holds by the definition of the Vietoris topology.

To prove (6.2.2) take $B \in \mathcal{C}$. For each $i \in \{1, \dots, r\}$, let $B_i = B \cap C_i$. Then $B_i \neq \varnothing$ and $B = B_1 \cup \dots \cup B_r$. Suppose that B_i has m_i elements. Let

$$C = C_1^{m_1} \times C_2^{m_2} \times \dots \times C_r^{m_r} \times C_r^{n-(m_1+\dots+m_r)}.$$

Then C is a connected subset of X^n. Let $\eta : X^n \to F_n(X)$ be given by $\eta(x_1, \dots, x_n) = \{x_1, \dots, x_n\}$. Clearly, η is continuous, $B, A \in \eta(C)$ and $\eta(C) \subset \mathcal{C}$. Therefore, $\mathcal{C}$ is connected.

The proof of (6.2.3) is similar to that of (6.2.2) (applying [**26**, Theorem 9, p. 201] as in the proof of Proposition 2.10), so it is left to the reader. □

Theorem 6.3. *Let X be a compact Hausdorff space, $n, r \in \mathbf{N}$ with $r \leq n$, and let $A = \{a_1, \dots, a_r\} \in F_n(X)$, where $a_1, \dots, a_r$ are different. Then the following hold.*

(6.3.1) *$F_n(X)$ is locally connected at A if and only if X is locally connected at each a_i.*

(6.3.2) *$F_n(X)$ is strongly locally connected at A if and only if X is strongly locally connected at each a_i.*

(6.3.3) *If X is locally arcwise connected at each a_i, then $F_n(X)$ is locally arcwise connected at A.*

(6.3.4) *If X is strongly locally arcwise connected at each a_i, then $F_n(X)$ is strongly locally arcwise connected at A.*

(6.3.5) *If X is metric, then the implications in* (6.3.3) *and* (6.3.4) *can be reversed.*

Proof. We only prove (6.3.2); proofs of the other assertions are similar.

First, assume that $F_n(X)$ is strongly locally connected at A. Let $U_1, \dots, U_r$ be pairwise disjoint open subsets of X such that $a_i \in U_i$ for each $i \in \{1, \dots, r\}$. We show that X is strongly locally connected at a_1 (for other points a_i the proof is the same). Let U be an open subset of X such that $a_1 \in U$. By the assumption there exists an open connected subset $\mathcal{C}$ of $F_n(X)$ such that $A \in \mathcal{C} \subset \langle U \cap U_1, U_2, \dots, U_r \rangle \cap F_n(X)$. By Lemma 6.1 the set $(\cup \mathcal{C}) \cap U_1$ is connected and open in X. Moreover, $a_1 \in (\cup \mathcal{C}) \cap U_1 \subset U_1$. Therefore X is strongly locally connected at a_1.

Second, assume that X is strongly locally connected at each a_i. Let $\mathcal{W} = \langle W_1, \dots, W_m \rangle \cap F_n(X)$ be such that $A \in \mathcal{W}$. For each $i \in \{1, \dots, r\}$ let $V_i = (\cap \{W_j : a_i \in W_j \text{ and } j \in \{1, \dots, m\}\}) \cap U_i$. Then V_i is open and $a_i \in V_i \subset U_i$. By assumption there exists an open connected subset C_i of X such that $a_i \in C_i \subset V_i$. Let

$$\mathcal{D} = \big\{ B \in F_n(X) : B \subset C_1 \cup \dots \cup C_r \text{ and } B \cap C_i \neq \varnothing \\ \text{for each } i \in \{1, \dots, r\} \big\}.$$

By Lemma 6.2, $\mathcal{D}$ is connected and open in $F_n(X)$. Clearly $A \in \mathcal{D} \subset \mathcal{W}$. Therefore $F_n(X)$ is strongly locally connected at A. The proof is then complete. □

Proofs of Lemmas 6.1 and 6.2 can easily be adapted for the space $F_\infty(X)$ instead of $F_n(X)$. Thus we have the following result.

Theorem 6.4. *Let X be a compact Hausdorff space, $r \in \mathbf{N}$ and $A = \{a_1, \dots, a_r\} \in F_\infty(X)$, where $a_1, \dots, a_r$ are different. Then the following hold.*

(6.4.1) *$F_\infty(X)$ is strongly locally connected at A if and only if X is strongly locally connected at each a_i.*

(6.4.2) *$F_\infty(X)$ is locally connected at A if and only if X is locally connected at each a_i.*

(6.4.3) *If X is strongly locally arcwise connected at each a_i, then $F_\infty(X)$ is strongly locally arcwise connected at A.*

(6.4.4) *If X is locally arcwise connected at each a_i, then $F_\infty(X)$ is locally arcwise connected at A.*

(6.4.5) *If X is metric, then the implications in* (6.4.3) *and* (6.4.4) *can be reversed.*

To prove (6.3.5) and (6.4.5), one has to use the fact that locally connected metric continua are arcwise connected. This implication is not true for the non-metric case, see [**22**, Theorem 1 and Corollary 1, p. 167]. So, we have the following question.

Question 6.5. Is metrizability of X an essential assumption in (6.3.5) and (6.4.5)?

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Instituto de Matemáticas, UNAM, Circuito Exterior, Ciudad Universitaria, 04510 México, D.F., México and Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

Instituto de Matemáticas, UNAM, Circuito Exterior, Ciudad Universitaria, 04510 México, D.F., México
E-mail address: illanes@math.unam.mx