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DIFFERENTIAL INCLUSIONS ON PROXIMATE RETRACTS OF SEPARABLE HILBERT SPACES

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ABSTRACT. New existence results are presented which guarantee the existence of viable solutions to differential inclusions in separable Hilbert spaces. Our results rely on the existence of maximal solutions for an appropriate differential equation in the real case.

1. Introduction. In this paper we discuss the existence of solutions $y : [0,T] \to K \subseteq H$ (so called viable solutions) to the differential inclusion

$$(1.1) \qquad \begin{cases} y'(t) \in \phi(t, y(t)) & \text{a.e.} \quad t \in [0,T] \\ y(0) = x_0 \in K. \end{cases}$$

Here $T > 0$ is fixed, K is a proximate retract (defined in Section 2) and H is a separable Hilbert space. Our existence theory relies on (i) solution set results for differential inclusions due to Cichon and Kubiaczyk [**2**], (ii) the existence of maximal solutions for appropriate differential equations in the real case, (iii) properties of the Bouligand cone and (iv) the Urysohn function. Our results extend and complement results in the literature (see [**3–5, 7, 8**] and the references therein).

For the convenience of the reader we recall the results in [**2**]. Consider the differential inclusion

$$(1.2) \qquad \begin{cases} y'(t) \in F(t, y(t)) & \text{a.e.} \quad t \in [0,T] \\ y(0) = x_0 \in H \end{cases}$$

where $F : [0,T] \times H \to C(H)$ (here $C(H)$ denotes the family of nonempty compact subsets of H) and H is a separable Hilbert space. We look for solutions to (1.2) in $W^{1,1}([0,T], H)$. Recall $W^{1,1}([0,T], H)$ denotes the Sobolev class of absolutely continuous functions on $[0,T]$. We assume F satisfies *some* of the following conditions, to be specified

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later:

$$(1.3)\quad \begin{cases} \text{(i)}\ \ t \mapsto F(t,x) \text{ is measurable for every } x \in H \\ \text{(ii)}\ x \mapsto F(t,x) \text{ is upper semi-continuous for a.e. } t \in [0,T] \end{cases}$$

$$(1.4)\quad \begin{cases} \text{(i)}\ \ t \mapsto F(t,x) \text{ is measurable for every } x \in H \\ \text{(ii)}\ x \mapsto F(t,x) \text{ is continuous for a.e. } t \in [0,T] \end{cases}$$

$$(1.5)\quad \begin{cases} \text{(i)}\ \ (t,x) \mapsto F(t,x) \text{ is } \mathcal{L} \otimes \mathcal{B} \text{ measurable} \\ \text{(ii)}\ x \mapsto F(t,x) \text{ is lower semi-continuous for a.e. } t \in [0,T] \end{cases}$$

$$(1.6)\quad \begin{cases} \text{there exists } h \in L^1[0,T] \text{ such that } |F(t,x)| \le h(t) \\ \text{for a.e. } t \in [0,T] \text{ and all } x \in H \end{cases}$$

$$(1.7)\quad \begin{cases} \text{for each } r > 0,\ \exists\ h_r \in L^1[0,T] \text{ such that } |F(t,x)| \le h_r(t) \\ \text{for a.e. } t \in [0,T] \text{ and every } x \in H \text{ with } |x| \le r \end{cases}$$

and

$$(1.8)\quad \begin{cases} \text{there exists } k \in L^1[0,T] \text{ with } \lim_{h\to 0^+} \alpha(F(J_{t,h} \times \Omega)) \\ \le k(t)\,\alpha(\Omega) \text{ for } t \in (0,T] \text{ and for any bounded} \\ \text{subset } \Omega \text{ of } H;\text{ here } J_{t,h} = [t-h,t] \cap [0,T]. \end{cases}$$

Remark 1.1. In this paper assumption (1.8) could be replaced by any of the assumptions in Lemma 4 of [**2**]. For example, we could replace (1.8) with

(1.9)

$$\begin{cases} \lim_{h\to 0^+} \alpha(F(J_{t,h} \times \Omega)) \le w(t, \alpha(\Omega)) \text{ a.e. on } [0,T] \text{ for any bounded} \\ \text{subset } \Omega \text{ of } H;\text{ here } w \text{ is a Kamke function } [\mathbf{2}, \text{ p. } 600]. \end{cases}$$

The following result is taken from [**2**].

Theorem 1.1. *Suppose* $F : [0,T] \times H \to CK(H)$ *satisfies* (1.3), (1.6) *and* (1.8) (*here* $CK(H)$ *denotes the family of nonempty compact convex subsets of* H). *Then* (1.2) *has a solution in* $W^{1,1}([0,T],H)$. *In fact the solution set of* (1.2) *is an* R_δ *subset of* $C([0,T],H)$.

Remark 1.2. The existence of one solution was established in [**3**, p. 117], [**7**], and the fact that the solution set is an R_δ set was established in [**2**]. Of course (1.8) in Theorem 1.1 could be replaced by (1.9).

Remark 1.3. In fact H could be replaced by any Banach space in Theorem 1.1 (H does not need to be separable either) if (1.3) (i) is replaced by

$$F(\,.\,,x) \quad \text{has a measurable selector for each} \quad x \in H. \tag{1.10}$$

Also in [**7**, Theorem 2.4], see also [**3**, p. 117], we established the following result.

Theorem 1.2. *Suppose* $F : [0,T] \times H \to C(H)$ *satisfies* (1.6)*,* (1.8) *and either* (1.4) *or* (1.5)*. Then* (1.2) *has a solution in* $W^{1,1}([0,T],H)$*.*

2. Viable solutions. In this section we study the existence of solutions $x : [0,T] \to K \subseteq H$ (so called viable solutions) to the differential inclusion

$$\begin{cases} x'(t) \in \phi(t,x(t)) & \text{a.e. } t \in [0,T] \\ x(0) = x_0 \in K; \end{cases} \tag{2.1}$$

here H is a separable Hilbert space. By a solution to (2.1) we mean a $x \in W^{1,1}([0,T],H)$ with $x' \in \phi(t,x)$ almost everywhere on $[0,T]$, $x(0) = x_0$ and $x(t) \in K$ for $t \in [0,T]$. Throughout this section we assume

$$K \quad \text{is a proximate retract.} \tag{2.2}$$

Definition 2.1 [**4, 5, 8**]. A nonempty closed subset K of H is said to be a proximate retract if there exists an open neighborhood U of K in H and a continuous (single valued) mapping $r : U \to K$ (called a metric retraction) such that the following two conditions are satisfied:

(i) $r(x) = x$ for all $x \in K$

(ii) $|r(x) - x| = \operatorname{dist}(x,K)$ for all $x \in U$.

Remark 2.1. Any closed, convex subset of H is a proximate retract.

Remark 2.2. Now since we can take a sufficiently small U (for example by restricting U to $U \cap \{y \in H : \operatorname{dist}(y, K) < \delta\}$ for some given $\delta > 0$) we may assume, and we do so, that $|r(u) - u| \le \delta$ for all $u \in U$.

Throughout this section we will assume ϕ satisfies either

$$(2.3) \quad \begin{cases} \phi : [0,T] \times K \to CK(H) \text{ satisfies (1.3) and (1.7)} \\ \text{(with } F \text{ replaced by } \phi \text{ and } H \text{ replaced by } K) \end{cases}$$

or

$$(2.4) \quad \begin{cases} \phi : [0,T] \times K \to C(H) \text{ satisfies (1.7) and either (1.4)} \\ \text{or (1.5) (with } F \text{ replaced by } \phi \text{ and } H \text{ replaced by } K). \end{cases}$$

Now let U be a fixed neighborhood of K, chosen as in Remark 2.2, and let λ be an Urysohn function for $(K, H \setminus U)$ with $\lambda(x) = 1$ if $x \in K$ and $\lambda(x) = 0$ if $x \notin U$. Let $r : U \to K$ be a metric retraction. Define $\tilde{\phi} : [0,T] \times H \to C(H)$ by

$$\tilde{\phi}(t,x) = \begin{cases} \lambda(x)\phi(t, r(x)) & \text{if } x \in U \\ \{0\} & \text{if } x \notin U. \end{cases}$$

Remark 2.3. If ϕ satisfies (2.3) then $\tilde{\phi}$ satisfies (1.3) and (1.7) (with F replaced by $\tilde{\phi}$). A similar remark applies if ϕ satisfies (2.4).

Next we assume that

$$(2.5) \qquad \phi(t,x) \subseteq T_K(x) \quad \text{for all} \quad x \in K \quad \text{and a.e.} \quad t \in [0,T]$$

where

$$T_K(x) = \left\{ v \in H : \liminf_{t \to 0^+} \frac{\operatorname{dist}(x + t\,v, K)}{t} = 0 \right\}$$

is the Bouligand tangent cone to K at x.

The following result can be found in [**5**].

Theorem 2.1. *Let $a > 0$ and assume* (2.5) *holds. If $x \in W^{1,1}([0,a], H)$ is such that $x'(t) \in \tilde{\phi}(t, x(t))$ for almost every $t \in [0,a]$ and $x(0) \in K$, then $x(t) \in K$ for all $t \in [0,a]$.*

We now concentrate our study on the differential inclusion

$$\begin{cases} x'(t) \in \tilde{\phi}(t, x(t)) & \text{a.e. } t \in [0,T] \\ x(0) = x_0 \in K. \end{cases} \tag{2.6}$$

Notice any solution of (2.6) is a viable solution of (2.1); to see this notice if x is a solution of (2.6) then $x(t) \in K$ for all $t \in [0,T]$ by Theorem 2.1 and so $\tilde{\phi}(t, x(t)) = \lambda(x(t))\phi(t, r(x(t))) = \phi(t, x(t))$. Conversely, if y is a viable solution of (2.1), then y is a solution of (2.6).

Next suppose there is a constant M with $|y|_0 = \sup_{t\in[0,T]} |y(t)| < M$ for any *possible* viable solution to (2.1). Let $\varepsilon > 0$ be given, and let $\tau_\varepsilon : H \to [0,1]$ be the Urysohn function for

$$\big(\overline{B}(0,M)\,,\, H \setminus B(0, M+\varepsilon)\big)$$

such that $\tau_\varepsilon(x) = 1$ if $|x| \le M$ and $\tau_\varepsilon(x) = 0$ if $|x| \ge M + \varepsilon$. Let $\tilde{\phi}_\varepsilon(t,x) = \tau_\varepsilon(x)\,\tilde{\phi}(t,x)$ and we now look at the differential inclusion

$$\begin{cases} x'(t) \in \tilde{\phi}_\varepsilon(t, x(t)) & \text{a.e. } t \in [0,T] \\ x(0) = x_0 \in K. \end{cases} \tag{2.7}$$

Theorem 2.2. *Let H be a separable Hilbert space and assume* (2.2) *and* (2.5) *hold. In addition suppose $\phi : [0,T] \times K \to C(H)$ satisfies either* (2.3) *or* (2.4). *Assume there is a constant M with $|y|_0 < M$ for any possible viable solution to* (2.1). *Let $\varepsilon > 0$ be given, and let τ_ε, $\tilde{\phi}_\varepsilon$ be as above and suppose $|w|_0 < M$ for any possible solution w to* (2.7). *Finally assume the following condition is satisfied*:

$$\begin{cases} \textit{there exists } k \in L^1[0,T] \textit{ with } \lim_{h\to 0^+} \alpha(\tilde{\phi}_\varepsilon(J_{t,h} \times \Omega)) \\ \le k(t)\,\alpha(\Omega) \textit{ for } t \in (0,T] \textit{ and for any bounded} \\ \textit{subset } \Omega \textit{ of } H\textit{; here } J_{t,h} = [t-h, t] \cap [0,T]. \end{cases} \tag{2.8}$$

Then (2.1) *has a viable solution* u *with* $|u|_0 < M$.

Remark 2.4. Of course (2.8) could be replaced by (1.9) with F replaced by $\tilde{\phi}_\varepsilon$. Also (2.8) could be replaced [**5**] by $\overline{\phi(t,\Omega)}$ is compact for almost everywhere $t \in [0,T]$ for any bounded subset Ω of K.

Proof. From Theorem 1.1 or Theorem 1.2 (note (1.6) is satisfied with F replaced by $\tilde{\phi}_\varepsilon$) we have immediately that (2.7) has a solution y. By assumption $|y|_0 < M$ and so by definition $\tilde{\phi}_\varepsilon(t, y(t)) = \tilde{\phi}(t, y(t))$. Thus y is a solution of (2.6). Now Theorem 2.1 implies $y(t) \in K$ for every $t \in [0,T]$ and so y is a solution of (2.1). $\square$

Corollary 2.3. *Let* H *be a separable Hilbert space,* K *a closed convex subset of* H *and assume* (2.5) *holds. In addition suppose* $\phi : [0,T] \times K \to C(H)$ *satisfies either* (2.3) *or* (2.4). *Assume there is a constant* M *with* $|y|_0 < M$ *for any possible viable solution to* (2.1). *Let* $\varepsilon > 0$ *be given and let* τ_ε, $\tilde{\phi}_\varepsilon$ *be as above and suppose* $|w|_0 < M$ *for any possible solution* w *to* (2.7). *Finally assume the following condition is satisfied:*

$$(2.9)\qquad \begin{cases} \text{there exists } k \in L^1[0,T] \text{ with } \lim_{h\to 0^+} \alpha(\phi(J_{t,h} \times \Omega)) \\ \leq k(t)\,\alpha(\Omega) \text{ for } t \in (0,T] \text{ and for any bounded} \\ \text{subset } \Omega \text{ of } K\text{; here } J_{t,h} = [t-h,t] \cap [0,T]. \end{cases}$$

Then (2.1) *has a viable solution* u *with* $|u|_0 < M$.

Remark 2.5. Of course (2.9) could be replaced by (1.9) with F replaced by ϕ.

Proof. The result follows from Theorem 2.2 once we show (2.8) is true. To see this notice r in this case is nonexpansive. Now if Ω is a bounded subset of H, then since

$$\begin{aligned} \tilde{\phi}_\varepsilon(J_{t,h} \times \Omega) &\subseteq \overline{\mathrm{co}}\left(\tilde{\phi}(J_{t,h} \times \Omega) \cup \{0\}\right) \\ &\subseteq \overline{\mathrm{co}}\,(\overline{\mathrm{co}}\,[\phi(J_{t,h} \times r(\Omega)) \cup \{0\}] \cup \{0\}) \end{aligned}$$

we have

$$\alpha(\tilde{\phi}_\varepsilon(J_{t,h} \times \Omega)) \leq \alpha(\phi(J_{t,h} \times r(\Omega))).$$

This together with (2.9) and the fact that r is nonexpansive yields

$$\begin{aligned}\lim_{h\to 0^+} \alpha(\tilde{\phi}_\epsilon(J_{t,h}\times\Omega)) &\le \lim_{h\to 0^+} \alpha(\phi(J_{t,h}\times r(\Omega))) \\ &\le k(t)\,\alpha(r(\Omega)) \le k(t)\,\alpha(\Omega). \qquad \square\end{aligned}$$

Now let $S_\phi(x_0;K)$ denote the solution set of viable solutions of (2.1), $S_{\tilde{\phi}}(x_0;K)$ the solution set of (2.6) and $S_{\tilde{\phi}_\varepsilon}(x_0;H)$ the solution set of (2.7).

Theorem 2.4. *Let H be a separable Hilbert space and assume* (2.2) *and* (2.5) *hold. In addition suppose $\phi:[0,T]\times K\to CK(H)$ satisfies* (2.3) *and* (2.8). *Assume there is a constant M with $|y|_0 < M$ for any possible viable solution to* (2.1). *Let $\varepsilon > 0$ be given and let τ_ε, $\tilde{\phi}_\varepsilon$ be as above and suppose $|w|_0 < M$ for any possible solution w to* (2.7). *Then $S_\phi(x_0;K)$ is an R_δ subset of $C([0,T],H)$.*

Proof. Previously we showed any viable solution of (2.1) is a solution of (2.6) and vice versa. Then it is enough to show that $S_{\tilde{\phi}}(x_0;K)$ is an R_δ set since $S_\phi(x_0;K) = S_{\tilde{\phi}}(x_0;K)$. Notice as well, by assumption, that any solution of (2.7) is a solution of (2.6) and vice versa. Thus

$$S_\phi(x_0;K) = S_{\tilde{\phi}}(x_0;K) = S_{\tilde{\phi}_\varepsilon}(x_0;H).$$

Theorem 1.1 guarantees that $S_{\tilde{\phi}_\varepsilon}(x_0;H)$ is an R_δ subset of $C([0,T],H)$.
$\square$

Remark 2.6. If in Theorem 2.4, K is convex, then (2.8) could be replaced by (2.9).

Remark 2.7. In Theorem 2.4, (2.8) could be replaced by (1.9) with F replaced by $\tilde{\phi}_\varepsilon$. Also (2.8) could be replaced by $\overline{\phi(t,\Omega)}$ is compact for almost every $t\in[0,T]$ for any bounded subset Ω of K.

Theorem 2.5. *Let H be a separable Hilbert space, K a closed convex subset of H and assume* (2.5) *holds. In addition, suppose $\phi : [0,T] \times K \to C(H)$ satisfies* (2.9) *and either* (2.3) *or* (2.4). *Also we assume the following conditions hold*:

$$(2.10) \quad \begin{cases} \text{there exists an } L^1\text{-Carathéodory function} \\ g : [0,T] \times [0,\infty) \to [0,\infty) \text{ such that } |\phi(t,x)| \le g(t,|x|) \\ \text{for a.e. } t \in [0,T] \text{ and all } x \in K \end{cases}$$

and

$$(2.11) \quad \begin{cases} \text{the problem} \\ \begin{cases} v'(t) = g(t,v(t)) \quad \text{a.e. } t \in [0,T] \\ v(0) = |x_0| \end{cases} \\ \text{has a maximal solution } m(t) \text{ on } [0,T]. \end{cases}$$

Then (2.1) *has a viable solution.*

Remark 2.8. Of course (2.9) could be replaced by (1.9) with F replaced by ϕ. In addition K convex in Theorem 2.5 could be replaced by (2.2) provided (2.9) is replaced by (i). Equation (2.8) for some $\varepsilon > 0$ and $M = m(T)+1$, or (ii). $\overline{\phi(t,\Omega)}$ is compact for almost every $t \in [0,T]$ for any bounded subset Ω of K.

Proof. Let $\varepsilon > 0$ be given and $M = \sup_{t\in[0,T]} m(t) + 1 = m(T) + 1$. We will show any possible viable solution y of (2.1) satisfies $|y|_0 < M$ and any possible solution u of (2.7) satisfies $|u|_0 < M$. If this is true then Corollary 2.3 guarantees the result. Suppose y is a possible viable solution of (2.1). Let $t \in [0,T]$ and we will show $|y(t)| < M$. If $|y(t)| \le |x_0|$ we are finished so it remains to discuss the case when $|y(t)| > |x_0|$. In this case since $|y(0)| = |x_0|$ there exists $a \in [0,t)$ with

$$|y(s)| > |x_0| \quad \text{for} \quad s \in (a,t] \quad \text{and} \quad |y(a)| = |x_0|.$$

Also

$$|y(s)|' \le |y'(s)| \le g(s,|y(s)|) \quad \text{a.e. on } (a,t)$$

so

$$\begin{cases} |y(s)|' \le g(s,|y(s)|) \quad \text{a.e. on } (a,t) \\ |y(a)| = |x_0|. \end{cases}$$

Now a standard comparison theorem for ordinary differential equations in the real case [**6**, Theorem 1.10.2] guarantees that $|y(s)| \leq m(s)$ for $s \in [a,t]$. In particular $|y(t)| \leq m(t)$. As a result $|y|_0 < M$. Next suppose u is a possible solution of (2.7). Let $t \in [0,T]$. If $|u(t)| \leq m(T) + (1/2)$, we are finished so it remains to discuss the case when $|u(t)| > m(T) + (1/2)$. Then there exists $t_0 \in [0,t)$ with $0 \leq |u(s)| < m(T) + (1/2)$ for $s \in [0,t_0)$ and $|u(t_0)| = m(T) + (1/2)$. Then u satisfies

$$\begin{cases} u'(s) \in \tilde{\phi}(s,u(s)) & \text{a.e. } s \in [0,t_0] \\ u(0) = x_0 \in K. \end{cases} \tag{2.12}$$

Also Theorem 2.1, with $a = t_0$, implies any solution w of (2.12) satisfies $w(s) \in K$ for $s \in [0,t_0]$. Thus, in particular, $u(s) \in K$ for $s \in [0,t_0]$. As a result

$$\begin{cases} u'(s) \in \phi(s,u(s)) & \text{a.e. } s \in [0,t_0] \\ u(0) = x_0. \end{cases}$$

Now since $|u(t_0)| = m(T) + (1/2) > |x_0|$ and $|u(0)| = |x_0|$ there exists $a \in [0,t_0)$ with

$$|u(s)| > |x_0| \quad \text{for} \quad s \in (a,t_0] \quad \text{and} \quad |u(a)| = |x_0|.$$

As a result

$$\begin{cases} |u(s)|' \leq g(s,|u(s)|) & \text{a.e. on } (a,t_0) \\ |u(a)| = |x_0|. \end{cases}$$

Now [**6**, Theorem 1.10.2] guarantees that $|u(s)| \leq m(s)$ for $s \in [a,t_0]$. In particular $|u(t_0)| \leq m(t_0) < m(T) + (1/2)$ and this contradicts $|u(t_0)| = m(T) + (1/2)$. As a result $|u(t)| \leq m(T) + (1/2)$ so $|u|_0 < M$. $\square$

The next result follows from the argument in Theorem 2.5 with Remark 2.6.

Theorem 2.6. *Let H be a separable Hilbert space, K a closed convex subset of H and assume* (2.5) *holds. In addition suppose $\phi : [0,T] \times K \to CK(H)$ satisfies* (2.3), (2.9), (2.10) *and* (2.11). *Then $S_\phi(x_0;K)$ is an R_δ subset of $C([0,T],H)$.*

Remark 2.9. Of course (2.9) could be replaced by (1.9) with F replaced by ϕ. In addition K convex in Theorem 2.6 could be replaced by (2.2) provided (2.9) is replaced by (i) equation (2.8) for some $\varepsilon > 0$ and $M = m(T)+1$, or (ii) $\overline{\phi(t,\Omega)}$ is compact for almost every $t \in [0,T]$ for any bounded subset Ω of K.

Next we replace condition (2.5) with

$$\phi(t,x) \cap T_K(x) \neq \varnothing \quad \text{for all } x \in K \quad \text{and a.e. } t \in [0,T]. \tag{2.13}$$

First we recall two results from the literature [**5**].

Theorem 2.7. *Let* (2.2) *hold. Then*

$$T_K(r(x)) \subseteq \{y \in \mathbf{R}^n : \langle y\,,\, x - r(x)\rangle \leq 0\} \quad \text{for all } x \in U;$$

here $\langle\,.\,,\,.\,\rangle$ *denotes the inner product in* H.

Theorem 2.8. *Let* (2.2) *hold, and let* $r : U \to K$ *be a metric retraction and* $N > 0$ *such that* $K \cap \overline{B}(0,N) \neq \varnothing$. *Then there exists* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon \leq \varepsilon_0$ *there exist subsets* $K \subset K_\varepsilon \subset U_\varepsilon \subset U$ *of* H, K_ε *closed and* U_ε *open, and a continuous retraction* $r_\varepsilon : U_\varepsilon \to K_\varepsilon$ *such that*

(i) $\cap_{\varepsilon \leq \varepsilon_0} K_\varepsilon = K$;

(ii) $|r_\varepsilon(u) - u| = \operatorname{dist}(u, K_\varepsilon)$ *for all* $u \in U_\varepsilon \cap \overline{B}(0,N)$;

(iii) $\{y \in H : \langle y\,,\, x - r(x)\rangle \leq 0\} \subseteq T_{K_\varepsilon}(x)$ *for all* $x \in K_\varepsilon \cap \overline{B}(0,N)$.

For the remainder of this section we assume (2.2), (2.3) and (2.13) hold. In addition, we assume

$$\begin{cases} \text{there exists a constant } M > 0 \text{ with } |y|_0 < M \\ \text{for any possible viable solution of (2.1).} \end{cases} \tag{2.14}$$

Let $N = M + 1$, and let $K_{1/n}, r_{1/n} : U_{1/n} \to K_{1/n}$ be given for each $n \in \{1, 2, \dots,\}$ as in Theorem 2.8 (and without loss of generality assume $K_{1/(n+1)} \subseteq K_{1/n}$). Fix $n \in \{1, 2, \dots,\}$ for the moment. Let $\tau_{1/n} : H \to [0,1]$ be the Urysohn function for

$$\left(\overline{B}(0,M)\,,\, H \Big\backslash B\left(0, M + \frac{1}{n}\right)\right)$$

with $\tau_{1/n}(x) = 1$ if $|x| \le M$ and $\tau_{1/n}(x) = 0$ if $|x| \ge M + (1/n)$. Define $\psi_{1/n} : [0,T] \times K_{1/n} \to H$ by

$$\psi_{1/n}(t,x) = \tau_{1/n}(x)\,\phi(t, r(x)) \cap G(x)$$

where $r : U \to K$ is the metric retraction and

$$G(x) = \{y \in H : \langle y\,,\, x - r(x)\rangle \le 0\}\,.$$

First we show $\varnothing \ne \psi_{1/n}(t,x)$ for $x \in U$ and almost every $t \in [0,T]$. To see this notice Theorem 2.7 implies $T_K(r(x)) \subseteq G(x)$ for all $x \in U$ and so since $\tau_{1/n}(x)\,\phi(t,r(x)) \cap T_K(r(x)) \ne \varnothing$ for almost every t, see (2.13) and [**3**, p. 32], we have that $\tau_{1/n}(x)\,\phi(t,r(x)) \cap G(x) \ne \varnothing$. Next we claim $\psi_{1/n}(t,x) \subseteq T_{K_{1/n}}(x)$ for all $x \in K_{1/n}$ and almost every t. This is immediate from Theorem 2.8 if $x \in K_{1/n} \cap \overline{B}(0, M+1)$ since $\psi_{1/n}(t,x) \subseteq G(x)$ if $x \in K_{1/n} \cap \overline{B}(0, M+1)$. On the other hand, if $x \in K_{1/n}$ and $x \notin \overline{B}(0, M+1)$, then $|x| > M+1$ and so once again our claim is true since $\psi_{1/n}(t,x) \subseteq \{0\}$. Consequently,

$$\varnothing \ne \psi_{1/n}(t,x) \subseteq T_{K_{1/n}}(x) \quad \text{for all } x \in K_{1/n} \quad \text{and a.e. } t \in [0,T].$$

It's easy to see that the graph of G is a closed subset of $K_{1/n} \times H$ and this together with [**1**, p. 470] implies the map $x \mapsto \psi_{1/n}(t,x)$ is upper semi-continuous for almost every $t \in [0,T]$. Thus, $\psi_{1/n} : [0,T] \times K_{1/n} \to CK(H)$ satisfies (1.3), with F replaced by $\psi_{1/n}$.

We now study the existence of viable solutions to the differential inclusion

$$\text{(2.15)} \qquad \begin{cases} x'(t) \in \psi_{1/n}(t, x(t)) & \text{a.e. } t \in [0,T] \\ x(0) = x_0 \in K \subseteq K_{1/n}. \end{cases}$$

Let $S_{\psi_{1/n}}(x_0; K_{1/n})$ be the solution set of viable solutions of (2.15). Keeping Theorem 2.4 in mind we also examine

$$\text{(2.16)} \qquad \begin{cases} x'(t) \in \tilde{\psi_{1/n}}(t, x(t)) & \text{a.e. } t \in [0,T] \\ x(0) = x_0 \in K; \end{cases}$$

here

$$\tilde{\psi_{1/n}}(t,x) = \begin{cases} \lambda_{1/n}(x)\,\psi_{1/n}(t, r_{1/n}(x)) & \text{if } x \in U_{1/n} \\ \{0\} & \text{if } x \notin U_{1/n} \end{cases}$$

and $\lambda_{1/n}$ is the Urysohn function for

$$\big(K_{1/n}\,,\,H\backslash U_{1/n}\big)$$

with $\lambda_{1/n}(x)=1$ if $x\in K_{1/n}$ and $\lambda_{1/n}(x)=0$ if $x\notin U_{1/n}$.

Now Theorem 2.1 implies any solution y of (2.16) is a viable solution of (2.15), i.e., $y(t)\in K_{1/n}$ for each $t\in[0,T]$, and vice versa. Thus

$$S_{\psi_{1/n}}(x_0;K_{1/n})=S_{\tilde{\psi_{1/n}}}(x_0;K_{1/n})$$

where $S_{\tilde{\psi_{1/n}}}(x_0;K_{1/n})$ is the solution set of (2.16). Let $\tau_{1/n}:H\to[0,1]$ be the Urysohn function described above and consider the differential inclusion

$$\text{(2.16)}\qquad \begin{cases} x'(t)\in\psi^{\star}_{1/n}(t,x(t)) & \text{a.e. } t\in[0,T]\\ x(0)=x_0\in K\subseteq K_{1/n}\end{cases}$$

where $\psi^{\star}_{1/n}(t,x) = \tau_{1/n}(x)\,\tilde{\psi_{1/n}}(t,x)$. Let $S_{\psi^{\star}_{1/n}}(x_0;H)$ denote the solution set of (2.17). Assume the following conditions hold:

$$\text{(2.18)}\quad \begin{cases} \text{for each } n\in\{1,2,\dots,\} \text{ assume any possible viable}\\ \text{solution } y \text{ of (2.15) satisfies } |y|_0<M \text{ and any possible}\\ \text{solution } u \text{ of (2.17) satisfies } |u|_0<M\end{cases}$$

and

$$\text{(2.19)}\quad \begin{cases} \text{for each } n\in\{1,2,\dots,\} \text{ there exists } k_n\in L^1[0,T] \text{ with}\\ \lim_{h\to 0^+}\alpha(\psi^{\star}_{1/n}(J_{t,h}\times\Omega))\le k_n(t)\,\alpha(\Omega) \text{ for } t\in(0,T]\\ \text{and for any bounded subset } \Omega \text{ of } H\text{; here}\\ J_{t,h}=[t-h,t]\cap[0,T].\end{cases}$$

Remark 2.10. Of course (2.19) could be replaced by a condition of type (1.9) with F replaced by $\psi^{\star}_{1/n}$. Also (2.19) could be replaced by $\overline{\phi(t,\Omega)}$ is compact for almost every $t\in[0,T]$ for any bounded subset Ω of K.

Apply Theorem 2.4 to deduce that

$$S_{\psi_{1/n}}(x_0; K_{1/n}) = S_{\tilde{\psi_{1/n}}}(x_0; K_{1/n}) = S_{\psi^\star_{1/n}}(x_0; H)$$

is an R_δ subset of $C([0,T],H)$. Now if $y \in S_\phi(x_0;K)$ then $y(t) \in K$ for $t \in [0,T]$ and by (2.14) we have $r(y(t)) = y(t)$ so $G(y(t)) = H$ and $\tau_{1/n}(y(t)) = 1$. Hence $y \in S_{\psi_{1/n}}(x_0; K_{1/n})$ for each $n \in \{1, 2, \dots, \}$ and so

$$S_\phi(x_0; K) \subseteq \cap_{n=1}^\infty S_{\psi_{1/n}}(x_0; K_{1/n}).$$

On the other hand, if $y \in S_{\psi_{1/n}}(x_0; K_{1/n})$ for each $n \in \{1, 2, \dots, \}$, then in particular $y(t) \in K_{1/n}$ for all $t \in [0,T]$ and for all $n \in \{1, 2, \dots, \}$. Now Theorem 2.8 (i) implies $y(t) \in K$ for all $t \in [0,T]$. In addition, (2.18) implies $|y|_0 < M$ and so

$$\cap_{n=1}^\infty S_{\psi_{1/n}}(x_0; K_{1/n}) \subseteq S_\phi(x_0; K).$$

Thus

$$S_\phi(x_0; K) = \cap_{n=1}^\infty S_{\psi_{1/n}}(x_0; K_{1/n})$$

is an R_δ subset of $C([0,T],H)$ by [**4**, p. 132].

Theorem 2.9. *Let H be a separable Hilbert space, and suppose* (2.2)*,* (2.3)*,* (2.13) *and* (2.14) *hold. For each $n \in \{1, 2, \dots, \}$ let $K_{1/n}$, $r_{1/n}$, $U_{1/n}$ be as above, and suppose* (2.18) *and* (2.19) *are satisfied. Then $S_\phi(x_0;K)$ is an R_δ subset of $C([0,T],H)$.*

Remark 2.11. In Theorem 2.9 of course (2.19) could be replaced by one of the conditions in Remark 2.10.

Theorem 2.10. *Let H be a separable Hilbert space and assume* (2.2) *and* (2.13) *hold. In addition, suppose $\phi : [0,T] \times K \to CK(H)$ satisfies* (2.3) *and* (2.10)*. Also assume the following conditions hold*:

$$\begin{cases} \overline{\phi(t,\Omega)} \text{ is compact} & \text{for a.e. } t \in [0,T] \\ \text{for any bounded subset } \Omega \text{ of } K \end{cases} \tag{2.20}$$

$$g(t,x) \quad \text{is nondecreasing in } x \text{ for a.e. } t \in [0,T] \tag{2.21}$$

and

$$(2.22)\qquad \begin{cases} \textit{there exists } \delta > 0 \textit{ such that the problem} \\ \begin{cases} v'(t) = g(t, v(t) + \delta) & \textit{a.e. } t \in [0, T] \\ v(0) = |x_0| \end{cases} \\ \textit{has a maximal solution } m(t) \textit{ on } [0,\mathrm{T}]. \end{cases}$$

Then $S_\phi(x_0; K)$ *is an* R_δ *subset of* $C([0, T], H)$.

Proof. Let $\delta > 0$ be chosen as in (2.22) and from Remark 2.2 we can assume

$$(2.23)\qquad |r(u)| \le |u| + \delta \quad \text{for all} \quad u \in U.$$

Let $M = m(T) + 1$. We will show any possible viable solution y of (2.1) satisfies $|y|_0 < M$ and also for each $n \in \{1, 2, \dots, \}$ we will show any possible viable solution v of (2.15) satisfies $|v|_0 < M$ and any possible solution u of (2.17) satisfies $|u|_0 < M$. If this is true, then Theorem 2.9 guarantees the result.

Let y be a possible viable solution of (2.1). Fix $t \in [0, T]$. If $|y(t)| \le |x_0|$ we are finished so it remains to discuss the case when $|y(t)| > |x_0|$. In this case since $|y(0)| = |x_0|$ there exists $a \in [0, t)$ with

$$|y(s)| > |x_0| \quad \text{for} \quad s \in (a, t] \quad \text{and} \quad |y(a)| = |x_0|.$$

Also

$$|y(s)|' \le |y'(s)| \le g(s, |y(s)|) \le g(s, |y(s)| + \delta) \quad \text{a.e. on} \quad (a, t)$$

so

$$\begin{cases} |y(s)|' \le g(s, |y(s)| + \delta) & \text{a.e. on } (a, t) \\ |y(a)| = |x_0|. \end{cases}$$

Now [**6**, Theorem 1.10.2] guarantees that $|y(s)| \le m(s)$ for $s \in [a, t]$ so in particular $|y(t)| \le m(t)$. As a result $|y|_0 < M$.

Next fix $n \in \{1, 2, \dots\}$, and let v be a possible viable solution of (2.15). Let $t \in [0, T]$. If $|v(t)| \le m(T) + (1/2)$ we are finished, so it remains to discuss the case when $|v(t)| > m(T) + (1/2)$. Then there exists $t_0 \in [0,t)$ with $0 \le |v(t)| < m(T) + (1/2)$ for $s \in [0,t_0)$ and $|v(t_0)| = m(T) + (1/2)$. Note $\tau_{(1/n)}(v(s)) = 1$ for $s \in [0,t_0]$ and

$$\psi_{1/n}(s,v(s)) = \phi(s,r(v(s))) \cap G(v(s)) \quad \text{for} \quad s \in [0,t_0]. \tag{2.24}$$

Now since $|v(t_0)| > |x_0|$ and $|v(0)| = |x_0|$ there exists $a \in [0,t_0)$ with

$$|v(s)| > |x_0| \quad \text{for} \quad s \in (a,t_0] \quad \text{and} \quad |v(a)| = |x_0|.$$

Also (2.10), (2.21) and (2.24) imply (note $v(x) \in K_{1/n} \subseteq U$ for $x \in [0,T]$)

$$|v(s)|' \le |v'(s)| = |\psi_{1/n}(s,v(s))| \le g(s,|r(v(s))|) \le g(s,|v(s)|+\delta)$$

almost everywhere on (a,t_0), and so

$$\begin{cases} |v(s)|' \le g(s,|v(s)|+\delta) & \text{a.e. on } (a,t_0) \\ |v(a)| = |x_0|. \end{cases}$$

Now [**6**, Theorem 1.10.2] guarantees that $|v(s)| \le m(s)$ for $s \in [a,t_0]$ so $|v(t_0)| \le m(t_0) \le M(T)$, and this contradicts $|v(t_0)| = m(T) + (1/2)$. As a result $|v(t)| \le m(T) + (1/2)$, so $|v|_0 < M$.

Finally fix $n \in \{1,2,\dots\}$, and let u be a possible solution of (2.17). Let $t \in [0,T]$. If $|u(t)| \le m(T) + (1/2)$ we are finished, so it remains to discuss the case when $|u(t)| > m(T)+(1/2)$. Then there exists $t_0 \in [0,t)$ with $0 \le |u(t)| < m(T)+(1/2)$ for $s \in [0,t_0)$ and $|u(t_0)| = m(T)+(1/2)$. Then u satisfies

$$\begin{cases} u'(s) \in \tilde{\psi_{1/n}}(s,u(s)) & \text{a.e. } s \in (0,t_0) \\ u(0) = x_0 \in K \subseteq K_{1/n}. \end{cases} \tag{2.25}$$

Also Theorem 2.1 (with $a = t_0$ and recall $\psi_{1/n}(s,x) \subseteq T_{K_{1/n}}(x)$ for $x \in K_{1/n}$ and almost every $s \in [0,T]$) implies any solution w of (2.25) satisfies $w(s) \in K_{1/n}$ for $s \in [0,t_0]$. In particular $u(s) \in K_{1/n}$ for $s \in [0,t_0]$. Thus

$$\begin{cases} u'(s) \in \psi_{1/n}(s,u(s)) & \text{a.e. } s \in (0,t_0) \\ u(0) = x_0. \end{cases}$$

Note $\tau_{1/n}(v(s)) = 1$ for $s \in [0, t_0]$ so

$$\begin{cases} u'(s) \in \phi(s, r(u(s))) \cap G(u(s)) & \text{a.e. } s \in (0, t_0) \\ u(0) = x_0. \end{cases}$$

Now since $|u(t_0)| > |x_0|$ and $|u(0)| = |x_0|$ there exists $a \in [0, t_0)$ with

$$|u(s)| > |x_0| \quad \text{for} \quad s \in (a, t_0] \quad \text{and} \quad |u(a)| = |x_0|.$$

As above,

$$\begin{cases} |u(s)|' \leq g(s, |u(s)| + \delta) & \text{a.e. on } (a, t_0) \\ |u(a)| = |x_0|, \end{cases}$$

so [**6**, Theorem 1.10.2] guarantees that $|u(s)| \leq m(s)$ for $s \in [a, t_0]$. Thus $|u(t_0)| \leq m(t_0) \leq M(T)$, and this contradicts $|u(t_0)| = m(T) + (1/2)$. As a result, $|u(t)| \leq m(T) + (1/2)$, so $|u|_0 < M$. □

Remark 2.12. If K is convex and $0 \in K$, then we could take $\delta = 0$ in (2.22) (note r is nonexpansive so $|r(x) - 0| \leq |x - 0|$ for $x \in U$).

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