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# OSCILLATION TESTS FOR CERTAIN SYSTEMS OF PARABOLIC DIFFERENTIAL EQUATIONS WITH NEUTRAL TYPE

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ABSTRACT. Sufficient conditions are established for the forced oscillation of a class of systems of neutral parabolic differential equations with deviating arguments. The main results are illustrated by some examples.

**1. Introduction.** In the past decades, the fundamental theory of partial functional differential equation(PFDE) has been investigated extensively. We refer the reader to the monograph by Wu [**12**]. Simultaneously, let us note that the oscillation theory for PFDE is an object of long standing interest.

In 1970, Domšlovk [**2**] introduced the concept of $H$-oscillation to study the oscillation of solutions of vector differential equations, where $H$ is a unit vector in $R^n$. But there are only a few papers [**9**–**11**] dealing with $H$-oscillation of vector partial differential equations. On the other hand, in recent years, some results on the oscillation theory for systems of PFDE were established in [**3**–**8**]. However, using the approach in these papers, it is impossible to obtain the forced oscillation of systems of PFDE. In this paper, we use a new technique to study the forced oscillation of systems of neutral parabolic differential equations with deviating arguments of the form

$$\frac{\partial}{\partial t}\Big(\delta_i(t)u_i(x,t)+\sum_{r=1}^{s}\lambda_{ir}(t)u_i(x,\rho_{ir}(t))\Big)$$
$$=\sum_{k=1}^{m}a_{ik}(t)\Delta u_k(x,t)+\sum_{k=1}^{m}b_{ik}(t)\Delta u_k(x,\tau_{ik}(t))$$

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$$
\begin{aligned}
&- c_i\big(x, t, (u_k(x,t))_{k=1}^m, (u_k(x,\sigma_{ik}(t)))_{k=1}^m\big)\\
&- \sum_{h=1}^{l} \int_a^b q_{ih}(x,t,\xi) u_i(x, g_{ih}(t,\xi))\, d\sigma(\xi) + f_i(x,t),\\
&\quad (x,t) \in \Omega \times [0,\infty) \equiv G, \quad i = 1, 2, \dots, m,
\end{aligned}
$$

where $\Omega$ is a bounded domain in $R^n$ with a piecewise smooth boundary $\partial\Omega$, $\Delta u_i(x,t) = \sum_{r=1}^n (\partial^2 u_i(x,t)/\partial x_r^2)$, $i = 1, 2, \dots, m$, and the integral in (1) is the Stieltjes integral.

We assume throughout this paper that

(H1) $\delta_i$, $\lambda_{ir} \in C^1([0,\infty);[0,\infty))$, $a_{ik}, b_{ik} \in C([0,\infty);R)$, $a_{ii}(t) > 0$, and $b_{ii}(t) > 0$, $i = 1, 2, \dots, m$; $k = 1, 2, \dots, m$; $r = 1, 2, \dots, s$;

(H2) $\rho_{ir}, \tau_{ik}, \sigma_{ik} \in C([0,\infty);R)$, $\rho_{ir}(t) \le t$, $\tau_{ik}(t) \le t$, $\sigma_{ik}(t) \le t$ and $\lim_{t\to\infty} \rho_{ir}(t) = \lim_{t\to\infty} \tau_{ik}(t) = \lim_{t\to\infty} \sigma_{ik}(t) = \infty$, $i = 1, 2, \dots, m$; $r = 1, 2, \dots, s$; $k = 1, 2, \dots, m$;

(H3) $c_i \in C(\overline{G} \times R^{2m}; R)$, and

$$
\begin{aligned}
&c_i(x, t, \xi_1, \dots, \xi_i, \dots, \xi_m, \eta_1, \dots, \eta_i, \dots, \eta_m)\\
&\qquad \begin{cases} \ge 0 & \text{if } \xi_i \text{ and } \eta_i \in (0,\infty),\\ \le 0 & \text{if } \xi_i \text{ and } \eta_i \in (-\infty, 0), \end{cases}\\
&\qquad\qquad i = 1, 2, \dots, m;
\end{aligned}
$$

(H4) $q_{ih} \in C(\overline{G} \times [a,b]; [0,\infty))$, $q_{ih}(t,\xi) = \min_{x\in\overline{\Omega}} q_{ih}(x,t,\xi)$, $i = 1, 2, \dots, m$; $h = 1, 2, \dots, l$;

(H5) $g_{ih} \in C([0,\infty) \times [a,b]; R)$, $g_{ih}(t,\xi) \le t$, $\xi \in [a,b]$ and $g_{ih}(t,\xi)$ are nondecreasing functions with respect to $t$ and $\xi$, respectively,

$$
\lim_{t\to\infty} \min_{\xi\in[a,b]} \{g_{ih}(t,\xi)\} = \infty, \quad i = 1, 2, \dots, m; \quad h = 1, 2, \dots, l;
$$

(H6) $\sigma \in ([a,b]; R)$ and $\sigma(\xi)$ are nondecreasing in $\xi$;

(H7) $f_i \in C(\overline{G}; R)$, $i = 1, 2, \dots, m$.

Consider the following two kinds of boundary conditions:

$$
(2) \qquad \frac{\partial u_i(x,t)}{\partial N} = \psi_i(x,t), \quad (x,t) \in \partial\Omega \times [0,\infty), \quad i = 1, 2, \dots, m,
$$

where $N$ is the unit exterior normal vector to $\partial\Omega$ and $\psi_i(x,t)$ is a continuous function on $\partial\Omega \times [0,\infty)$, $i = 1, 2, \dots, m$, and

$$u_i(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,\infty), \quad i = 1, 2, \dots, m. \tag{3}$$

**Definition 1.1.** The vector function $u(x,t) = \{u_1(x,t), u_2(x,t), \dots, u_m(x,t)\}^T$ is said to be a solution of the problem (1), (2) (or (1), (3)) if it satisfies (1) in $G = \Omega \times [0,\infty)$ and boundary condition (2) (or (3)).

**Definition 1.2.** The vector solution $u(x,t) = \{u_1(x,t), u_2(x,t), \dots, u_m(x,t)\}^T$ of the problem (1), (2) (or (1), (3)) is said to oscillate in the domain $G = \Omega \times [0,\infty)$ if at least one of its nontrivial component oscillates in $G$. Otherwise, the vector solution $u(x,t)$ is said to be nonoscillatory.

**Definition 1.3.** The vector solution $u(x,t) = \{u_1(x,t), u_2(x,t), \dots, u_m(x,t)\}^T$ of the problem (1), (2) (or (1), (3)) is said to oscillate strongly in the domain $G = \Omega \times [0,\infty)$ if each of its nontrivial component oscillates in $G$.

**2. Main results.** Firstly, we introduce the following fact [**1**]:

The smallest eigenvalue $\alpha_0$ of the Dirichlet problem

$$\begin{cases} \Delta\omega(x) + \alpha\omega(x) = 0 & \text{in } \Omega \\ \omega(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

is positive and the corresponding eigenfunction $\varphi(x)$ is positive in $\Omega$.

For convenience, we will use the following notations:

$$U_i(t) = \int_\Omega u_i(x,t)\,dx, \quad \Psi_i(t) = \int_{\partial\Omega} \psi_i(x,t)\,dS, \quad F_i(t) = \int_\Omega f_i(x,t)\,dx,$$

$$H_i(t) = F_i(t) + \sum_{k=1}^m a_{ik}(t)\Psi_k(t) + \sum_{k=1}^m b_{ik}(t)\Psi_k(\tau_{ik}(t)),$$

$$\widetilde{U}_i(t) = \int_\Omega u_i(x,t)\varphi(x)\,dx, \quad E_i(t) = \int_\Omega f_i(x,t)\varphi(x)\,dx,$$

$$t \geq 0, \quad i = 1, 2, \dots, m,$$

where $dS$ is the surface element on $\partial\Omega$.

**Lemma 2.1.** *Suppose that* $u(x,t)=\{u_1(x,t),u_2(x,t),\dots,u_m(x,t)\}^T$ *is a solution of the problem* (1), (2) *in* $G$. *If there exists some* $i_0\in\{1,2,\dots,m\}$ *such that* $u_{i_0}(x,t)>0$, $t\ge t_0\ge 0$, *then* $U_{i_0}(t)$ *satisfies the neutral differential inequality*

$$\begin{aligned}\Big(\delta_{i_0}(t)V(t)+\sum_{r=1}^{s}\lambda_{i_0r}(t)V(\rho_{i_0r}(t))\Big)' \\ +\sum_{h=1}^{l}\int_a^b q_{i_0h}(t,\xi)V(g_{i_0h}(t,\xi))\,d\sigma(\xi)\le H_{i_0}(t).\end{aligned} \tag{4}$$

*Proof.* From the conditions (H2) and (H5), we easily obtain that there exists a number $t_1\ge t_0$ such that $u_{i_0}(x,t)>0$, $u_{i_0}(x,\rho_{i_0r}(t))>0$, $u_{i_0}(x,\sigma_{i_0}k(t))>0$, $u_{i_0}(x,\sigma_{i_0k}(t))>0$ and $u_{i_0}(x,g_{i_0h}(t,\xi))>0$ in $\Omega\times[t_1,\infty)$, $k=1,2,\dots,m$; $r=1,2,\dots,s$, $h=1,2,\dots,l$.

Consider the following equation

$$\begin{aligned}&\frac{\partial}{\partial t}\Big(\delta_{i_0}(t)u_{i_0}(x,t)+\sum_{r=1}^{s}\lambda_{i_0r}(t)u_{i_0}(x,\rho_{i_0r}(t))\Big)\\ &=\sum_{k=1}^{m}a_{i_0k}(t)\Delta u_k(x,t)+\sum_{k=1}^{m}b_{i_0k}(t)\Delta u_k(x,\tau_{i_0k}(t))-c_{i_0}\big(x,t,(u_k(x,t))_{k=1}^m,\\ &\quad(u_k(x,\sigma_{i_0k}(t)))_{k=1}^m\big)-\sum_{h=1}^{l}\int_a^b q_{i_0h}(x,t,\xi)u_{i_0}(x,g_{i_0h}(t,\xi))\,d\sigma(\xi)\\ &\qquad+f_{i_0}(x,t),\quad (x,t)\in\Omega\times[0,\infty)\equiv G.\end{aligned} \tag{5}$$

Integrating (5) with respect to $x$ over the domain $\Omega$, we have

$$\begin{aligned}&\frac{d}{dt}\Big(\delta_{i_0}(t)\int_\Omega u_{i_0}(x,t)\,dx+\sum_{r=1}^{s}\lambda_{i_0r}(t)\int_\Omega u_{i_0}(x,\rho_{i_0r}(t))\,dx\Big)\\ &\quad=\sum_{k=1}^{m}a_{i_0k}(t)\int_\Omega \Delta u_k(x,t)\,dx+\sum_{k=1}^{m}b_{i_0k}(t)\int_\Omega\Delta u_k(x,\tau_{i_0k}(t))\,dx\end{aligned} \tag{6}$$

$$- \int_\Omega c_{i_0}\big(x,t,(u_k(x,t))_{k=1}^m,(u_k(x,\sigma_{i_0k}(t)))_{k=1}^m\big)\,dx$$
$$- \sum_{h=1}^{l}\int_\Omega\int_a^b q_{i_0h}(x,t,\xi)u_{i_0}(x,g_{i_0h}(t,\xi))\,d\sigma(\xi)\,dx+\int_\Omega f_{i_0}(x,t)\,dx,$$
$$t\ge t_1.$$

Green's formula and (2) yield

$$\int_\Omega \Delta u_k(x,t)\,dx=\int_{\partial\Omega}\frac{\partial u_k(x,t)}{\partial N}\,dS=\int_{\partial\Omega}\psi_k(x,t)\,dS=\Psi_k(t), \tag{7}$$

and

(8)
$$\int_\Omega \Delta u_k(x,\tau_{i_0k}(t))\,dx=\int_{\partial\Omega}\frac{\partial u_k(x,\tau_{i_0k}(t))}{\partial N}\,dS=\int_{\partial\Omega}\psi_k(x,\tau_{i_0k}(t))\,dS$$
$$=\Psi_k(\tau_{i_0k}(t)),\quad t\ge t_1,\quad k=1,2,\dots,m.$$

Noting that

$$\int_\Omega\int_a^b q_{i_0h}(x,t,\xi)u_{i_0}(x,g_{i_0h}(t,\xi))\,d\sigma(\xi)\,dx$$
$$=\int_a^b\int_\Omega q_{i_0h}(x,t,\xi)u_{i_0}(x,g_{i_0h}(t,\xi))\,dx\,d\sigma(\xi),$$
$$t\ge t_1,\quad h=1,2,\dots,l,$$

then from condition (H4), we have

(9)
$$\int_\Omega\int_a^b q_{i_0h}(x,t,\xi)u_{i_0}(x,g_{i_0h}(t,\xi))\,d\sigma(\xi)\,dx$$
$$\ge\int_a^b q_{i_0h}(t,\xi)\int_\Omega u_{i_0}(x,g_{i_0h}(t,\xi))\,dx\,d\sigma(\xi),\quad h=1,2,\dots,l.$$

Using the condition (H3), we have $c_{i_0}\big(x,t,(u_k(x,t))_{k=1}^m,(u_k(x,\sigma_{i_0k}(t)))_{k=1}^m\big)>0$, then combining (6)–(9), we have

$$\Big(\delta_{i_0}(t)U_{i_0}(t)+\sum_{r=1}^{s}\lambda_{i_0r}(t)U_{i_0}(\rho_{i_0r}(t))\Big)'$$
$$+\sum_{h=1}^{l}\int_a^b q_{i_0h}(t,\xi)U_{i_0}(g_{i_0h}(t,\xi))\,d\sigma(\xi)$$
$$\le F_{i_0}(t)+\sum_{k=1}^{m}a_{i_0k}(t)\Psi_k(t)+\sum_{k=1}^{m}b_{i_0k}(t)\Psi_k(\tau_{i_0k}(t)),\quad t\ge t_1,$$

which shows that $U_{i_0}(t) > 0$ is a positive solution of the inequality (4). The proof is complete. $\square$

Using a similar way, we easily obtain the following lemma.

**Lemma 2.2.** *Suppose that* $u(x,t)=\{u_1(x,t),u_2(x,t),\dots,u_m(x,t)\}^T$ *is a solution of the problem* (1), (2) *in* $G$. *If there exists some* $i_0 \in \{1,2,\dots,m\}$ *such that* $u_{i_0}(x,t) < 0$, $t \ge t_0 \ge 0$, *then* $U_{i_0}(t)$ *satisfies the neutral differential inequality*

$$\Big(\delta_{i_0}(t)V(t) + \sum_{r=1}^{s} \lambda_{i_0 r}(t)V(\rho_{i_0 r}(t))\Big)' + \sum_{h=1}^{l} \int_a^b q_{i_0 h}(t,\xi)V(g_{i_0 h}(t,\xi))\, d\sigma(\xi) \ge H_{i_0}(t). \tag{10}$$

**Theorem 2.1.** *If there exists some* $i_0 \in \{1,2,\dots,m\}$ *such that the inequality* (4) *has no eventually positive solutions and the inequality* (10) *has no eventually negative solutions, then every solution of the problem* (1), (2) *is oscillatory in* $G$.

*Proof.* Suppose to the contrary that there is a nonoscillatory solution $u(x,t) = \{u_1(x,t),u_2(x,t),\dots,u_m(x,t)\}^T$ of the problem (1), (2). It is obvious that $|u_i(x,t)| > 0$ for $t \ge t_0 \ge 0$, $i = 1,2,\dots,m$; then $u_{i_0}(x,t) > 0$ or $u_{i_0}(x,t) < 0$, $t \ge t_0$.

If $u_{i_0}(x,t) > 0$, $t \ge t_0$, using Lemma 2.1 we obtain that $U_{i_0}(t) > 0$ is a solution of inequality (4), which is a contradiction.

If $u_{i_0}(x,t) < 0$, $t \ge t_0$, using Lemma 2.2 we obtain that $U_{i_0}(t) < 0$ is a solution of inequality (10), which is a contradiction. This completes the proof. $\square$

**Theorem 2.2.** *If there exists some* $i_0 \in \{1,2,\dots,m\}$ *such that*

$$\liminf_{t\to\infty} \int_{t_1}^{t} H_{i_0}(s)\, ds = -\infty, \quad t_1 \ge t_0, \tag{11}$$

and

$$\limsup_{t\to\infty}\int_{t_1}^{t} H_{i_0}(s)\,ds=\infty,\quad t_1\geq t_0, \tag{12}$$

*hold. Then every solution of the problem* (1), (2) *is oscillatory in* $G$.

*Proof*. We prove that the inequality (4) has no eventually positive solutions and the inequality (10) has no eventually negative solutions.

Assume to the contrary that (4) has a positive solution $U_{i_0}(t)$; then there exists $t_0 \geq 0$ such that $U_{i_0}(t) > 0$, $U_{i_0}(\rho_{i_0 r}(t)) > 0$, $U_{i_0}(g_{i_0 h}(t,\xi)) > 0$, $t \geq t_0$, $h = 1,2,\dots,l$, $r = 1,2,\dots,s$. Then from (4) we have

$$\Big(\delta_{i_0}(t)U_{i_0}(t)+\sum_{r=1}^{s}\lambda_{i_0 r}(t)U_{i_0}(\rho_{i_0 r}(t))\Big)'\leq H_{i_0}(t). \tag{13}$$

Integrating (13) over the interval $[t_1, t]$, $t_1 \geq t_0$, we have

$$\delta_{i_0}(t)U_{i_0}(t)+\sum_{r=1}^{s}\lambda_{i_0 r}(t)U_{i_0}(\rho_{i_0 r}(t))\leq C+\int_{t_1}^{t}H_{i_0}(s)\,ds, \tag{14}$$

where $C$ is a constant. Taking $t \to \infty$, from (14) we have

$$\liminf_{t\to\infty}\Big[\delta_{i_0}(t)U_{i_0}(t)+\sum_{r=1}^{s}\lambda_{i_0 r}(t)U_{i_0}(\rho_{i_0 r}(t))\Big]=-\infty,$$

which contradicts the assumption that $U_{i_0}(t) > 0$.

Assume that (10) has a negative solution $\overline{U}_{i_0}(t)$. Noting that condition (12) and using the above mentioned method, we easily obtain a contradiction. The proof is complete. □

Using the above oscillation results, it is not difficult to derive the following strong oscillation conclusions.

**Theorem 2.3.** *Suppose that for all* $i \in \{1,2,\dots,m\}$,

$$\begin{aligned}&\Big(\delta_i(t)V(t)+\sum_{r=1}^{s}la_{ir}(t)V(\rho_{ir}(t))\Big)'\\&\qquad+\sum_{h=1}^{l}\int_a^b q_{ih}(t,\xi)V(g_{ih}(t,\xi))\,d\sigma(\xi)\leq H_i(t)\end{aligned} \tag{15}$$

*has no eventually positive solutions and*

$$
\begin{aligned}
&\Big(\delta_i(t)V(t)+\sum_{r=1}^{s}\lambda_{ir}(t)V(\rho_{ir}(t))\Big)' \\
&\qquad+\sum_{h=1}^{l}\int_a^b q_{ih}(t,\xi)V(g_{ih}(t,\xi))\,d\sigma(\xi)\ge H_i(t)
\end{aligned}
\tag{16}
$$

*has no eventually negative solutions.*

*Then every solution of the problem* (1), (2) *oscillates strongly in* $G$.

**Theorem 2.4.** *Suppose that for all* $i\in\{1,2,\dots,m\}$,

$$
\liminf_{t\to\infty}\int_{t_1}^{t}H_i(s)\,ds=-\infty,\quad t_1\ge t_0,
\tag{17}
$$

and

$$
\limsup_{t\to\infty}\int_{t_1}^{t}H_i(s)\,ds=\infty,\quad t_1\ge t_0,
\tag{18}
$$

*hold. Then every solution of the problem* (1), (2) *oscillates strongly in* $G$.

Next, we study the oscillation of the problem (1), (3).

**Lemma 2.3.** *Assume that* $u(x,t)=\{u_1(x,t),u_2(x,t),\dots,u_m(x,t)\}^T$ *is a solution of the problem* (1), (3) *in* $G$, *and the following hypothesis* (H8) *is satisfied*:

(H8) $a_{ik}(t)=b_{ik}(t)=0$, $i\ne k$, $i=1,2,\dots,m$; $k=1,2,\dots,m$.

*If there exists some* $i_0\in\{1,2,\dots,m\}$ *such that* $u_{i_0}(x,t)>0$, $t\ge t_0\ge 0$, *then* $\widetilde{U}_{i_0}(t)$ *satisfies the neutral differential inequality*

$$
\begin{aligned}
&\Big(\delta_{i_0}(t)V(t)+\sum_{r=1}^{s}\lambda_{i_0r}(t)V(\rho_{i_0r}(t))\Big)' \\
&\qquad+\alpha_0a_{i_0i_0}(t)V(t)+\alpha_0b_{i_0i_0}(t)V(\tau_{i_0i_0}(t)) \\
&\qquad+\sum_{h=1}^{l}\int_a^b q_{i_0h}(t,\xi)V(g_{i_0h}(t,\xi))\,d\sigma(\xi)\le E_{i_0}(t).
\end{aligned}
\tag{19}
$$

*Proof*. As in the proof of Lemma 2.1, consider equation (5). Multiplying both sides of (5) by $\varphi(x)$ and integrating with respect to $x$ over the domain $\Omega$, and noting the hypothesis (H8), we have

(20)
$$\begin{aligned}\frac{d}{dt}\Big(\delta_{i_0}(t)\int_\Omega u_{i_0}(x,t)\varphi(x)\,dx+\sum_{r=1}^{s}\lambda_{i_0r}(t)\int_\Omega u_{i_0}(x,\rho_{i_0r}(t))\varphi(x)\,dx\Big)\\ =a_{i_0i_0}(t)\int_\Omega \Delta u_{i_0}(x,t)\varphi(x)\,dx+b_{i_0i_0}(t)\int_\Omega\Delta u_{i_0}(x,\tau_{i_0i_0}(t))\varphi(x)\,dx\\ -\int_\Omega c_{i_0}(x,t,(u_k(x,t))_{k=1}^m,(u_k(x,\sigma_{i_0k}(t)))_{k=1}^m)\varphi(x)\,dx\\ -\sum_{h=1}^{l}\int_\Omega\int_a^b q_{i_0h}(x,t,\xi)u_{i_0}(x,g_{i_0h}(t,\xi))\,d\sigma(xi)\varphi(x)\,dx\\ +\int_\Omega f_{i_0}(x,t)\varphi(x)\,dx,\quad t\ge t_1.\end{aligned}$$

Using Green's formula and the boundary condition (3), we obtain

(21)
$$\begin{aligned}\int_\Omega\Delta u_{i_0}(x,t)\varphi(x)\,dx&=\int_\Omega u_{i_0}(x,t)\Delta\varphi(x)\,dx\\ &=-\alpha_0\int_\Omega u_{i_0}(x,t)\varphi(x)\,dx,\end{aligned}$$

and

(22)
$$\begin{aligned}\int_\Omega\Delta u_{i_0}(x,\tau_{i_0i_0}(t))\varphi(x)\,dx&=\int_\Omega u_{i_0}(x,\tau_{i_0i_0}(t))\Delta\varphi(x)\,dx\\ &=-\alpha_0\int_\Omega u_{i_0}(x,\tau_{i_0i_0}(t))\varphi(x)\,dx,\quad t\ge t_1.\end{aligned}$$

It is easy to see that

(23)
$$\begin{aligned}&\int_\Omega\int_a^b q_{i_0h}(x,t,\xi)u_{i_0}(x,g_{i_0h}(t,\xi))\varphi(x)\,d\sigma(\xi)\,dx\\ &\quad=\int_a^b\int_\Omega q_{i_0h}(x,t,\xi)u_{i_0}(x,g_{i_0h}(t,\xi))\varphi(x)\,dx\,d\sigma(\xi)\\ &\quad\ge\int_a^b q_{i_0h}(t,\xi)\int_\Omega u_{i_0}(x,g_{i_0h}(t,\xi))\varphi(x)\,dx\,d\sigma(\xi),\quad h=1,2,\dots,l,\end{aligned}$$

and

$$c_{i_0}\Big(x,t,(u_k(x,t))_{k=1}^m,\big(u_k(x,\sigma_{i_0k}(t))\big)_{k=1}^m\Big)\varphi(x)>0. \tag{24}$$

Therefore,

$$\begin{aligned}\Big(\delta_{i_0}(t)\widetilde{U}_{i_0}(t)&+\sum_{r=1}^s\lambda_{i_0r}(t)\widetilde{U}_{i_0}(\rho_{i_0r}(t))\Big)'\\&+\alpha_0a_{i_0i_0}(t)\widetilde{U}_{i_0}(t)+\alpha_0b_{i_0i_0}(t)\widetilde{U}_{i_0}(\tau_{i_0i_0}(t))\\&+\sum_{h=1}^l\int_a^b q_{i_0h}(t,\xi)\widetilde{U}_{i_0}(g_{i_0h}(t,\xi))\,d\sigma(\xi)\le E_{i_0}(t),\end{aligned}$$

which shows that $\widetilde{U}_{i_0}(t)>0$ is a positive solution of the inequality (19). This completes the proof. □

Similarly, we also have the following lemma.

**Lemma 2.4.** *Assume that* $u(x,t)=\{u_1(x,t),u_2(x,t),\dots,u_m(x,t)\}^T$ *is a solution of the problem* (1), (3) *in* $G$, *and the hypothesis* (H8) *holds. If there exists some* $i_0\in\{1,2,\dots,m\}$ *such that* $u_{i_0}(x,t)<0$, $t\ge t_0\ge 0$, *then* $\widetilde{U}_{i_0}(t)$ *satisfies the neutral differential inequality*

$$\begin{aligned}\Big(\delta_{i_0}(t)V(t)&+\sum_{r=1}^s\lambda_{i_0r}(t)V(\rho_{i_0r}(t))\Big)'\\&+\alpha_0a_{i_0i_0}(t)V(t)+\alpha_0b_{i_0i_0}(t)V(\tau_{i_0i_0}(t))\\&+\sum_{h=1}^l\int_a^b q_{i_0h}(t,\xi)V(g_{i_0h}(t,\xi))\,d\sigma(\xi)\ge E_{i_0}(t).\end{aligned} \tag{25}$$

Using Lemmas 2.3 and 2.4, we easily establish the following results.

**Theorem 2.5.** *Assume that the hypothesis* (H8) *holds. If there exists some* $i_0\in\{1,2,\dots,m\}$ *such that the inequality* (19) *has no eventually* *positive solutions and the inequality* (25) *has no eventually negative solutions, then every solution of the problem* (1), (3) *is oscillatory in* $G$.

**Theorem 2.6.** *Assume that the hypothesis* (H8) *holds. If there exists some* $i_0 \in \{1, 2, \dots, m\}$ *such that*

$$\liminf_{t\to\infty} \int_{t_1}^{t} E_{i_0}(s)\, ds = -\infty, \quad t_1 \geq t_0, \tag{26}$$

*and*

$$\limsup_{t\to\infty} \int_{t_1}^{t} E_{i_0}(s)\, ds = \infty, \quad t_1 \geq t_0, \tag{27}$$

*hold. Then every solution of the problem* (1), (3) *is oscillatory in* $G$.

**Theorem 2.7.** *Assume that the hypothesis* (H8) *holds. If for all* $i \in \{1, 2, \dots, m\}$,

$$\begin{aligned} \Big(\delta_i(t)V(t) + \sum_{r=1}^{s} \lambda_{ir}(t)V(\rho_{ir}(t))\Big)' \\ + \alpha_0 a_{ii}(t)V(t) + \alpha_0 b_{ii}(t)V(\tau_{ii}(t)) \\ + \sum_{h=1}^{l} \int_a^b q_{ih}(t,\xi)V(g_{ih}(t,\xi))\, d\sigma(\xi) \leq E_i(t) \end{aligned} \tag{28}$$

*has no eventually positive solutions and*

$$\begin{aligned} \Big(\delta_i(t)V(t) + \sum_{r=1}^{s} \lambda_{ir}(t)V(\rho_{ir}(t))\Big)' \\ + \alpha_0 a_{ii}(t)V(t) + \alpha_0 b_{ii}(t)V(\tau_{ii}(t)) \\ + \sum_{h=1}^{l} \int_a^b q_{ih}(t,\xi)V(g_{ih}(t,\xi))\, d\sigma(\xi) \geq E_i(t) \end{aligned} \tag{29}$$

*has no eventually negative solutions.*

*Then every solution of the problem* (1), (3) *oscillates strongly in* $G$.

**Theorem 2.8.** *Suppose that the hypothesis* (H8) *holds, and for all* $i \in \{1, 2, \dots, m\}$

$$\liminf_{t\to\infty} \int_{t_1}^{t} E_i(s)\,ds = -\infty, \quad t_1 \ge t_0, \tag{30}$$

*and*

$$\limsup_{t\to\infty} \int_{t_1}^{t} E_i(s)\,ds = \infty, \quad t_1 \ge t_0, \tag{31}$$

*hold. Then every solution of the problem* (1), (3) *oscillates strongly in* $G$.

**3. Examples.** In this section, we give some illustrative examples.

**Example 3.1.** Consider the system of neutral parabolic differential equations

(32)

$$\begin{cases} \dfrac{\partial}{\partial t}\,[u_1(x,t)+u_1(x,t-\pi)] = \Delta u_1(x,t) + \dfrac{1}{3}\,\Delta u_1(x,t-\pi) + e^t \Delta u_2(x,t) \\ +\dfrac{2}{3}\,\Delta u_2\Big(x, t-\Big(3\,\dfrac{\pi}{2}\Big)\Big) \\ -(u_1(x,t)+u_1(x,t-\pi))\exp\{u_2(x,t)+u_2(x,t-\pi)\} \\ -\displaystyle\int_{-\pi}^{-\pi/2} e^t u_1(x,t+\xi)\,d\xi - e^t \sin t \cos x - e^t(\sin t + \cos t), \\ \dfrac{\partial}{\partial t}[u_2(x,t)+u2(x,t-\pi)] = 2\Delta u_1(x,t) + \dfrac{5}{3}\,\Delta u_1(x,t-\pi) + e^t \Delta u_2(x,t) \\ +\dfrac{1}{3}\,\Delta u_2\Big(x, t-\Big(3\,\dfrac{\pi}{2}\Big)\Big) \\ -(u_2(x,t)+u_2(x,t-\pi))\exp\{u_1(x,t)+u_1(x,t-\pi)\} \\ -\displaystyle\int_{-\pi}^{-\pi/2} e^t u_2(x,t+\xi)\,d\xi + e^t \sin t \cos x + e^t(\sin t - \cos t), \\ (x,t) \in (0,\pi) \times [0,\infty), \end{cases}$$

with boundary condition

$$\frac{\partial u_i(0,t)}{\partial x} = \frac{\partial ui(\pi,t)}{\partial x} = 0, \quad t \geq 0, \quad i = 1, 2. \tag{33}$$

Here $\Omega = (0, \pi)$, $n = 1$, $m = 2$, $s = 1$, $l = 1$, $\delta_1(t) = \delta_2(t) = 1$, $\lambda_{11}(t) = \lambda_{21}(t) = 1$, $\rho_{11}(t) = \rho_{21}(t) = t - \pi$, $a_{11}(t) = 1$, $a_{12}(t) = e^t$, $a_{21}(t) = 2$, $a_{22}(t) = e^t$, $b_{11}(t) = 1/3$, $b_{12}(t) = 2/3$, $b_{21}(t) = 5/3$, $b_{22}(t) = 1/3$, $\tau_{11}(t) = \tau_{21}(t) = t - \pi$, $\tau_{12}(t) = \tau_{22}(t) = t - (3\pi/2)$,

$$\begin{aligned} &c_1\big(x, t, u_1(x,t), u_2(x,t), u_1(x, \sigma_{11}(t)), u_2(x, \sigma_{12}(t))\big) \\ &\qquad = \big(u_1(x,t) + u_1(x, \sigma_{11}(t))\big) \exp\{u_2(x,t) + u_{12}(x, \sigma_{12}(t))\}, \end{aligned}$$

$$\begin{aligned} &c_2\big(x, t, u_1(x,t), u_2(x,t), u_1(x, \sigma_{21}(t)), u_2(x, \sigma_{22}(t))\big) \\ &\qquad = \big(u_2(x,t) + u_2(x, \sigma_{22}(t))\big) \exp\{u_1(x,t) + u_1(x, \sigma_{21}(t))\}, \end{aligned}$$

$\sigma_{11}(t) = \sigma_{12}(t) = \sigma_{21}(t) = \sigma_{22}(t) = t - \pi$, $q_{11}(x,t,\xi) = q_{21}(x,t,\xi) = e^t$, $a = -\pi$, $b = -\pi/2$, $g_{11}(t,\xi) = g_{21}(t,\xi) = t + \xi$, $\psi_1(x,t) = \psi_2(x,t) = 0$, $f_1(x,t) = -e^t \sin t \cos x - e^t(\sin t + \cos t)$, $f_2(x,t) = e^t \sin t \cos x + e^t(\sin t - \cos t)$.

It is obvious that $\Psi_1(t) = \Psi_2(t) = 0$, $\Psi_1(\tau_{11}(t)) = \Psi_1(\tau_{21}(t)) = 0$, $\Psi_2(\tau_{12}(t)) = \Psi_2(\tau_{22}(t)) = 0$, $\Phi_1(t) = \Phi_2(t) = 0$, $\Phi_1(\tau_{11}(t)) = \Phi_1(\tau_{21}(t)) = 0$, $\Phi_2(\tau_{12}(t)) = \Phi_2(\tau_{22}(t)) = 0$, then

$$\begin{aligned} H_1(t) = F_1(t) &= \int_\Omega f_1(x,t)\,dx = \int_0^\pi f_1(x,t)\,dx = -\pi e^t(\sin t + \cos t), \\ H_2(t) = F_2(t) &= \int_\Omega f_2(x,t)\,dx = \int_0^\pi f_2(x,t)\,dx = \pi e^t(\sin t - \cos t). \end{aligned}$$

Hence

$$\liminf_{t\to\infty} \int_{t_1}^t H_1(s)\,ds = -\infty, \quad \limsup_{t\to\infty} \int_{t_1}^t H_1(s)\,ds = \infty,$$

and

$$\liminf_{t\to\infty} \int_{t_1}^t H_2(s)\,ds = -\infty, \quad \limsup_{t\to\infty} \int_{t_1}^t H_2(s)\,ds = \infty,$$

which shows that all the conditions of Theorem 2.4 are fulfilled. Then every solution of the problem (32), (33) oscillates strongly in $(0,\pi)\times[0,\infty)$. In fact, $u_1(x,t)=(1+\cos x)\sin t$, $u_2(x,t)=(1+\cos x)\cos t$ is such a solution.

**Example 3.2.** Consider the system of neutral parabolic differential equations

(34)

$$\begin{cases} \dfrac{\partial}{\partial t}\,[u_1(x,t)+u_1(x,t-\pi)]=(e^t+1)\Delta u_1(x,t)+\Delta u_1(x,t-\pi) \\ +\Delta u_2(x,t)+(-1)\Delta u_2\Big(x,t-\Big(\dfrac{\pi}{2}\Big)\Big)-u_1(x,t)-u_1(x,t-\pi) \\ -\displaystyle\int_{-\pi}^{-\pi/2} e^t u_1(x,t+\xi)\,d\xi+\Big(\Big(\dfrac{\pi}{2}\Big)-e^t\cos t\Big)\cos x-e^t(\sin t+\cos t), \\ \dfrac{\partial}{\partial t}\Big[tu_2(x,t)+2u_2\Big(x,t-\Big(\dfrac{\pi}{4}\Big)\Big)\Big]=\Delta u_1(x,t)+\Delta u_1(x,t-\pi) \\ +\Delta u_2(x,t)+3\Delta u_2\Big(x,t-\Big(\dfrac{\pi}{3}\Big)\Big)-u_2(x,t)-u_2\Big(x,t-\Big(\dfrac{\pi}{3}\Big)\Big) \\ -\displaystyle\int_{-\pi}^{-\pi/2}\dfrac{8}{\pi}\,u_2(x,t+\xi)\,d\xi+\Big(12t-\Big(13\,\dfrac{\pi}{3}\Big)+3\Big)\cos x, \\ (x,t)\in(0,\pi)\times[0,\infty), \end{cases}$$

with the boundary condition (33).

It is easy to see that $H_1(t)=-\pi e^t(\sin t+\cos t)$, $H_2(t)=0$. Therefore,

$$\liminf_{t\to\infty}\int_{t_1}^{t} H_1(s)\,ds=-\infty,$$

$$\limsup_{t\to\infty}\int_{t_1}^{t} H_1(s)\,ds=\infty.$$

Then, using Theorem 2.2, we obtain that every solution of the problem (34), (33) oscillates in $(0,\pi)\times[0,\infty)$. In fact, $u_1(x,t)=(1+\cos x)\sin t$, $u_2(x,t)=t\cos x$ is such a solution.

**Example 3.3.** Consider the system of neutral parabolic differential equations

(35)

$$\begin{cases} \dfrac{\partial}{\partial t}\Big[2u_1(x,t)+\dfrac{1}{3}\,u_1(x,t-\pi)\Big]=e^t\Delta u_1(x,t)+\dfrac{5}{3}\,\Delta u_1\Big(x,t-\Big(\dfrac{\pi}{2}\Big)\Big) \\ -(u_1(x,t)+u_1(x,t-\pi))\exp\Big\{u_2(x,t)+u_2\Big(x,t-\Big(\dfrac{\pi}{2}\Big)\Big)\Big\} \\ -\displaystyle\int_{-\pi}^{-\pi/2} e^t u_1(x,t+\xi)\,d\xi+e^t\sin t\sin x, \\ \dfrac{\partial}{\partial t}\Big[\dfrac{1}{3}\,u_2(x,t)+\Big(7\,\dfrac{\pi}{4}\Big)u_2\Big(x,t-\Big(\dfrac{\pi}{3}\Big)\Big)\Big] \\ =2\Delta u_2(x,t)+4\Delta u_2\Big(x,t-\Big(\dfrac{\pi}{4}\Big)\Big) \\ -\Big(u_2(x,t)+u_2\Big(x,t-\dfrac{1}{3}\Big)\Big)\exp\{u_1(x,t)+u_1(x,t-\pi)\} \\ -\displaystyle\int_{-\pi}^{-\pi/2}\dfrac{2}{\pi}\,u_2(x,t+\xi)\,d\xi+9t\sin x, \\ (x,t)\in(0,\pi)\times[0,1), \end{cases}$$

with the boundary condition

$$u_i(0,t)=u_i(\pi,t)=0,\quad t\ge 0,\quad i=1,2. \tag{36}$$

Here $f_1(x,t)=e^t\sin t\sin x$, $f_2(x,t)=9t\sin x$. We easily see that $\alpha_0=1$, $\varphi(x)=\sin x$. Let $i_0=1$, then

$$E_{i_0}(t)=E_1(t)=\int_\Omega f_1(x,t)\varphi(x)\,dx=\int_0^\pi e^t\sin t\sin^2 x\,dx=\frac{\pi}{2}\,e^t\sin t.$$

Hence,

$$\liminf_{t\to\infty}\int_{t_1}^t E_1(s)\,ds=-1,\quad \limsup_{t\to\infty}\int_{t_1}^t E_1(s)\,ds=\infty,$$

then using Theorem 2.6, we obtain that every solution of the problem (35), (36) oscillates in $(0,\pi)\times[0,\infty)$. In fact, $u_1(x,t)=\cos t\sin x$, $u_2(x,t)=t\sin x$ is such a solution.

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## REFERENCES

**1.** R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. I, Interscience, New York, 1966.

**2.** Ju.I. Domšlovk, *On the oscillation of solutions of vector differential equations*, Soviet Math. Dokl. **11** (1970), 839–841.

**3.** Y.K. Li, *Oscillation of systems of hyperbolic differential equations with deviating arguments*, Acta Math. Sinica **40** (1997), 100–105 (in Chinese).

**4.** W.N. Li, *Oscillation properties for systems of hyperbolic differential equations of neutral type*, J. Math. Anal. Appl. **248** (2000), 369–384.

**5.** W.N. Li and B.T. Cui, *Oscillation for systems of parabolic equations of neutral type*, Southeast Asian Bull. Math. **23** (1999), 447–456.

**6.** ———, *Oscillation for systems of neutral delay hyperbolic differential equations*, Indian J. Pure Appl. Math. **31** (2000), 933–948.

**7.** W.N. Li and L. Debnath, *Oscillation of a system of delay hyperbolic differential equations*, Int. J. Appl. Math. **2** (2000), 417–431.

**8.** W.N. Li and F.W. Meng, *Oscillation for systems of neutral partial differential equations with continuous distributed deviating arguments*, Demonstratio Math. **34** (2001), 619–633.

**9.** E. Minchev and N. Yoshida, *Oscillation of solutions of vector differential equations of parabolic type with functional arguments*, J. Comput. Appl. Math. **151** (2003), 107–117.

**10.** E.S. Noussair and C.A. Swanson, *Oscillation theorems for vector differential equations*, Util. Math. **1** (1972), 97–109.

**11.** ———, *Oscillation of nonlinear vector differential equations*, Ann. Mat. Pura Appl. **109** (1976), 305–315.

**12.** J. Wu, *Theory and applications of partial functional differential equations*, Springer-Verlag, New York, 1996.

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