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FROM N PARAMETER FRACTIONAL BROWNIAN MOTIONS TO N PARAMETER MULTIFRACTIONAL BROWNIAN MOTIONS

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ABSTRACT. Multifractional Brownian motion is an extension of the well-known fractional Brownian motion where the Hölder regularity is allowed to vary along the paths. In this paper, two kinds of multi-parameter extensions of mBm are studied: one is isotropic while the other is not. For each of these processes, a moving average representation, a harmonizable representation, and the covariance structure are given.

The Hölder regularity is then studied. In particular, the case of an irregular exponent function H is investigated. In this situation, the almost sure pointwise and local Hölder exponents of the multi-parameter mBm are proved to be equal to the correspondent exponents of H. Eventually, a local asymptotic self-similarity property is proved. The limit process can be another process than fBm.

1. Introduction. In many applications, fractional Brownian motion (f Bm) seems to fit very well to random phenomena. Recall that one-dimensional f Bm can be defined by one of the following four properties. Let $H \in (0,1)$ (H is sometimes called the Hurst parameter).

- B^H is a centered Gaussian process such that

$$\forall\, s,t \in \mathbf{R}_+; \quad E\left[B_s^H B_t^H\right] = \frac{1}{2}\left[s^{2H} + t^{2H} - |t-s|^{2H}\right]$$

- the process B^H such that

$\forall\, t \in \mathbf{R}_+;$

$$B_t^H = \int_{-\infty}^0 \left[(t-u)^{H-1/2} - (-u)^{H-1/2}\right].\mathbf{W}(du) + \int_0^t (t-u)^{H-1/2}.\mathbf{W}(du)$$

is an f Bm,

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• the process B^H such that

$$\forall\, t \in \mathbf{R}_+; \quad B_t^H = \int_{\mathbf{R}} \frac{e^{it\xi} - 1}{|\xi|^{H+1/2}} \,.\widehat{\mathbf{W}}(d\xi)$$

is an f Bm,

• B^H is the unique self-similar Gaussian process with stationary increments.

Its efficiency has already been shown in simulation of traffic on the Internet or in finance. This induced some recent progress such as stochastic integration against f Bm.

However, the main limitation of f Bm is that the Hölder regularity is constant along the paths.

Multifractional Brownian motion (mBm) has been independently introduced in [**4, 13**]. This process is a generalization of fractional Brownian motion where the Hurst parameter H is substituted by a function $t \mapsto H(t)$. As a consequence the Hölder exponent is allowed to vary along trajectories. The different definitions by the two groups of authors provided two different representations of mBm.

Peltier and Lévy-Véhel ([**13**]) defined the mBm from the moving average definition of the fractional Brownian motion

$$X_t = \int_{-\infty}^{0} \left[(t-u)^{H(t)-1/2} - (-u)^{H(t)-1/2} \right] .\mathbf{W}(du) + \int_0^t (t-u)^{H(t)-1/2} .\mathbf{W}(du)$$

where $t \mapsto H(t)$ is a Hölder function.

Benassi, Jaffard and Roux [**4**] defined the mBm from the harmonizable representation of the f Bm

$$X_t = \int_{\mathbf{R}} \frac{e^{it\xi} - 1}{|\xi|^{H(t)+1/2}} \,.\widehat{\mathbf{W}}(d\xi).$$

These two definitions were proved to be equivalent up to a multiplicative deterministic function [**6**].

Moreover, in [**1**] the covariance function of this Gaussian process has been proved to be

$$E\left[X_s X_t\right] \\ = D\left(H(s), H(t)\right)\left[|s|^{H(s)+H(t)} + |t|^{H(s)+H(t)} - |t-s|^{H(s)+H(t)}\right]$$

where D is a known deterministic function.

The goal of this paper is to study some multi-parameter extension of the multifractional Brownian motion, i.e., a stochastic process indexed by $\mathbf{R}_+^N$, which is an mBm when $N = 1$. One extension has already been considered in [**4**].

2D extension of fractional Brownian motion has been already used in various applications such as underwater terrain modeling [**14**]. It may be more realistic to allow local regularity to vary at each point: our extension of mBm in $\mathbf{R}^2$ may be used for this kind of application.

2. Multi-parameter extension of the fractional Brownian motion.

Since multifractional Brownian motion is an extension of fractional Brownian motion, we start with a review of the existing extensions of fBm. Most of the results in this section are well known, but we give new proofs based only on the covariance functions.

In the same way as Brownian motion has two main multi-parameter extensions, Levy Brownian motion and Brownian sheet, two different multi-parameter extensions of fractional Brownian motion have been defined.

2.1 Levy fractional Brownian motion.

The Levy fractional Brownian motion is defined to be a centered Gaussian process of covariance function

$$E\left[X_s X_t\right] = \frac{1}{2}\left[\|s\|^{2H} + \|t\|^{2H} - \|t-s\|^{2H}\right]. \tag{1}$$

There are several definitions of this process by its trajectories. Among these, it can be defined as integral against white noise. Lindstrom stated the following, see [**12**].

Proposition 1. *The process defined by*

$$X_t = \int_{\mathbf{R}^N} \left[\|t-u\|^{H-N/2} - \|u\|^{H-N/2} \right] .\mathbf{W}(du) \tag{2}$$

is a Levy fractional Brownian motion up to a multiplicative constant.

The harmonizable representation of fractional Brownian motion can also be generalized.

Proposition 2. *The process defined by*

$$X_t = \int_{\mathbf{R}^N} \frac{e^{i\langle t,\xi\rangle} - 1}{\|\xi\|^{H+(N/2)}} \,.\, \widehat{\mathbf{W}}(d\xi) \tag{3}$$

where $\widehat{\mathbf{W}}$ *is the Fourier transform of white noise in* $\mathbf{R}^N$*, is a Levy fractional Brownian motion up to a multiplicative constant.*

Proof. As will be done for multifractional Brownian field, the Fourier transform of the kernel of representation (2) could be directly computed. But as this representation defines a real centered Gaussian process, it is enough to show that the covariance function has the form (1).

For all $t \in \mathbf{R}^N$, let us denote by f_t the function $\xi \mapsto (e^{i<t,\xi>} - 1)/(\|\xi\|^{H+(N/2)})$ and consider the centered Gaussian process

$$X = \left\{ X_t = \widehat{W}(f_t);\ t \in \mathbf{R}_+^N \right\}$$

where $\widehat{W}$ denotes the Fourier transform of the complex isonormal process.

First, we remark easily that, for all t, almost surely, $\widehat{W}(f_t) \in \mathbf{R}$.

The covariance function of the real process X is

$$\begin{aligned} E\left[X_s X_t\right] &= E\Big[\widehat{W}(f_s)\overline{\widehat{W}(f_t)}\ \Big] \\ &= \int_{\mathbf{R}^N} \frac{(e^{i<s,\xi>} - 1)(e^{-i<t,\xi>} - 1)}{\|\xi\|^{2H+N}} \,.d\xi \\ &= \int_{\mathbf{R}^N} \frac{e^{i<s-t,\xi>} - e^{i<s,\xi>} - e^{-i<t,\xi>} + 1}{\|\xi\|^{2H+N}} \,.d\xi. \end{aligned}$$

Then we have to consider three integrals of the form

$$\int_{\mathbf{R}^N} \frac{(1-e^{i<t,\xi>})}{\|\xi\|^{2H+N}}.d\xi.$$

For $t \in \mathbf{R}^N$, $t \neq 0$, fixed, consider the change of variables from $\mathbf{R}^N$ into itself, $u = \phi(\xi)$ where ϕ is the linear application which maps the canonic basis of $\mathbf{R}^N$ to the orthonormal basis $(e_1 = t/\|t\|, e_2, \dots, e_N)$.

Then, we get

$$\int_{\mathbf{R}^N} \frac{1-e^{i<t,\xi>}}{\|\xi\|^{2H+N}}.d\xi = \int_{\mathbf{R}^N} \frac{1-e^{i\|t\|.u_1}}{\|u\|^{2H+N}}.du.$$

After the second change of variables

$$\begin{aligned} v &= \|t\|.u = \|t\| Id.u \\ dv &= \|t\|^N.du, \end{aligned}$$

we get

$$\int_{\mathbf{R}^N} \frac{1-e^{i<t,\xi>}}{\|\xi\|^{2H+N}}.d\xi = \frac{\|t\|^{2H+N}}{\|t\|^N} \underbrace{\int_{\mathbf{R}^N} \frac{1-e^{iv_1}}{\|v\|^{2H+N}}.dv}_{C_{N,H}>0}.$$

Proceeding the same way for the two other integrals, we can conclude

$$E[X_s X_t] = C_{N,H}\left[\|s\|^{2H} + \|t\|^{2H} - \|t-s\|^{2H}\right]$$

which shows that the process $\left\{1/(\sqrt{C_{N,H}})\widehat{W}(f_t), t \in \mathbf{R}_+^N\right\}$ is a Levy fractional Brownian motion. □

2.2 Fractional Brownian sheet. On the contrary to the Levy fractional Brownian motion, this process is not isotropic. In particular, we can have different Hurst parameters in each of the N directions.

The fractional Brownian sheet (fBs) is defined to be a centered Gaussian process of covariance function

$$E[X_s X_t] = \prod_{i=1}^{N} \frac{1}{2}\left(s_i^{2H_i} + t_i^{2H_i} - |t_i - s_i|^{2H_i}\right). \tag{4}$$

As in the isotropic case, this process has two different representations by its trajectories.

Proposition 3. *The process defined by*

$$X_t = \int_{\mathbf{R}^N} \prod_{i=1}^{N} \left[|t_i - u_i|^{H_i - 1/2} - |u_i|^{H_i - 1/2} \right] .\mathbf{W}(du)$$

is a fractional Brownian sheet, up to a multiplicative constant.

Remark 1. In [**8**], Pontier/Léger introduced another moving average representation of fractional Brownian sheet.

$$X_t = \int_{\mathbf{R}^N} \prod_{i=1}^{N} \left[(t_i - u_i)_+^{H_i - 1/2} - (-u_i)_+^{H_i - 1/2} \right] .\mathbf{W}(du).$$

Proof. This process is obviously Gaussian and centered. Thus, we only need to show that its covariance function has the expected form. We compute

$$E\left[X_s X_t\right] = \prod_{i=1}^{N} \int_{\mathbf{R}} \left[|s_i - u_i|^{H_i - 1/2} - |u_i|^{H_i - 1/2} \right] \cdot \left[|t_i - u_i|^{H_i - 1/2} - |u_i|^{H_i - 1/2} \right] .du_i.$$

We can see that the factor corresponding to each i, is the covariance of an fBm with Hurst parameter H_i (or a Levy fractional Brownian motion with $N = 1$). Then we have

$$E\left[X_s X_t\right] = \prod_{i=1}^{N} K_{1,H_i} \left[|s_i|^{2H_i} + |t_i|^{2H_i} - |t_i - s_i|^{2H_i} \right]. \qquad \square$$

This process also has a harmonizable representation, using the Fourier transform of the white noise in $\mathbf{R}^N$ as in the previous paragraph.

Proposition 4. *For all $t = (t_i)$, consider the function ϕ_t such that for all $\xi = (\xi_i)$,*

$$\phi_t(u) = \prod_{m=1}^{N} \frac{e^{it_m\xi_m} - 1}{|\xi_m|^{H_m+1/2}}.$$

The process defined by

$$X_t = \widehat{W}\,(\phi_t) = \int_{\mathbf{R}^N} \prod_{m=1}^{N} \frac{e^{it_m\xi_m} - 1}{|\xi_m|^{H_m+1/2}}\,.\widehat{\mathbf{W}}(d\xi)$$

is a fractional Brownian sheet, up to a multiplicative constant.

Proof. As in the previous proposition, let us compute the covariance function of this process.

$$\begin{aligned} E\left[X_s X_t\right] &= \prod_{m=1}^{N} \int_{\mathbf{R}} \frac{\left(e^{is_m\xi_m} - 1\right)\left(e^{-it_m\xi_m} - 1\right)}{|\xi_m|^{2H_m+1}}\,.d\xi_m \\ &= \prod_{m=1}^{N} C_{1,H_m}\left[|s_m|^{2H_m} + |t_m|^{2H_m} - |t_m - s_m|^{2H_m}\right] \end{aligned}$$

using the same argument of the previous proposition. □

Remark 2. The processes defined in Propositions 3 and 4 are proved to have the same law. In fact, as a particular case of Proposition 10, they are indistinguishable.

2.3 Stationarity of increments and self similarity. Let us start by recalling the notion of increments in $\mathbf{R}_+^N$.

For a function $f : [0,1]^N \to \mathbf{R}$ and $h \in \mathbf{R}$, one usually defines the progressive difference in direction ε_i by

$$\Delta_{h,i} f(x) = \begin{cases} f(x + h\varepsilon_i) - f(x) & \text{if } x, x + h\varepsilon_i \in [0,1]^N \\ 0 & \text{either} \end{cases}$$

and, for $h \in \mathbf{R}^N$ and $A = (i_1, \ldots, i_k)$,

$$\Delta_{h,A} f = \Delta_{h_{i_1},i_1} f \circ \cdots \circ \Delta_{h_{i_k},i_k} f.$$

Despite the temptation to define the increments by $X_t - X_s$ as in one dimension, it is better to set

$$\begin{aligned} \Delta X_{s,t} &= \Delta_{t-s,(1,\dots,N)} X_s \\ &= \sum_{r\in\{0,1\}^N} (-1)^{N-\sum_l r_l} X_{[s_i+r_i(t_i-s_i)]_i} . \end{aligned} \tag{5}$$

If there exists $i \in \{1,\dots,N\}$ such that $s_i = t_i$, we have $\Delta X_{s,t} = 0$. Then, we consider

$$I = \{i = 1,\dots,N;\ s_i \neq t_i\}$$

and

$$\Delta_{t-s,I} X_s = \sum_{r\in\{0,1\}^{\#I}} (-1)^{\#I-\sum_l r_l} X_{[s_i+r_i(t_i-s_i)]_{i\in I}} .$$

2.3.1 *Isotropic case.* In the isotropic case, the following extension of fBm's properties are well known, see [**12**].

Proposition 5. *Let $X = \left\{X_t;\ t \in \mathbf{R}_+^N\right\}$ be a Levy fractional Brownian motion. We have the following two properties for all $h \in \mathbf{R}_+^N$ and $a > 0$,*

$$\left\{X_{t+h} - X_h;\ t \in \mathbf{R}_+^N\right\} \stackrel{(d)}{=} \left\{X_t - X_0;\ t \in \mathbf{R}_+^N\right\}$$
$$\left\{X_{at};\ t \in \mathbf{R}_+^N\right\} \stackrel{(d)}{=} \left\{a^H X_t;\ t \in \mathbf{R}_+^N\right\}$$

where $\stackrel{(d)}{=}$ means equality of finite dimensional distributions.

Proposition 5 implies the stationarity of increments (5).

Proposition 6. *The increments of Levy fractional Brownian are stationary, i.e., for all $h \in \mathbf{R}_+^N$,*

$$\left\{\Delta X_{h,t+h};\ t \in \mathbf{R}_+^N\right\} \stackrel{(d)}{=} \left\{\Delta X_{0,t};\ t \in \mathbf{R}_+^N\right\} .$$

Proof. We fix $h \in \mathbf{R}_+^N$ and write

$$\Delta X_{h,t+h} = \sum_{r\in\{0,1\}^N-\{0\}} (-1)^{N-\sum_l r_l} \left(X_{[h_i+r_it_i]_i} - X_h\right);$$

then in the development of $E\left[\Delta X_{h,s+h}\Delta X_{h,t+h}\right]$, we only have terms of the form

$$E\left[\left(X_{[h_i+r_is_i]_i} - X_h\right)\left(X_{[h_i+\rho_it_i]_i} - X_h\right)\right] = E\left[X_{[r_is_i]_i} X_{[\rho_it_i]_i}\right]$$

using the previous proposition. Therefore, we have

$$E\left[\Delta X_{h,s+h}\Delta X_{h,t+h}\right] = E\left[\Delta X_{0,s}\Delta X_{0,t}\right]. \qquad \square$$

2.3.2 *Non-isotropic case.* In the non-isotropic case, the properties of self-similarity and stationarity of increments have been stated by Léger/Pontier, cf. [**11**]. Here, we give another proof based on the covariance function rather than the moving average representation.

Proposition 7. *Let $X = \left\{X_t; t \in \mathbf{R}_+^N\right\}$ be a fractional Brownian sheet. We have the following two properties for all $h \in \mathbf{R}_+^N$ and $a > 0$*

$$\left\{\Delta X_{h,t+h}; t \in \mathbf{R}_+^N\right\} \overset{(d)}{=} \left\{\Delta X_{0,t};\ t \in \mathbf{R}_+^N\right\}$$
$$\left\{X_{at};\ t \in \mathbf{R}_+^N\right\} \overset{(d)}{=} \left\{a^{\sum_i H_i}\ X_t;\ t \in \mathbf{R}_+^N\right\}.$$

Proof. We consider N independent fBm $X^{(1)}, \dots X^{(N)}$ of Hurst parameter H_i, and the process $Y = \left\{Y_t; t \in \mathbf{R}_+^N\right\}$ such that $Y_t = \prod_{i=1}^N X_{t_i}^{(i)}$. We can see easily that X and Y have the same covariance function. The same result follows for the increments $\left\{\Delta X_{h,t+h};\ t \in \mathbf{R}_+^N\right\}$ and $\left\{\Delta Y_{h,t+h};\ t \in \mathbf{R}_+^N\right\}$. As a consequence, from

$$\begin{aligned}\Delta Y_{h,t+h} &= \sum_{r\in\{0,1\}^N} (-1)^{N-\sum_l r_l} \prod_{i=1}^N X_{h_i+r_it_i}^{(i)} \\ &= \prod_{i=1}^N \left[X_{t_i+h_i}^{(i)} - X_{h_i}^{(i)}\right]\end{aligned}$$

we get

$$E\left[\Delta X_{h,s+h}\Delta X_{h,t+h}\right] = \prod_{i=1}^{N} \underbrace{E\left[\left(X^{(i)}_{s_i+h_i} - X^{(i)}_{h_i}\right)\left(X^{(i)}_{t_i+h_i} - X^{(i)}_{h_i}\right)\right]}_{E\left[X^{(i)}_{s_i} X^{(i)}_{t_i}\right]}$$
$$= E\left[\Delta X_{0,s}\Delta X_{0,t}\right].$$

For self-similarity, we verify easily that, for all $a > 0$

$$E\left[X_{as}X_{at}\right] = E\left[a^{\sum_i H_i}\ X_s\ a^{\sum_i H_i}\ X_t\right]. \qquad \square$$

Therefore, we can conclude that both extensions of fBm satisfy the properties of self-similarity and stationarity of increments.

3. The multifractional Brownian motion's case. Once again, we can consider two different kinds of multi-parameter extension of mBm: isotropic and anisotropic extension. Note, first of all, that mBm already has a multi-parameter extension. Indeed, the formulation of Benassi/Jaffard/Roux in [**4**] was done for $t \in \mathbf{R}^N$. We will see that it can be considered as an isotropic extension.

3.1 Isotropic extension. To define an isotropic extension of the mBm, the natural way is to substitute the constant H of the moving average representation of the Levy fractional Brownian motion, with a function.

Definition 1. Let $H : \mathbf{R}^N \to (0,1)$ be a measurable function. The process $\left\{X_t;\ t \in \mathbf{R}^N_+\right\}$ such that

$$X_t = \int_{\mathbf{R}^N} \left[\|t-u\|^{H(t)-(N/2)} - \|u\|^{H(t)-(N/2)}\right].\mathbf{W}(du) \tag{6}$$

is called multifractional Brownian field.

We will show that this process *is the same* as the process defined by Benassi/Jaffard/Roux. This result generalizes on the equivalence stated in the case $N = 1$ in [**6**].

Proposition 8. *Let $H : \mathbf{R}^N \to (0,1)$ be a measurable function. The process defined by*

$$X_t = \int_{\mathbf{R}^N} \frac{e^{i\langle t,\xi\rangle} - 1}{\|\xi\|^{H(t)+(N/2)}} \, .\widehat{\mathbf{W}}(d\xi) \tag{7}$$

is indistinguishable, up to a multiplicative deterministic function, from the process defined by (6)*. This formulation is the harmonizable representation of the multifractional Brownian field.*

Proof. First of all, let us compute the Fourier transform of the function $\|.\|^\alpha$, $-N < \alpha < 0$.

$$\begin{aligned}\langle \mathcal{T}\|.\|^\alpha, \varphi\rangle &= \langle \|.\|^\alpha, \hat{\varphi}\rangle \\ &= \int_{\mathbf{R}^N} \|t\|^\alpha \left(\int_{\mathbf{R}^N} e^{-i<w,t>} \varphi(w).dw \right) .dt\end{aligned}$$

we consider the change of variables

$$\begin{aligned}\mathbf{R}^N \times \mathbf{R}^N &\longrightarrow \mathbf{R}^N \times \mathbf{R}^N \\ (w,t) &\longmapsto (w, \lambda = \phi(t))\end{aligned}$$

where ϕ is the linear application which maps the canonic basis of $\mathbf{R}^N$ to the orthonormal basis $(e_1 = w/\|w\|, e_2, \ldots, e_N)$. We get

$$\begin{aligned}\langle \mathcal{T}\|.\|^\alpha, \varphi\rangle &= \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} \|\lambda\|^\alpha e^{i\lambda_1 \|w\|} \varphi(w).dw\, .d\lambda \\ &= \int_{\mathbf{R}^N} \int_{\mathbf{R}^N} \frac{\|u\|^\alpha}{\|w\|^\alpha} \, e^{-iu_1} \varphi(w) \, \frac{dw.du}{\|w\|^N}\end{aligned}$$

using the change of variables $(w, \lambda) \mapsto (w, u = \|w\|\lambda)$. Then we have

$$\langle \mathcal{T}\|.\|^\alpha, \varphi\rangle = \underbrace{\left(\int_{\mathbf{R}^N} \|u\|^\alpha e^{-iu_1} \, .du \right)}_{\lambda_\alpha} \int_{\mathbf{R}^N} \frac{1}{\|w\|^{\alpha+N}} \, \varphi(w) \, .dw.$$

Thus, for $-N < \alpha < 0$,

$$\mathcal{T}\|.\|^\alpha(w) = \frac{\lambda_\alpha}{\|w\|^{\alpha+N}}.$$

From this result, an elementary computation gives the Fourier transform of $\|t-.\|^{\alpha}-\|.\|^{\alpha}$. We get

$$\mathcal{T}\left[\|t-.\|^{\alpha}-\|.\|^{\alpha}\right](v)=\left[e^{-i<t,v>}-1\right]\frac{\lambda_{\alpha}}{\|v\|^{\alpha+N}}.$$

We deduce that for all $t\in\mathbf{R}^N$, almost surely,

$$\int_{\mathbf{R}^N}\left[\|t-u\|^{H(t)-(N/2)}-\|u\|^{H(t)-(N/2)}\right].\mathbf{W}(du) = \lambda_{H(t)}\int_{R^N}\frac{e^{i\langle t,\xi\rangle}-1}{\|\xi\|^{H(t)+(N/2)}}.\widehat{\mathbf{W}}(d\xi)$$

using the fact we saw previously that the second integral is almost surely real. Therefore, by an argument of continuity, the result follows. □

This process is obviously a centered Gaussian process. It is thus of interest to study its covariance function. The following proposition is an extension of the case $N=1$ stated in [**1**].

Proposition 9. *Let* $\left\{X_t;\ t\in\mathbf{R}_+^N\right\}$ *be a multifractional Brownian field. There exists a deterministic function* $D_N^f:\mathbf{R}\to\mathbf{R}$ *such that the covariance function of* X *can be written*

$$E\left[X_sX_t\right]=D_N^f(H(s)+H(t)) \times\left[\|s\|^{H(s)+H(t)}+\|t\|^{H(s)+H(t)}-\|t-s\|^{H(s)+H(t)}\right] \tag{8}$$

Proof. The easiest way to show this result is to use the harmonizable representation. By definition of $\widehat{\mathbf{W}}$, we have

$$E\left[X_sX_t\right]=\int_{\mathbf{R}^N}\frac{\left(e^{i<s,\xi>}-1\right)\left(e^{-i<t,\xi>}-1\right)}{\|\xi\|^{H(s)+H(t)+N}}.d\xi.$$

This integral has already been calculated for a Levy fractional Brownian motion with a parameter $H = (H(s) + H(t))/2$. Then we have

$$E\left[X_s X_t\right] = \underbrace{\left(\int_{\mathbf{R}^N} \frac{1-e^{iu_1}}{\|u\|^{H(s)+H(t)+N}}.du\right)}_{D_N^f(H(s)+H(t))} \times \left[\|s\|^{H(s)+H(t)} + \|t\|^{H(s)+H(t)} - \|t-s\|^{H(s)+H(t)}\right],$$

with $D_N^f(x) = \int_{\mathbf{R}^N} (1-e^{iu_1})/(\|u\|^{x+N}).du$. □

3.2 Non isotropic extension. Another way to extend the multifractional Brownian motion for a set of index included in $\mathbf{R}_+^N$, is to copy the definition of the Brownian sheet.

Definition 2. Let $H : \mathbf{R}_+^N \to (0,1)^N$ be a measurable function. The process $\left\{X_t; t \in \mathbf{R}_+^N\right\}$ such that

$$X_t = \int_{\mathbf{R}^N} \prod_{i=1}^{N} \left[|t_i - u_i|^{H_i(t)-1/2} - |u_i|^{H_i(t)-1/2}\right] \mathbf{W}(du)$$

where $\mathbf{W}$ is the white noise, is called multifractional Brownian sheet (mBs).

As in the case of the isotropic extension, there also exists a harmonizable representation of the mBs.

Proposition 10. *Let $H : \mathbf{R}_+^N \to (0,1)^N$ be a measurable function. For all $t = (t_i)_{i\in\{1;\dots;N\}}$, we consider the function ϕ_t such that for all $\xi = (\xi_i)$,*

$$\phi_t(u) = \prod_{m=1}^{N} \frac{e^{it_m\xi_m} - 1}{|\xi_m|^{H_m(t)+(1/2)}}.$$

The process defined by

$$X_t = \widehat{W}(\phi_t) = \int_{\mathbf{R}^N} \prod_{m=1}^{N} \frac{e^{it_m\xi_m} - 1}{|\xi_m|^{H_m(t)+(1/2)}} \widehat{\mathbf{W}}(d\xi)$$

is indistinguishable, up to a multiplicative deterministic function, from the process defined previously. This formulation is the harmonizable representation of the multifractional Brownian sheet.

Proof. We have already seen that for each $m \in \{1, \ldots, N\}$

$$\mathcal{T}\left[|t_m - .|^{H_m(t)-(1/2)} - |.|^{H_m(t)-(1/2)}\right](\xi_m) = \lambda_{H_m(t)} \overline{\left(\frac{e^{it_m\xi_m} - 1}{|\xi_m|^{H_m(t)+(1/2)}}\right)}.$$

By an easy computation,

$$\mathcal{T}\left(\prod_{m=1}^{N}\left[|t_m - .|^{H_m(t)-(1/2)} - |.|^{H_m(t)-(1/2)}\right]\right)(\xi) = \prod_{m=1}^{N}\mathcal{T}\left[|t_m - .|^{H_m(t)-(1/2)} - |.|^{H_m(t)-(1/2)}\right](\xi_m).$$

Therefore,

$$\underbrace{\left(\prod_{i=1}^{N}\lambda_m(t)\right)}_{\lambda(t)} \widehat{W} \overline{\left(\prod_{m=1}^{N}\frac{e^{it_m\cdot} - 1}{|.|^{H_m(t)+(1/2)}}\right)} = W\left(\prod_{m=1}^{N}\left[|t_m - .|^{H_m(t)-1/2} - |.|^{H_m(t)-1/2}\right]\right).$$

We use the same arguments as in Proposition 8 to conclude. □

The following proposition shows that the covariance structure of multifractional Brownian sheet, is a generalization of the fBs's one.

Proposition 11. *Let $\left\{X_t;\ t \in \mathbf{R}_+^N\right\}$ be a multifractional Brownian sheet. There exists a deterministic function $D^s : \mathbf{R}^N \to \mathbf{R}$ such that*

$$E\left[X_s X_t\right] = D^s\left(H(s) + H(t)\right)\prod_{m=1}^{N}\Big[|s_m|^{H_m(s)+H_m(t)} + |t_m|^{H_m(s)+H_m(t)} - |t_m - s_m|^{H_m(s)+H_m(t)}\Big]. \tag{9}$$

Proof. As usual, we use the harmonizable representation of the process

$$E\left[X_s X_t\right] = \prod_{m=1}^{N} \int_{\mathbf{R}} \frac{\left(e^{is_m\xi_m} - 1\right)\left(e^{-it_m\xi_m} - 1\right)}{|\xi_m|^{H_m(s)+H_m(t)+1}} . d\xi_m .$$

We remark that the factor corresponding to each m is the covariance of a multifractional Brownian motion, which has already been calculated. Therefore, we have

$$\begin{aligned} &E\left[X_s X_t\right] \\ &\quad = \prod_{m=1}^{N} D_1^f \left(H_m(s)+H_m(t)\right) \Big[|s_m|^{H_m(s)+H_m(t)} + |t_m|^{H_m(s)+H_m(t)} \\ &\qquad\qquad - |t_m - s_m|^{H_m(s)+H_m(t)} \Big] . \quad \square \end{aligned}$$

Remark 3. The form of the previous covariance function gives the idea to consider the process $Y = \left\{Y_t;\ t \in \mathbf{R}_+^N\right\}$ defined from N independent multifractional Brownian motions $X^{(i)}$ with parameter H_i by

$$Y_t = X^{(1)}_{t^{(1)}} \dots X^{(N)}_{t^{(N)}} .$$

Although Y is not a Gaussian process, it is easily seen that it has the same covariance function as a multifractional Brownian sheet. This remark will be often used in the following.

4. Regularity. A lot of properties are known about the regularity of the trajectories of Brownian motion and fractional Brownian motion. As we will see, in the case of the multi-parameter extension of the mBm, we have to make some assumptions about the regularity of H before studying the continuity of trajectories. In the definitions of mBm, cf. [**2, 4**], the function H is supposed to be Hölder continuous.

4.1 Existence of a continuous modification. As usual, the quantity $E\left[|X_t - X_s|^2\right]$ is studied for $s, t \in [a, b]$ where $a \preceq b$ to use Kolmogorov's criterion, cf. [**10**]. The following paragraphs show that in both isotropic and anisotropic cases, under Hölder regularity assumptions for H, we have

$$E\left[X_t - X_s\right]^2 \leq K\|t-s\|^{\alpha}$$

for some $\alpha > 0$. As usual, in the Gaussian case, we can write, for each integer n

$$E\left[X_t - X_s\right]^{2n} \leq \lambda_n \, K\|t-s\|^{n.\alpha}$$

and choose n such that $n.\alpha > N$.

Then, a classical patching argument is used to extend to $\mathbf{R}_+^N$ the existence of a continuous modification of the two processes.

4.1.1 *Isotropic case.*

Lemma 1. *For all η and μ such that $0 < \eta < \mu < 1$, the multiplicative factor D_N^f of covariance function in (9), is positive and belongs to $C^{\infty}([\eta,\mu])$. Moreover, the order n derivative is given by*

$$D_N^{f\,(n)}(x) = \int_{\mathbf{R}^N} \frac{1-e^{iu_1}}{\|u\|^{x+N}} \, \ln^n \, \frac{1}{\|u\|} \, .du. \tag{10}$$

Proof. As the integral of a positive function, D_N^f is positive. By an argument of uniform convergence of integrals (10) on $[\eta,\mu]$, D_N^f is $C^{\infty}([\eta,\mu])$. □

Proposition 12. *For all $s,t \in [a,b]$, we have*

(11)

$$\begin{aligned}\frac{1}{2}E\left[X_t - X_s\right]^2 = {} & D\left[H(s)+H(t)\right] \times \|t-s\|^{H(s)+H(t)} \\ & + \frac{1}{2}\left[\frac{\partial^2\varphi}{\partial x^2}\left(H(s)+H(t);\|s\|\right) + \frac{\partial^2\varphi}{\partial x^2}\left(H(s)+H(t);\|t\|\right)\right] \\ & \times \left(H(t)-H(s)\right)^2 \\ & + O_{a,b}\left[\left(H(t)-H(s)\right)\left(\|t\|-\|s\|\right)\right] + o_{a,b}\left(H(t)-H(s)\right)^2,\end{aligned}$$

where $\varphi(x,y) = D(x)y^x$.

Proof. Using the covariance function of the multifractional Brownian field, we have

(12)

$$\frac{1}{2}E\left[|X_s-X_t|^2\right] = D\left[2H(s)\right]\,\|s\|^{2H(s)} - D\left[H(s)+H(t)\right]\,\|s\|^{H(s)+H(t)} + D\left[2H(t)\right]\,\|t\|^{2H(t)} - D\left[H(s)+H(t)\right]\,\|t\|^{H(s)+H(t)} + D\left[H(s)+H(t)\right]\,\|t-s\|^{H(s)+H(t)}.$$

We have to get a second order expansion of this expression.

We introduce the function φ defined by

$$\varphi(x,y) = D(x)y^x.$$

We can write

$$\begin{aligned}\frac{1}{2}\,E\left[|X_s-X_t|^2\right] &= \varphi(2H(s),\|s\|) - \varphi(H(s)+H(t),\|s\|)\\ &\quad + \varphi(2H(t),\|t\|) - \varphi(H(s)+H(t),\|t\|)\\ &\quad + D\left[H(s)+H(t)\right]\,\|t-s\|^{H(s)+H(t)}.\end{aligned} \tag{13}$$

We use the second order expansion

$$\begin{aligned}&\varphi(2H(s),\|s\|) - \varphi(H(s)+H(t),\|s\|)\\ &\quad = (H(s)-H(t)) \times \frac{\partial\varphi}{\partial x}\,(H(s)+H(t),\|s\|) + \frac{(H(s)-H(t))^2}{2}\\ &\qquad \times \frac{\partial^2\varphi}{\partial x^2}\,(H(s)+H(t),\|s\|) + o_{a,b}\,(H(s)-H(t))^2\,.\end{aligned}$$

An inversion of roles between s and t provides the expansion of

$$\varphi(2H(t),\|t\|) - \varphi(H(s)+H(t),\|t\|).$$

Then (13) becomes

$$\begin{aligned}&\frac{1}{2}\,E\left[|X_s-X_t|^2\right]\\ &= (H(t)-H(s))\left[\frac{\partial\varphi}{\partial x}\,(H(s)+H(t),\|t\|) - \frac{\partial\varphi}{\partial x}\,(H(s)+H(t),\|s\|)\right]\\ &\quad + \frac{(H(t)-H(s))^2}{2}\left[\frac{\partial^2\varphi}{\partial x^2}\,(H(s)+H(t),\|s\|) + \frac{\partial^2\varphi}{\partial x^2}\,(H(s)+H(t),\|t\|)\right]\\ &\quad + D\left[H(s)+H(t)\right]\,\|t-s\|^{H(s)+H(t)} + o_{a,b}\,(H(t)-H(s))^2\,.\end{aligned}$$

Since

$$(H(t)-H(s))\times\left[\frac{\partial\varphi}{\partial x}\,(H(s)+H(t),\|t\|)-\frac{\partial\varphi}{\partial x}\,(H(s)+H(t),\|s\|)\right]$$

is $O_{a,b}\left[(H(t)-H(s))\,(\|t\|-\|s\|)\right]$, the result follows. □

Corollary 1. *For all* $s,t\in[a,b]$, *we have*

$$\begin{aligned}\frac{1}{2}\,E\left[X_t-X_s\right]^2 &= D\left[2H(t)\right]\times\|t-s\|^{2H(t)}\\ &\quad+\frac{\partial^2\varphi}{\partial x^2}\,(2H(t);\|t\|)\times(H(t)-H(s))^2\\ &\quad+o_{a,b}\,(H(t)-H(s))^2+o_{a,b}\left(\|t-s\|^{2H(t)}\right)\end{aligned}\tag{14}$$

where $\varphi(x,y)=D(x)y^x$.

Proof. Using the expansion of $D\left[H(s)+H(t)\right]$ and

$$\begin{aligned}\|t-s\|^{H(s)+H(t)} &= \|t-s\|^{2H(t)}-(H(t)-H(s))\,\|t-s\|^{2H(t)}\ln\|t-s\|\\ &\quad+o_{a,b}\,(H(t)-H(s))^2,\end{aligned}$$

we get

$$\begin{aligned}&D\left[H(s)+H(t)\right]\times\|t-s\|^{H(s)+H(t)}\\ &\quad= D\left[2H(t)\right]\times\|t-s\|^{2H(t)}+o_{a,b}\left(\|t-s\|^{2H(t)}\right)+o_{a,b}\,(H(t)-H(s))^2.\end{aligned}\tag{15}$$

Moreover, as $H(t)<1$ for all $t\in[a,b]$, we have $\varepsilon=1-H(t)>0$ and

$$\begin{aligned}&2\,(H(t)-H(s))\,(\|t\|-\|s\|)\\ &\quad= 2\,(H(t)-H(s))\,(\|t\|-\|s\|)^{\varepsilon/2}\times(\|t\|-\|s\|)^{1-(\varepsilon/2)}\\ &\quad\le (H(t)-H(s))^2\,(\|t\|-\|s\|)^{\varepsilon}+(\|t\|-\|s\|)^{2-\varepsilon}\end{aligned}$$

that implies

$$(H(t)-H(s))\,(\|t\|-\|s\|)=o_{a,b}\,(H(t)-H(s))^2+o_{a,b}\left(\|t-s\|^{2H(t)}\right).\tag{16}$$

We conclude by (11), (15) and (16) using first order expansion of $\partial^2\varphi/\partial x^2$ in x and y. □

Using the continuity of D, D' and D'', we can state from the previous proposition

Corollary 2. *There exist positive constants K and L such that*

$$\forall\, s,t \in [a,b]; \quad E\left[X_t - X_s\right]^2 \leq K\, \|t-s\|^{2H(t)} + L\, |H(t)-H(s)|^2 . \tag{17}$$

Corollary 3. *Suppose H is β-Hölder continuous. There exists a constant M such that*

$$\forall\, s,t \in [a,b]; \quad E\left[X_t - X_s\right]^2 \leq M\, \|t-s\|^{2(\beta\wedge H(t))}. \tag{18}$$

4.1.2 *Non-isotropic case.*

Lemma 2. *There exists positive constants K and L such that*

$$\forall\, s,t \in [a,b]; \quad E\left[|X_t - X_s|^2\right] \leq K\, \|t-s\|^{2\min_i H_i(t)} + L\, \|H(t)-H(s)\|^2. \tag{19}$$

Proof. By Remark 3, we have

$$\begin{aligned} E\left[X_s - X_t\right]^2 &= E\left[\prod_{i=1}^N X^{(i)}_{s^{(i)}} - \prod_{i=1}^N X^{(i)}_{t^{(i)}}\right]^2 \\ &= E\left[\left(\prod_{i=1}^N X^{(i)}_{s^{(i)}} - X^{(1)}_{t^{(1)}} \prod_{i>1} X^{(i)}_{s^{(i)}}\right)\right. \\ &\quad + \left(X^{(1)}_{t^{(1)}} \prod_{i>1} X^{(i)}_{s^{(i)}} - X^{(1)}_{t^{(1)}} X^{(2)}_{t^{(2)}} \prod_{i>2} X^{(i)}_{s^{(i)}}\right) \\ &\quad + \cdots + \left.\left(\left(\prod_{i=1}^{N-1} X^{(i)}_{t^{(i)}}\right) X^{(N)}_{s^{(N)}} - \prod_{i=1}^N X^{(i)}_{t^{(i)}}\right)\right]^2 . \end{aligned}$$

Then

$$E\left[X_s - X_t\right]^2 \le N \left\{ E\left[\prod_{i>1} X^{(i)}_{s^{(i)}}\right]^2 E\left[X^{(1)}_{s^{(1)}} - X^{(1)}_{t^{(1)}}\right]^2 + \cdots \right.$$
$$\left. + E\left[X^{(N)}_{s^{(N)}} - X^{(N)}_{t^{(N)}}\right]^2 E\left[\prod_{i=1}^{N-1} X^{(i)}_{s^{(i)}}\right]^2 \right\}$$

and

$$E\left[X_s - X_t\right]^2 \le N M^{n-1} \sum_{i=1}^{N} E\left[X^{(i)}_{s^{(i)}} - X^{(i)}_{t^{(i)}}\right]^2 \tag{20}$$

with $M = M_{a,b} = \sup_{i,t} E\left[X^{(i)}_{t^{(i)}}\right]^2$.

Using

$$E\left[X^{(i)}_{s^{(i)}} - X^{(i)}_{t^{(i)}}\right]^2 \le K_i |s^{(i)} - t^{(i)}|^{2H_i(t)} + L_i \left(H_i(s) - H_i(t)\right)^2$$
$$\forall\, i = 1, \ldots, N,$$

(20) implies

$$E\left[X_t - X_s\right]^2$$
$$\le N M^{n-1} \left[\left(\sum_{i=1}^{N} K_i\right) \|t-s\|^{2 \min_i H_i(t)} + \left(\sum_{i=1}^{N} L_i\right) \|H(t) - H(s)\|^2 \right]. \quad \square$$

Corollary 4. *Suppose H is β-Hölder continuous. There exists a positive constant M such that*

$$\forall\, s, t \in [a,b]; \quad E\left[X_t - X_s\right]^2 \le M \|t - s\|^{2(\beta \wedge \min_i H_i(t))}. \tag{21}$$

4.2 Hölder exponents. The notion of Hölder function is well known. It is interesting to consider a localized version of this notion.

For the paths of a process X, one usually define two kinds of exponent, see [**2, 3**]:

• the pointwise Hölder exponent

$$\alpha(t_0) = \sup\left\{\alpha;\ \lim_{h\to 0}\frac{|X_{t_0+h}-X_{t_0}|}{\|h\|^\alpha}=0\right\}$$
$$= \sup\left\{\alpha;\ \limsup_{\rho\to 0}\frac{\sup_{s,t\in B(t_0,\rho)}|X_t-X_s|}{\rho^\alpha}<\infty\right\}.$$

• the local Hölder exponent

$$\tilde{\alpha}(t_0) = \sup\left\{\alpha;\ \limsup_{\rho\to 0}\sup_{s,t\in B(t_0,\rho)}\frac{|X_t-X_s|}{\|t-s\|^\alpha}<\infty\right\}.$$

We can see easily that, for all t_0, we have

$$\tilde{\alpha}(t_0)\le\alpha(t_0). \tag{22}$$

A study of these exponents, in the case of 1D mBm, is made in [**3**].

Remark 4. If H is β-Hölder continuous, then the local Hölder exponent $\tilde{\beta}(t)$ of H at every point is not smaller than β.

Conversely, suppose that the local Hölder exponent of H at every point of a compact $[a,b]$ is positive. Then H is β-Hölder continuous on $[a,b]$ with $\beta=\inf_{t\in[a,b]}\tilde{\beta}(t)$.

In the same way, one may define directional pointwise and local Hölder exponents in the direction $u\in\mathcal{U}=\{u\in\mathbf{R}^N;\ \|u\|=1\}$ by

$$\alpha_u(t_0) = \sup\left\{\alpha;\ \lim_{\rho\to 0}\frac{|X_{t_0+\rho.u}-X_{t_0}|}{\rho^\alpha}=0\right\}$$

and

$$\tilde{\alpha}_u(t_0) = \sup\left\{\alpha;\ \limsup_{\rho\to 0}\sup_{\substack{s,t\in B(t_0,\rho)\\ s,t\in t_0+\mathbf{R}.u}}\frac{|X_t-X_s|}{\|t-s\|^\alpha}<\infty\right\}.$$

As previously, for all $u \in \mathcal{U}$, we have

$$\tilde{\alpha}_u(t_0) \le \alpha_u(t_0). \tag{23}$$

Moreover, we can see easily that for all $u \in \mathcal{U}$, we have

$$\alpha(t_0) \le \alpha_u(t_0) \quad \text{and} \quad \tilde{\alpha}(t_0) \le \tilde{\alpha}_u(t_0). \tag{24}$$

In the following, we suppose that H admits positive local Hölder exponent $\tilde{\beta}(t_0)$ at every point t.

Proposition 13. *Let $X = \{X_t;\, t \in \mathbf{R}_+^N\}$ be a multifractional Brownian field, respectively sheet. For all $t_0 \in \mathbf{R}_+^N$, the local Hölder exponent of X at t_0 is almost surely given by*

$$\tilde{\alpha}(t_0) = \tilde{\beta}(t_0) \wedge H(t_0) \quad (\text{resp. } \tilde{\beta}(t_0) \wedge \min_i H_i(t_0)) \tag{25}$$

and the pointwise Hölder exponent of X at t_0 satisfies almost surely

$$\alpha(t_0) = \beta(t_0) \wedge H(t_0) \quad (\text{resp. } \beta(t_0) \wedge \min_i H_i(t_0)) \tag{26}$$

where $\beta(t_0)$ and $\tilde{\beta}(t_0)$ denote the pointwise and local Hölder exponents of H at t_0.

As a consequence of this result, if H satisfies

$$\forall\, t \in \mathbf{R}_+^N; \quad \beta(t) < H(t)$$

the Hölder regularity of multifractional Brownian field of parameter function H is given by the regularity of H (and not by the value of H). This point is developed in [**7**].

The proof of Proposition 13 is detailed in the following three paragraphs.

4.2.1 *Lower bound for the local Hölder exponent.* A lower bound for the local Hölder exponent is directly given by Kolmogorov's theorem. Indeed, for X a multifractional Brownian field or a multifractional Brownian sheet indexed by $[a, b]$, by Corollaries 3 and 4, Kolmogorov's theorem states that there exists a modification of X, which is q-Hölder continuous for all $q \in (0,\alpha)$, with $\alpha = \inf_{[a,b]}(\tilde{\beta} \wedge H)$ or $\alpha = \inf_{[a,b]}(\tilde{\beta} \wedge \min_i H_i)$.

Then, localizing this result, we get

- in the isotropic case,

$$\tilde{\alpha}(t_0) \geq \tilde{\beta}(t_0) \wedge H(t_0) \tag{27}$$

- in the non-isotropic case,

$$\tilde{\alpha}(t_0) \geq \tilde{\beta}(t_0) \wedge \min_i H_i(t_0). \tag{28}$$

4.2.2 *Lower bound for the pointwise Hölder exponent.* By (22), paragraph 4.2.1 provides a lower bound for the pointwise Hölder exponent. However, it can be improved in the case $\tilde{\beta}(t_0) < \beta(t_0)$.

Let $X = \left\{X_t; t \in \mathbf{R}_+^N\right\}$ be a multifractional Brownian field. By Corollary 2, there exist positive constants K and L such that for all $s,t \in \mathbf{R}_+^N$,

$$E\left[X_t - X_s\right]^2 \leq K\,\|t-s\|^{2H(t)} + L\,|H(t)-H(s)|^2$$

and by Corollary 3, there exist positive constants α and M such that

$$\forall\, s,t \in [a,b]; \quad E\left[X_t - X_s\right]^2 \leq M\|t-s\|^{\alpha}.$$

Therefore, using Kolmogorov's criterion, there exists a modification of X, which is ν-Hölder continuous for all $\nu \in (0,\alpha/2)$. In the following, we consider such a ν with $1/\nu \in \mathbf{N}$.

For any $\varepsilon > 0$, there exist $\rho_0 > 0$ and $\widehat{M} > 0$ such that for all $\rho < \rho_0$ and all $t \in B(t_0,\rho)$

$$E\left[\frac{X_t - X_{t_0}}{\rho^{\beta(t_0)\wedge H(t_0)-\varepsilon}}\right]^2 \leq \widehat{M}\,\rho^{\varepsilon}.$$

Then, setting $\gamma = \beta(t_0) \wedge H(t_0) - \varepsilon$, for any $p \in \mathbf{N}^*$,

$$P\left\{|X_t - X_{t_0}| > \rho^{\gamma}\right\} \leq E\left[\frac{X_t - X_{t_0}}{\rho^{\gamma}}\right]^{2p} \leq M_p\,\rho^{p\varepsilon}.$$

Let $\rho = 2^{-n}$ and for all $m \in \mathbf{N}$,

$$D_m = \left\{ t_0 + k.2^{-(n+m)};\ k \in \{0, \pm 1, \dots, \pm 2^m\}^N \right\},$$

let us compute

$$\begin{aligned} P\left\{ \max_{k \in \{\pm 1, \dots, \pm 2^m\}^N} \frac{|X_{t_0 + k.2^{-(m+n)}} - X_{t_0}|}{2^{-\gamma n}} > 1 \right\} & \\ &\leq \sum_{k \in \{\pm 1, \dots, \pm 2^m\}^N} P\left\{ |X_{t_0 + k.2^{-(n+m)}} - X_{t_0}| > 2^{-\gamma n} \right\} \\ &\leq M_p\, 2^{(m+1)N}\, 2^{-p\varepsilon n}. \end{aligned}$$

Let us take $m = (1 + \lfloor \gamma \rfloor / \nu) n = \kappa n$ and $p \in \mathbf{N}$ such that $N(1 + \lfloor \gamma \rfloor / \nu) - p\varepsilon < 0$. By the Borel-Cantelli lemma, there exists a finite random variable n^* such that almost surely,

$$\forall\, n \geq n^*; \quad \max_{k \in \{0, \dots, \pm 2^{\kappa n}\}^N} |X_{t_0 + k.2^{-(1+\kappa)n}} - X_{t_0}| \leq 2^{-\gamma n}. \tag{29}$$

From (29), we show that for all $n \geq n^*$, almost surely, for all $m \in \mathbf{N}$, we have

$$\forall\, t \in D_m; \quad |X_t - X_{t_0}| \leq C\, 2^{-\gamma n} \tag{30}$$

- if $0 \leq m \leq \kappa n$, (30) follows directly from (29)
- if $m > \kappa n$, for any $t \in D_m$, let

$$C^{\kappa n}_{t_0, t} = \{ x \in D_{\kappa n};\ \forall\, i, (t_0)_i \leq x_i \leq t_i \}.$$

Then consider $\hat{t} \in B(t, 2^{-(1+\kappa)n}) \cap C^{\kappa n}_{t_0, t}$.

As the paths of X are ν-Hölder continuous, we have

$$|X_{\hat{t}} - X_t| \leq \widetilde{C} 2^{-\nu(1+\kappa)n} \leq \widetilde{C} 2^{-\gamma n}$$

and by (29),

$$|X_{\hat{t}} - X_{t_0}| \leq 2^{-\gamma n}.$$

Using the triangular inequality, the result follows.

Therefore, (30) leads to

$$\forall\, m \in \mathbf{N};\ \forall\, s,t \in D_m; \quad |X_t - X_s| \le 2C\; 2^{-\gamma n}.$$

Using the continuity of X and $m \to +\infty$, we get

$$\sup_{s,t \in B(t_0, 2^{-n})} |X_t - X_s| \le 2C\; 2^{-\gamma n}$$

and therefore, almost surely,

$$\limsup_{\rho \to 0} \sup_{s,t \in B(t_0,\rho)} \frac{|X_t - X_s|}{\rho^{\gamma}} < +\infty. \tag{31}$$

By (31), for all $\varepsilon > 0$, almost surely

$$\alpha(t_0) \ge \beta(t_0) \wedge H(t_0) - \varepsilon.$$

Taking $\varepsilon \in \mathbf{Q}_+$, we have almost surely

$$\alpha(t_0) \ge \beta(t_0) \wedge H(t_0). \tag{32}$$

For a multifractional Brownian sheet X, by Lemma 2, we get in the same way that, almost surely

$$\alpha(t_0) \ge \beta(t_0) \wedge H_i(t_0) \tag{33}$$

for all $i = 1, \ldots, N$.

4.2.3 *Upper bound for the pointwise Hölder exponent.* The main result getting the upper bound for the Hölder exponents is the following lemma, a direct consequence of Proposition 12 using continuity of D, D' and D''.

Lemma 3. *Let $X = \{X_t;\ t \in \mathbf{R}_+^N\}$ be a multifractional Brownian field. For all $[a,b] \subset \mathbf{R}_+^N$, there exist positive constants k_1, k_2, l_1, l_2 such that*

(34)

$$\forall\, s,t \in [a,b]; \quad E\,[X_t - X_s]^2 \ge k_1\, \|t-s\|^{2H(t)} - l_1\,(H(t) - H(s))^2$$

$$E\,[X_t - X_s]^2 \ge k_2\,(H(t) - H(s))^2 - l_2\, \|t-s\|^{2H(t)}. \tag{35}$$

Proof. We only have to study the multiplicative factors of $\|t-s\|^{2H(t)}$ and $(H(t)-H(s))^2$ in (11). The proof only relies on continuity and positivity of the two functions $t \mapsto D[2H(t)]$ and

$$t \mapsto \|t\|^{2H(t)} \times \left\{ D[2H(t)] \ln^2 \|t\| - 2D'[2H(t)] \ln \|t\| + D''[2H(t)] \right\}. \quad \square$$

Lemma 4. *Let $X = \{X_t;\ t \in \mathbf{R}_+^N\}$ be a multifractional Brownian sheet. For all $[a,b] \subset \mathbf{R}_+^N$, there exist positive constants k_1, k_2, l_1, l_2 such that*

$$\forall\, s,t \in [a,b];\ t-s \in \mathbf{R}_+.\varepsilon_i$$

$$E[X_t - X_s]^2 \geq k_1 \|t-s\|^{2H_i(t)} - l_1 (H_i(t) - H_i(s))^2 \tag{36}$$

$$E[X_t - X_s]^2 \geq k_2 (H_i(t) - H_i(s))^2 - l_2 \|t-s\|^{2H_i(t)}. \tag{37}$$

Proof. For all s, t such that $t-s \in \mathbf{R}_+.\varepsilon_i$, using Lemma 3, we have

$$\begin{aligned} E[X_t - X_s]^2 &= E\left[X^{(i)}_{t^{(i)}} - X^{(i)}_{s^{(i)}}\right]^2 \prod_{j \neq i} E\left[X^{(j)}_{t^{(j)}}\right]^2 \\ &\geq k_1 |t_i - s_i|^{2H_i(t)} - l_1 (H_i(t) - H_i(s))^2 \end{aligned}$$

and

$$E[X_t - X_s]^2 \geq k_2 (H_i(t) - H_i(s))^2 - l_2 |t_i - s_i|^{2H_i(t)}. \quad \square$$

From this result, the upper bound for the pointwise exponent is a consequence of the following lemma whose proof is the same as the case $N=1$, see [**2**].

Lemma 5. *Let $X = \{X_t;\ t \in \mathbf{R}_+^N\}$ be a Gaussian process. Assume there exists $\mu \in (0,1)$ such that for all $\varepsilon > 0$, there exist a sequence $(h_n)_{n \in \mathbf{N}}$ of $\left(\mathbf{R}_+^N\right)^*$ converging to 0 and a constant $c > 0$ such that*

$$\forall\, n \in \mathbf{N}; \quad E[X_{t+h_n} - X_t]^2 \geq c \|h_n\|^{2\mu+\varepsilon}.$$

Then we have almost surely

$$\alpha(t) \le \mu.$$

Let $X = \left\{X_t;\ t \in \mathbf{R}_+^N\right\}$ be a multifractional Brownian field, respectively multifractional Brownian sheet. Let $\beta(t_0)$ be the pointwise Hölder exponent of H at t_0. We consider the two cases:

• if $H(t_0) < \beta(t_0)$, respectively $H_i(t_0) < \beta(t_0)$, by definition of $\beta(t_0)$, we have

$$\lim_{h\to 0} \frac{\|H(t_0+h) - H(t_0)\|}{\|h\|^{H(t_0)}} = 0.$$

Hence, by (34), respectively (36), there exists a positive constant C such that

$$E\left[X_{t_0+h} - X_{t_0}\right]^2 \ge C\,\|h\|^{2H(t_0)}.$$

Then, by Lemma 5,

$$\alpha(t_0) \le H(t_0) \quad (\text{respectively } H_i(t_0)) \tag{38}$$

• if $H(t_0) > \beta(t_0)$, respectively $H_i(t_0) > \beta(t_0)$, we consider $\alpha \in (\beta(t_0); H(t_0))$, respectively $\alpha \in (\beta(t_0); H_i(t_0))$. There exists a positive constant C and a sequence $(h_n)_{n\in\mathbf{N}}$ converging to 0 such that

$$\forall\, n \in \mathbf{N};\quad \|H(t_0+h_n) - H(t_0)\| > C\,\|h_n\|^{\alpha}.$$

Then, by (35), respectively (37),

$$\begin{aligned}\forall\, n \in \mathbf{N};\quad E\left[X_{t_0+n} - X_{t_0}\right]^2 &> k_2\, C\,\|h_n\|^{2\alpha} - l_2\,\|h_n\|^{2H(t_0)}\\ &\ge C'\,\|h_n\|^{2\alpha};\end{aligned}$$

hence, by Lemma 5,

$$\alpha \ge \alpha(t_0),$$

and therefore

$$\alpha(t_0) \le \beta(t_0). \tag{39}$$

We can restate the upper bounds (38) and (39) of the pointwise Hölder exponent of X at t_0

$$\alpha(t_0) \leq \beta(t_0) \wedge H(t_0) \quad (\text{resp. } \beta(t_0) \wedge H_i(t_0)). \tag{40}$$

4.2.4 *Upper bound for the local Hölder exponent.* By (22), any upper bound for the pointwise Hölder exponent is an upper bound for the local Hölder exponent. But we can improve on this result in the case $\tilde{\beta}(t_0) < H(t_0)$. We first give an analogous of Lemma 5 for the local exponent, whose proof is very similar.

Lemma 6. *Let $X = \{X_t;\ t \in \mathbf{R}_+^N\}$ be a Gaussian process. Assume there exists $\mu \in (0,1)$ such that for all $\varepsilon > 0$, there exist two sequences $(h_n)_{n\in\mathbf{N}}$ and $(l_n)_{n\in\mathbf{N}}$ of $(\mathbf{R}_+^N)^*$ converging to 0, and a constant $c > 0$ such that*

$$\forall\, n \in \mathbf{N}; \quad E\left[X_{t_0+h_n} - X_{t_0+l_n}\right]^2 \geq c\,\|h_n - l_n\|^{2\mu+\varepsilon}.$$

Then we have almost surely

$$\tilde{\alpha}(t_0) \leq \mu.$$

Let $\alpha \in (\tilde{\beta}(t_0), H(t_0))$, respectively $\alpha \in (\tilde{\beta}(t_0), H_i(t_0))$. As

$$\limsup_{\rho\to 0} \sup_{s,t\in B(t_0,\rho)} \frac{|H(t)-H(s)|}{\|t-s\|^{\alpha}} = +\infty$$

for all $M > 0$, there exists $\rho_0 > 0$ such that

$$\forall\, \rho < \rho_0; \quad \exists s,t \in B(t_0,\rho); \quad |H(t)-H(s)| > M\,\|t-s\|^{\alpha}.$$

Therefore we can construct two sequences (h_n) and (l_n) converging to 0 such that

$$\forall\, n \in \mathbf{N}; \quad |H(t_0+h_n) - H(t_0+l_n)| > M\,\|h_n - l_n\|^{\alpha}.$$

By Lemma 6, we can deduce

$$\tilde{\alpha}(t_0) \leq \tilde{\beta}(t_0). \tag{41}$$

5. Locally asymptotic self-similarity. Extending fBm into multifractional Brownian motion implies the loss of the two properties of self-similarity and stationarity of increments. However, a weak form of self-similarity remains, called locally asymptotic self-similarity, see [**2, 4**]. As we will see, this property still holds for the two kinds of extension of mBm in $\mathbf{R}^N$.

Theorem 1. *Let* $X = \{X_t;\ t \in \mathbf{R}_+^N\}$ *be a multifractional Brownian field.*

For all $t_0 \in \mathbf{R}_+^N$*, the law of the process*

$$Y^{\alpha}(\rho) = \left\{ Y_u^{\alpha}(\rho) = \frac{X_{t_0+\rho u} - X_{t_0}}{\rho^{\alpha}};\ u \in \mathbf{R}_+^N \right\}$$

converges weakly if one of the following two conditions holds

1. $\alpha = H(t_0)$ *and* $H(t_0) < \inf_{u,v} \beta_{uv}(t_0)$ *where*

$$\beta_{uv}(t_0) = \sup \left\{ \alpha;\ \lim_{\rho \to 0} \frac{|H(t_0+\rho u) - H(t_0+\rho v)|}{\rho^{\alpha}} = 0 \right\}.$$

Then, the limit measure is the law of a fractional Brownian field with parameter $H(t_0)$*.*

2. $\alpha = \inf_{u,v} \beta_{uv}(t_0)$*,* $H(t_0) > \inf_{u,v} \beta_{uv}(t_0)$ *and for all* $u, v \in \mathbf{R}_+^N$*, the following limit exists*

$$\lim_{\rho \to 0} \frac{|H(t_0+\rho u) - H(t_0+\rho v)|}{\rho^{\inf_{u,v} \beta_{uv}(t_0)}} = \Gamma(u,v)$$

with $(u,v) \mapsto (\Gamma(u,v))/\|u-v\|^{2\beta}$ *bounded on* $[a,b]^2$ *for some* $\beta > 0$*.*

The limit measure is the law of a Gaussian process $Y^{\inf_{u,v} \beta_{uv}(t_0)}$ *such that*

$$E\left[Y_u^{\inf_{u,v} \beta_{uv}(t_0)} - Y_v^{\inf_{u,v} \beta_{uv}(t_0)} \right]^2 = K_{t_0} \left[\Gamma(u,v) \right]^2.$$

Remark 5. As in the Levy fBm's case in Proposition 6, the same result as Theorem 1 can be stated for the increments ΔX defined in section 2.3. The law of the process

$$Y^{\alpha}(\rho) = \left\{Y_u^{\alpha}(\rho) = (\Delta X_{t_0,t_0+\rho u})/\rho^{\alpha};\ u \in \mathbf{R}_+^N\right\}$$

converges weakly under the same assumptions.

In the case $N = 1$, for all $u, v \in \mathbf{R}_+$, we have $\beta_{uv}(t_0) = \beta(t_0)$. Therefore, Theorem 1 has a simpler statement. The two cases to be considered, depend of the comparison between $H(t_0)$ and the pointwise exponent $\beta(t_0)$ of H.

The following example shows that the limit considered in the second case, can be non trivial.

Example 1. In the case $N=1$, let $H(t)=3/4+t^{1/2}$ for $t \in [0, 1/16]$.

For $t_0 = 0$, we compute, for all u, v and $\rho > 0$

$$\frac{|H(\rho.u) - H(\rho.v)|}{\rho^{1/2}} = |u^{1/2} - v^{1/2}| < |u - v|^{1/2}.$$

The limit measure is the law of a centered Gaussian process Y such that

$$E\left[Y_u - Y_v\right]^2 = K_0 \left(u^{1/2} - v^{1/2}\right)^2,$$

i.e.,

$$E\left[Y_u Y_v\right] = K_0\, u^{1/2} v^{1/2}.$$

For a multifractional Brownian sheet, as the two conditions of Theorem 1 can be mixed up, the LASS property takes the following statement.

Theorem 2. *Let* $X = \left\{X_t;\ t \in \mathbf{R}_+^N\right\}$ *be a multifractional Brownian sheet.*

The law of the process

$$Y^{\alpha}(\rho) = \left\{Y_u^{\alpha}(\rho) = \frac{\Delta X_{t_0,t_0+\rho u}}{\rho^{\sum_i \alpha_i}};\ u \in \mathbf{R}_+^N\right\}$$

converges weakly if, for all $i \in \{1, \dots, N\}$*, one of the following two conditions holds*

1. $\alpha_i = H_i(t_0)$ *and* $H_i(t_0) < \inf_{u,v} \beta^i_{uv}(t_0)$ *where* $\beta^i_{uv}(t_0) = \sup\{\alpha;\ \lim_{\rho\to 0} |H_i(t_0+\rho u) - H_i(t_0+\rho v)|/\rho^\alpha = 0\}$.
2. $\alpha_i = \inf_{u,v} \beta^i_{uv}(t_0)$, $H_i(t_0) > \inf_{u,v} \beta^i_{uv}(t_0)$ *and*

$$\lim_{\rho\to 0} \frac{|H_i(t_0+\rho u) - H_i(t_0+\rho v)|}{\rho^{\inf_{u,v} \beta^i_{uv}(t_0)}} = \Gamma_i(u,v)$$

with $(u,v) \mapsto \Gamma_i(u,v)/\|u-v\|^{2\beta_i}$ *bounded on* $[a,b]^2$ *for some* $\beta_i > 0$.

As usual, the proof of weak convergence proceeds in two steps. First, we need to show finite dimensional convergence, and then, use a tightness argument. Lemma 14.2 and Theorem 14.3 in [**8**], for instance, allow then to conclude.

5.1 Finite dimensional convergence. As the considered processes are Gaussian, we only have to show the convergence of covariance functions.

The only case considered is the multifractional Brownian field's one. For the multifractional Brownian sheet, we proceed in the same way.

By (11), we compute

$$\begin{aligned}(42)\quad & \rho^{2\alpha} E\left[Y^\alpha_u(\rho) - Y^\alpha_v(\rho)\right]^2\\ &= E\left[X_{t_0+\rho u} - X_{t_0+\rho v}\right]^2\\ &= D\left[H(t_0+\rho u) + H(t_0+\rho v)\right] \times \|\rho.(u-v)\|^{H(t_0+\rho u)+H(t_0+\rho v)}\\ &\quad + \frac{\partial^2\varphi}{\partial x^2}\left(2H(t_0+\rho u); \|t_0+\rho u\|\right) \times \left(H(t_0+\rho u) - H(t_0+\rho v)\right)^2\\ &\quad + o\left(\|\rho.(u-v)\|^2\right) + o\left(H(t_0+\rho u) - H(t_0+\rho v)\right)^2.\end{aligned}$$

We can show that $\rho^{H(t_0+\rho u)+H(t_0+\rho v)} \sim \rho^{2H(t_0)}$ in the neighborhood of $\rho = 0$. For this, we study for $\alpha < \beta(t_0)$

$$\begin{aligned}&[H(t_0+\rho u) + H(t_0+\rho v) - 2H(t_0)] \ln\rho\\ &= \frac{H(t_0+\rho u) - H(t_0)}{\|\rho.u\|^\alpha} \times \|\rho.u\|^\alpha \ln\rho + \frac{H(t_0+\rho v) - H(t_0)}{\|\rho.v\|^\alpha} \times \|\rho.v\|^\alpha \ln\rho.\end{aligned}$$

As $(u;\rho) \mapsto \|\rho.u\|^\alpha \ln\rho$ is bounded on $[a,b]\times[0,1]$ and

$$\lim_{\rho\to 0} \frac{H(t_0+\rho u)-H(t_0)}{\|\rho.u\|^\alpha} = 0$$

for all $u \in [a,b]$, we have

$$[H(t_0+\rho u)+H(t_0+\rho v)-2H(t_0)]\,\ln\rho \stackrel{\rho\to 0}{\longrightarrow} 0.$$

Therefore, in the neighborhood of $\rho = 0$, the first term of (42) is equivalent to

$$D\,[2H(t_0)]\,\|u-v\|^{2H(t_0)} \times \rho^{2H(t_0)}$$

and the second to

$$\frac{\partial^2\varphi}{\partial x^2}\,(2H(t_0);\|t_0\|) \times (H(t_0+\rho u)-H(t_0+\rho v))^2\,.$$

Let $\beta_{uv}(t_0) = \sup\{\alpha; \lim_{\rho\to 0}|H(t_0+\rho u)-H(t_0+\rho v)|/\rho^\alpha = 0\}$. We have to distinguish the following two cases

- if $H(t_0) < \inf_{u,v}\beta_{uv}(t_0)$, by definition of $\beta_{uv}(t_0)$,

$$\forall\, u,v \in \mathbf{R}_+^N; \quad \lim_{\rho\to 0}\frac{|H(t_0+\rho u)-H(t_0+\rho v)|}{\rho^{H(t_0)}} = 0.$$

Therefore

$$\forall\, u,v \in \mathbf{R}_+^N;\quad E\left[Y_u^{H(t_0)}(\rho)-Y_v^{H(t_0)}(\rho)\right]^2 \stackrel{\rho\to 0}{\longrightarrow} \underbrace{D\,[2H(t_0)]\,\|u-v\|^{2H(t_0)}}_{E\left[B_u^{H(t_0)}-B_v^{H(t_0)}\right]^2}$$

where $B^{H(t_0)}$ denotes fractional Brownian field of parameter $H(t_0)$.

- if $H(t_0) > \inf_{u,v}\beta_{uv}(t_0)$, for all $\alpha < \inf_{u,v}\beta_{uv}(t_0)$, as

$$\forall\, u,v \in \mathbf{R}_+^N; \quad \lim_{\rho\to 0}\frac{|H(t_0+\rho u)-H(t_0+\rho v)|}{\rho^{\alpha}} = 0$$

we have

$$\forall\, u,v \in \mathbf{R}_+^N; \quad \frac{1}{\rho^{2\alpha}}\,E\,[X_{t_0+\rho u}-X_{t_0+\rho v}]^2 \stackrel{\rho\to 0}{\longrightarrow} 0.$$

Moreover, since there exists $u, v \in \mathbf{R}_+^N$ such that $H(t_0) > \beta_{uv}(t_0)$, we can consider $\alpha \in (\beta_{uv}(t_0); H(t_0))$. The limit

$$\limsup_{\rho \to 0} \frac{|H(t_0 + \rho u) - H(t_0 + \rho v)|}{\rho^\alpha} = +\infty$$

implies

$$\exists u, v \in \mathbf{R}_+^N; \quad \limsup_{\rho \to 0} \frac{1}{\rho^{2\alpha}} E\left[X_{t_0+\rho u} - X_{t_0+\rho v}\right]^2 = +\infty.$$

Therefore $E\left[Y_u^\alpha(\rho) - Y_v^\alpha(\rho)\right]^2$ admits a limit for all $u, v \in \mathbf{R}_+^N$ when $\rho \to 0$ *if and only if*

$$\begin{cases} \alpha = \inf_{u,v} \beta_{uv}(t_0) \quad \text{and} \\ \lim_{\rho \to 0} |H(t_0 + \rho u) - H(t_0 + \rho v)| / \rho^{\inf_{u,v} \beta_{uv}(t_0)} = \Gamma(u, v) \in \mathbf{R}_+^*. \end{cases}$$

Remark 6. We can see easily that

$$\beta_{u/\|u\|}(t_0) \wedge \beta_{v/\|v\|}(t_0) \le \beta_{uv}(t_0) \tag{43}$$

hence

$$\inf_{u \in \mathcal{U}} \beta_u(t_0) \le \inf_{u,v} \beta_{uv}(t_0). \tag{44}$$

Conversely, assume there exist $u, v \in \mathcal{U}$ such that $\beta_u(t_0) < \beta_v(t_0)$, and let $\alpha \in (\beta_u(t_0);\ \beta_v(t_0))$. By the triangular inequality, we get

$$\limsup_{\rho \to 0} \frac{|H(t_0 + \rho u) - H(t_0 + \rho v)|}{\rho^\alpha} = +\infty,$$

and therefore $\alpha > \beta_{uv}(t_0)$. Then $\inf_{u,v} \beta_{uv}(t_0) \le \inf_{u \in \mathcal{U}} \beta_u(t_0)$, which gives

$$\inf_{u,v} \beta_{uv}(t_0) = \inf_{u \in \mathcal{U}} \beta_u(t_0). \tag{45}$$

5.2 Tightness of laws. The study of weak convergence is well known for stochastic processes indexed by $\mathbf{R}_+$. A comprehensive review was made by Billingsley, cf. [**5**], for a compact set of index ($[0,1]$). In [**11**], Karatzas and Shreve stated the same kind of results for the whole $\mathbf{R}_+$. The case of $\mathbf{R}_+^N$ can be found in [**8**] whose Corollary 14.9 provides

Proposition 14. *Consider a sequence of continuous processes* $\left(X^{(n)}\right)_{n\in\mathbf{N}}$ *with* $X^{(n)} = \left\{X_t^{(n)}; t \in \mathbf{R}_+^N\right\}$ *on* $(\Omega, \mathcal{F}, P)$ *such that*

1. *there exists a positive constant* ν *such that*

$$\sup_{n\geq 1} E\left|X_0^{(n)}\right|^{\nu} < \infty$$

2. *for all* $T > 0$, *there exist positive constants* α, β *and* C_T *such that*

$$\forall\, s,t \in [0,T]^N; \quad \sup_{n\geq 1} E\left|X_t^{(n)} - X_s^{(n)}\right|^{\alpha} \leq C_T \left\|t-s\right\|^{N+\beta}.$$

Then the probability measures $P_n \triangleq P.\left(X^{(n)}\right)^{-1}$ *on*

$$\left(C\left(\mathbf{R}_+^N\right), \mathcal{B}\left(C\left(\mathbf{R}_+^N\right)\right)\right)$$

form a tight sequence.

We verify the conditions of Proposition 14, in the case of mBm. As for finite dimensional convergence, we only consider the multifractional Brownian field's case.

By (17), there exist positive constants K_T and L_T such that, for all u, v in $[0,T]^N$,

$$\begin{aligned}\rho^{2\alpha} E\left[Y_u^{\alpha}(\rho) - Y_v^{\alpha}(\rho)\right]^2 &= E\left[X_{t_0+\rho u} - X_{t_0+\rho v}\right]^2 \\ &\leq K_T \left\|\rho.(u-v)\right\|^{2H(t_0+\rho u)} \\ &\quad + L_T \left|H(t_0+\rho u) - H(t_0+\rho v)\right|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}E\left[Y_u^{\alpha}(\rho) - Y_v^{\alpha}(\rho)\right]^2 \leq\ & K_T'\, \rho^{2(H(t_0)-\alpha)}. \left\|(u-v)\right\|^{2H(t_0)} \\ &+ L_T\, \frac{\left|H(t_0+\rho u) - H(t_0+\rho v)\right|^2}{\rho^{2\alpha}}\end{aligned}$$

• In the case $H(t_0) < \inf_{u,v} \beta_{uv}(t_0)$, there exists $M_T > 0$ such that

$$E\left[Y_u^{H(t_0)}(\rho) - Y_v^{H(t_0)}(\rho)\right]^2 \leq M_T \, \|u-v\|^{2H(t_0)}.$$

• In the case $H(t_0) > \inf_{u,v} \beta_{uv}(t_0)$, under the assumption

$$\lim_{\rho \to 0} \frac{|H(t_0+\rho u) - H(t_0+\rho v)|}{\rho^{\inf_{u,v} \beta_{uv}(t_0)}} = \Gamma(u,v)$$

with $(u,v) \mapsto \Gamma(u,v)/\|u-v\|^{2\beta}$ bounded on $[a,b]^2$, there exists $M_T > 0$ such that

$$E\left[Y_u^{\inf_{u,v} \beta_{uv}(t_0)}(\rho) - Y_v^{\inf_{u,v} \beta_{uv}(t_0)}(\rho)\right]^2 \leq M_T \, \|u-v\|^{2(\beta \wedge H(t_0))}.$$

Since the process Y^α is Gaussian, we get an exponent greater than N in the usual way. Then we can conclude by Proposition 14 that the laws of Y^α are tight.

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