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ON SOME MEASURES ASSOCIATED TO THE GEODESIC FLOW

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ABSTRACT. We generalize a previous result in [**4**] concerning some measures associated to the geodesic flows on compact negatively curved Riemannian manifolds and give also an application of the result in [**5**] to Anosov flows.

1. Introduction. Let (M,g) be a compact Riemannian manifold with strictly negative curvature. Its geodesic flow is of Anosov type and the corresponding decomposition creates four foliations: stable, strongly stable, unstable and strongly unstable.

According to [**7**], one can consider harmonic measures on SM, the unit tangent bundle of (M,g), associated to these foliations. One also has some natural measures on SM which are invariant under the action of the geodesic flow, namely the Liouville measure m or the Bowen-Margulis measure μ. The various relationships between these measures have been extensively studied, see for example [**4, 10**–**14**].

In [**4**], we considered the special case when $\omega^{su} = \omega^{ss}$, where ω^{su}, respectively ω^{ss}, denotes harmonic measure on SM associated to the strongly unstable, respectively strongly stable, foliation. This equality can be written in terms of equalities between Patterson-Sullivan measures $\{\mu_x\}_{x\in\widetilde{M}}$ on $\partial\widetilde{M}$, where $\widetilde{M}$ denotes the universal covering space of M. More precisely, we obtain flip invariance of these measures, i.e., for all $x\in\widetilde{M}$ one has:

$$d\mu_x(\xi) = d\mu_x(-_x\,\xi), \quad \text{for all} \quad \xi\in\partial\widetilde{M}$$

where $-_x\,\xi$ denotes the symmetric image of ξ with respect to x. This condition of flip invariance of Patterson-Sullivan measures implies in particular that $m=\mu$. A famous conjecture of Katok, see [**2**], confirms that in this case when $m=\mu$, our manifold is locally symmetric. In

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[**4**], using the strong result of [**1**] when $\dim M \geq 3$, we showed that this is indeed the case.

In this note, by a modification of the proof presented in [**4**], we generalize the result in the following sense. If the Liouville measure m and the Bowen-Margulis measure μ on SM coincide, then it is easy to see, [**11**], that the measures $\mu_x(\xi)$ and $\mu_x(-_x\,\xi)$ are equivalent. This means that there is a function $U : SM \to \mathbf{R}$ such that, see [**11**]:

$$d\mu_x(-v) = \exp(U(v))\, d\mu_x(v)$$

where the sphere $S_x M$ is identified with $\partial\widetilde{M}$. Now Katok's conjecture confirms that (M, g) must be locally symmetric. The function U does not need in general to have high regularity. In this note we suppose that this function satisfies some extra regularity condition and with the same method used in [**4**] we show the following.

Theorem 1. *Let (M, g) be a compact Riemannian manifold with strictly negative curvature and $\dim M \geq 3$. Suppose that $m = \mu$, and let $U : SM \to \mathbf{R}$ be the function with the property*

$$d\mu_x(-v) = \exp(U(v))\, d\mu_x(v).$$

If the function U is in the class C_s^∞, i.e., smooth on the stable leaves, then (M, g) is locally symmetric.

We give the proof in the next section. In the last section, we give an application of our result in [**5**] to Anosov flows. We are interested in conjugacy of geodesic flows and consider the special case of conformal metrics. The result is the following.

Theorem 2. *Let (M, g) be a compact Riemannian manifold whose geodesic flow is of Anosov type. Then any Riemannian metric in the conformal class of g, whose geodesic flow is C^0-conjugate to that of g, is equal to g.*

2. Proof of Theorem 1. We use some well-known formulas concerning harmonic measures, see for example [**10, 12, 13**].

For any function $\varphi : SM \to \mathbf{R}$ integrable with respect to ω^{su} and differentiable in the direction of the geodesic flow X, one has:

$$\int_{SM} (X\varphi)(v)\, d\omega^{su}(v) = \int_{SM} (h - \operatorname{tr} U^+(v))\varphi(v)\, d\omega^{su}(v)$$

where $U^+(v)$ denotes the second fundamental form of the unstable horosphere associated to v, in the basepoint of v and h is the topological entropy of the geodesic flow. For the measure ω^{ss}, one has also:

$$\int_{SM} (X\phi)(v)\, d\omega^{ss}(v) = \int_{SM} (\operatorname{tr} U^+(-v) - h)\phi(v)\, d\omega^{ss}(v).$$

We use the description of the measures ω^{su} and ω^{ss} by the family $\{\mu_x\}_{x\in\widetilde{M}}$ and we obtain $\omega^{su}(v) = \exp(U(v))\ \omega^{ss}(v)$. Now we replace $\phi(v)$ in the second formula by $\phi(v)\exp(U(v))$ and we obtain:

$$\begin{aligned}
&\int_{SM} (\operatorname{tr} U^+(-v) - h)\phi(v)\, d\omega^{su}(v) \\
&\qquad = \int_{SM} (\operatorname{tr} U^+(-v) - h)\phi(v)\exp(U(v))\, d\omega^{ss}(v) \\
&\qquad = \int_{SM} (X(\phi\exp(U)))(v)\, d\omega^{ss}(v) \\
&\qquad = \int_{SM} (X\phi)(v)\, d\omega^{su}(v) + \int_{SM} \phi(v)(XU)(v)\, d\omega^{su} \\
&\qquad = \int_{SM} (h - \operatorname{tr} U^+(v) + (XU)(v))\phi(v)\, d\omega^{su}(v).
\end{aligned}$$

Now the support of ω^{su} is the whole of SM, see [**7**]. Hence, for all $v \in SM$,

$$2h = \operatorname{tr} U^+(v) + \operatorname{tr} U^+(-v) - (XU)(v).$$

The function $\operatorname{tr} U^+$ on the unstable leaves is C^∞. The above equality implies that this function is C^∞ on the stable leaves too. So we can use the result of [**8**] to obtain that the function $\operatorname{tr} U^+$ is C^∞ on SM.

Now a well-known improvement of [**6**] can be applied. This generalization, originally observed by Foulon and Labourie and communicated to Katok and finally appeared in [**15**], implies that the geodesic flow of (M, g) is C^∞-conjugate to the geodesic flow of a rank 1 symmetric space. Hence the result of [**1**] ensures that the Riemannian manifold (M, g) itself is locally symmetric.

3. Application to Anosov flows. In this section, we present an application of the result in [**5**] to Anosov flows. Let us first recall the result of [**5**]. Let M be a compact manifold and g, g' two Riemannian metrics without conjugate points on M which are conformally equivalent. Suppose that the convex closure of Dirac measures in the unit tangent bundle is dense in the set of probabilities which are invariant under the geodesic flow for two metrics. We showed that, if the marked length spectrum of g is equal to that of g', then $g = g'$. We present now an application of this result.

Let (M, g) be a compact Riemannian manifold whose geodesic flow is of Anosov type. By a result of [**9**], we know that in this case the metric g has no conjugate points. The density condition of Dirac measures is also satisfied for g because we have a hyperbolic flow. Now let g' be any conformally equivalent metric to g whose geodesic flow is C^0-conjugate to that of g. It is clear that the density condition of Dirac measures holds for g' as well. To apply our result, we just need to prove that g' has no conjugate points. But this is actually proved in [**3**]. Hence we obtain Theorem 2.

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