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NEW CONVERGENCE THEOREMS FOR CERTAIN ITERATIVE SCHEMES IN BANACH SPACES

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ABSTRACT. In this paper, we first review some convergence results for certain iterative schemes for a certain class of operators and discuss essential relations between the old results and new results. Then we establish several general convergence principles for certain iterative schemes for accretive operators, and show how our convergence principles can be applied to Mann's and Ishikawa's methods. Finally, four open problems are also given.

1. Introduction. The main purpose of this paper is to present a version for the convergence theorems due to Nevanlinna and Reich [**35**], Bruck and Reich [**6**]. As consequences of the revision, most of the recent results can be deduced from our convergence principle.

Early in 1979, Nevanlinna and Reich [**35**] studied discrete implicit and explicit schemes for finding the zeros of accretive operators that satisfy the convergence condition ([**35**]). Soon afterwards, Bruck and Reich [**6**] generalized the results of Nevanlinna and Reich [**35**] by establishing the general convergence principle for the implicit and explicit schemes for a certain class of operators that satisfy the so called condition 2.1 in [**6**].

Let $A : D(A) \subset E \to 2^E$ be an m-accretive operator in a Banach space E. The following implicit scheme with errors and the explicit scheme were considered by Nevanlinna and Reich [**35**]:

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$$x_{n+1} + \lambda_{n+1} A x_{n+1} \ni x_n + e_{n+1}, \quad n \geq 0, \tag{1.1}$$

$$x_{n+1} \in x_n - \lambda_n A x_n, \quad n \geq 0, \tag{1.2}$$

respectively, where $x_0 \in E$ and $\{\lambda_n\}$ is a positive real sequence.

Recently, concerning the convergence problems of the Ishikawa and Mann iterative processes have been extensively studied by various authors, see Chidume [**11, 12**], Osilike [**36**], Chang [**7, 8**], Chang et al. [**9**], Liu [**31**], Ding [**26**] and Zhou [**50**].

Let E be a real Banach space, K a nonempty convex subset of E and $T : K \to K$ a self-mapping from K into itself. For arbitrary initial value $x_0 \in K$, the sequence $\{x_n\}$ iteratively defined by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, & n \geq 0, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, & n \geq 0, \end{cases} \tag{1.3}$$

is called the Ishikawa iterative process, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ satisfying certain conditions. In particular, if $\beta_n = 0$ for all $n \geq 0$ in (1.3), then the sequence $\{x_n\}$ defined by (1.3) is called the Mann iterative process.

Observe now that the implicit scheme (1.1) is equivalent to the so called resolvent iterative scheme $\{x_n\}$ defined by

$$x_{n+1} = J_{n+1}(x_n + e_{n+1}), \quad n \geq 0, \tag{1.4}$$

and, setting $A = I - T$ in (1.2), the scheme (1.2) is equivalent to the following:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0, \tag{1.5}$$

respectively, where I denotes the identity mapping and $J_n = (I + \lambda_n A)^{-1}$ for $n \geq 0$ are resolvents of A.

Observe also that the Mann and Ishikawa iteration processes can be viewed as approximations of the corresponding resolvent iterative schemes for a certain class of nonlinear operators, respectively.

Evidently, it is of interest and importance to study the following iterative schemes $\{x_n\}$ with perturbations:

$$x_{n+1} + \lambda_n A x_{n+1} \ni x_n + e_n + o(\lambda_n), \quad n \geq 0, \tag{1.6}$$

$$x_{n+1} \in x_n - \lambda_n A x_n + e_n + o(\lambda_n), \quad n \geq 0, \tag{1.7}$$

respectively, where $x_0 \in E$, $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and $\{\lambda_n\}$ is a positive real sequence.

In this paper we will establish a general convergence principle for the iterative schemes defined by (1.6) and (1.7) and provide a unified treatment for the previously established results by several authors.

2. A review of the recent convergence results. Let E be a real Banach space with norm $\|\cdot\|$ and E^* the dual space of E. The normalized duality mapping from E to the family of subset of E^* is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. A multivalued operator T with domain $D(T)$ and range $R(T)$ in E is said to be accretive if, for each x, $y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that, for all $u \in Tx$ and $v \in Ty$,

$$\langle u - v, j(x-y) \rangle \geq 0. \tag{2.1}$$

Furthermore, the operator T is called strongly accretive if, for every x, $y \in D(T)$, there exists a constant $k > 0$ and there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y) \rangle \geq k||x-y||^2 \tag{2.2}$$

for all $u \in Tx$ and $v \in Ty$. The operator T is said to be ϕ-strongly accretive if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and, for every x, $y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y) \rangle \geq \phi(||x-y||)||x-y|| \tag{2.3}$$

for all $u \in Tx$ and $v \in Ty$. Let $N(T) = \{x \in D(T) : 0 \in Tx\}$. If $N(T) \neq \varnothing$ and the inequalities (2.1), (2.2) and (2.3) hold for any $x \in D(T)$ and $y \in N(T)$, then the corresponding operator T is called quasi-accretive, strongly quasi-accretive and ϕ-strongly quasi-accretive, respectively. It was shown in [**16**] that the class of strongly accretive operators is a proper subclass of ϕ-strongly accretive operators.

A class of mappings closely related to accretive operators is the class of pseudo-contractive operators. A mapping $T : D(T) \subset E \to E$ is called pseudo-contractive (respectively strongly pseudo-contractive, ϕ-strongly pseudo-contractive, ϕ-hemicontractive) if and only if $(I - T)$ is accretive (respectively, strongly accretive, ϕ-strongly accretive, ϕ-strongly quasi-accretive), where I denotes the identity operator on E. An accretive operator T is said to be m-accretive if $R(I + rT) = E$ for all $r > 0$.

Such operators have been extensively studied and applied by various researchers, see [**5**–**56**]. The interest and importance of such operators stem mainly from the fact that many physically significant problems can be modeled in terms of an initial value problem of the form

$$\begin{cases} u'(t) + Tu(t) = 0, \\ u(0) = u_0, \end{cases} \tag{2.4}$$

where T is an accretive operator in an appropriate Banach space. In this case, a zero of T corresponds to an equilibrium of the system (2.4).

Recall that a quasi-accretive operator A is said to satisfy the condition (I) if, for any $x \in D(A)$, $p \in N(A)$ and $j(x-p) \in J(x-p)$, the equality $< Ax, j(x-p) >= 0$ holds if and only if $Ax = Ap = 0$.

Lemma 2.1 (Reich [**37**]). *Let E be a real uniformly smooth Banach space. Then there exists a nondecreasing continuous function*

$$b : [0, \infty) \longrightarrow [0, \infty)$$

satisfying the following conditions:

(i) $b(ct) \le cb(t)$ *for all* $c \ge 1$,

(ii) $\lim_{t \to 0^+} b(t) = 0$,

(iii) $||x + y||^2 \le ||x||^2 + 2\langle y, j(x)\rangle + \max\{||x||, 1\}||y||b(||y||)$ *for all* $x, y \in E$.

The inequality (iii) is called Reich's inequality.

We denote the distance between a point $x \in E$ and a set $V \subset E$ by $d(x, V)$. Recall that a point $z \in V$ is said to be a best approximation to $x \in E$ if

$$\|x - z\| = d(x, V).$$

A set $V \subset X$ is said to be a sun if, whenever $z \in V$ is a best approximation to $x \in E$, z is also a best approximation to $z + t(x - z)$ for all $t \geq 0$. It is well known that every convex set is a sun. If V is a sun and $z \in V$ is a best approximation to $x \in E$, then there exists $j(x - z) \in J(x - z)$ such that

$$\langle y - z, j(x - z)\rangle \leq 0$$

for all $y \in V$. The set V is said to be proximinal if every $x \in X$ has at least one best approximation in V. It is also well known that, if E is a reflexive Banach space and V is a nonempty closed and convex subset of E, then V is proximinal. Thus, for all $x \in E$, there exists a nearest point mapping $P : E \to 2^V$ and $j(x - z) \in J(x - z)$ such that

$$\langle y - z, j(x - z)\rangle \leq 0$$

for all $y \in V$ and $z \in Px$.

In 1979, Nevanlinna and Reich [**35**] established their convergence results for the implicit and explicit schemes for m-accretive operators that satisfy the convergence condition in the setting that both E and E^* are uniformly convex Banach spaces.

Let E and E^* be both uniformly convex Banach spaces and $A : D(A) \subset E \to 2^E$ be an $m-$accretive operator. Then $A^{-1}0$ is closed and convex. Furthermore, the normalized duality mapping $J : E \to E^*$ is single-valued and continuous. If C is a nonempty closed convex subset of E, then also the nearest point mapping $P : E \to C$ defined by

$$\|x - Px\| = \inf\{\|x - y\| : y \in C\}$$

is single-valued and continuous.

It is well known that every nonempty closed convex subset of E is proximinal. Thus, if $0 \in R(A)$, then $A^{-1}0$ is nonempty closed and convex. Hence, $A^{-1}0$ is proximinal and convex. Consequently, for each $x \in E$, there is a point $J(x - Px)$ such that

$$\langle y - Px, J(x - Px)\rangle \leq 0$$

for all $y \in A^{-1}0$. Let $P : E \to A^{-1}0$ be the nearest point mapping. The operator A is said to satisfy the convergence condition if $[x_n, y_n] \in A$, $\|x_n\| \leq M$, $\|y_n\| \leq M$ and

$$\lim_{n\to\infty} \langle y_n, J(x_n - Px_n)\rangle = 0$$

imply that

$$\liminf_{n\to\infty} \|x_n - Px_n\| = 0.$$

It is obvious that every strongly accretive operator A satisfies the convergence condition.

Nevanlinna and Reich [**35**] proved the following results:

Theorem NR1. *Let A be an m-accretive operator in a Banach space E with a zero and $\{x_n\}$ a sequence defined by* (1.1)*. If the operator A satisfies the convergence condition, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sum_{n=1}^{\infty} \|e_n\| < \infty$, then the sequence $\{x_n\}$ converges strongly to a zero of A.*

Theorem NR2. *Let A be an m-accretive operator with a zero and $\{\lambda_n\}$ a positive real sequence such that $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\sum_{n=0}^{\infty} \lambda_n b(\lambda_n) < \infty$. Assume that a sequence $\{x_n\}$ in a Banach space E satisfies* (1.2) *and that $\{(x_n - x_{n+1})/\lambda_n\}$ is bounded. If the operator A satisfies the convergence condition, then the sequence $\{x_n\}$ converges strongly to a zero of A.*

In 1980, Bruck and Reich [**6**] generalized the results of [**35**] by establishing a general convergence principle for the explicit and implicit schemes for a certain class of operators that satisfy the Bruck-Reich condition in Banach spaces.

Consider two mappings f and F such that $D(f) \subset D(F) \subset E$, $f : D(f) \to [0, \infty)$ and $F : D(F) \to E^*$. The pair (f, F) is said to be admissible if these mappings satisfy either

$$f(x+y) \geq f(x) + \langle y, F(x)\rangle - b(x,y)\|y\| \tag{2.5}$$

or

$$f(x+y) \leq f(x) + \langle y, F(x)\rangle + b(x,y)\|y\| \tag{2.6}$$

for all appropriate x and y, where $0 \leq b(x,y) \to 0$ as $y \to 0$ uniformly for x.

A multivalued mapping $A : D(A) \subset E \to 2^E$ is said to satisfy the Bruck-Reich condition if, for each $K > 0$, there is an increasing function $g : R^+ \to R^+$ such that $g(r) > 0$ for $r > 0$ and

$$\langle y, F(x)\rangle \geq g(f(x))\|y\| \tag{2.7}$$

for all $x \in D(A)$, $y \in Ax$ and $\|x\| \leq K$, $\|y\| \leq K$.

Recall that a set $A \subset E \times E$ is said to be accretive in the sense of Browder if, for all $[x_i, y_i] \in A,\ i = 1, 2$,

$$\langle y_1 - y_2, j(x_1 - x_2)\rangle \geq 0$$

for all $j \in J(x_1 - x_2)$. Let $A \subset E \times E$ be an accretive operator in the sense of Browder with $0 \in R(A)$ and assume that $A^{-1}0$, the kernel, is proximinal and convex. If P is a selection of the nearest point mapping onto $A^{-1}0$, let $J_p(x-Px)$ denote an element in $J(x-Px)$ that satisfies

$$\langle y - Px, J_p(x - Px)\rangle \leq 0$$

for all $y \in A^{-1}0$. Recall that, in this setting, A is said to satisfy the convergence condition [**35**] if there is a selection P of the nearest point mapping onto $A^{-1}0$ such that, if $[x_n, y_n] \in A$, $\|x_n\| \leq K$, $\|y_n\| \leq K$ and

$$\lim_{n\to\infty} \langle y_n, J_p(x_n - Px_n)\rangle = 0,$$

then $\lim_{n\to\infty} \|x_n - Px_n\| = 0$.

Bruck and Reich [**6**] have shown that, if one takes

$$f(x) = (1/2)d(x, N(A))^2 \quad \text{and} \quad F(x) = J_p(x - Px),$$

then (f, F) satisfies (2.5) with $b = 0$ in any Banach spaces and that it satisfies (2.6) if E is a uniformly smooth Banach space.

We remark in passing that the definition of accretive operator in the sense of Browder is stronger than that appearing in Section 1. If E^* is strictly convex, then the normalized duality mapping J is single-valued and then the two definitions are uniform.

By virtue of above concepts, Bruck and Reich [**6**] established the following general convergence principle:

Theorem BR1. *Suppose that* (f, F) *satisfies* (2.6)*, the operator* A *satisfies the Bruck-Reich condition and the sequence* $\{x_n\}$ *in a Banach space* E *satisfies* (1.2)*. If the sequences* $\{x_n\}$ *and* $\{y_n\}$ *remain bounded and* $\lim_{n\to\infty} z_n = 0$*, then either*

$$\lim_{n\to\infty} f(x_n) = 0 \quad \text{or} \quad \sum_{n=0}^{\infty} \|z_n\| < \infty.$$

Theorem BR2. *Suppose that the pair* (f, F) *satisfies* (2.5)*, the operator* A *satisfies the Bruck-Reich condition and the sequence* $\{x_n\}$ *in a Banach space* E *satisfies* (1.1) *with* $e_n \equiv 0$*. If the sequences* $\{x_n\}$ *and* $\{y_n\}$ *remain bounded and* $\lim_{n\to\infty} z_n = 0$*, then either*

$$\lim_{n\to\infty} f(x_n) = 0 \quad \text{or} \quad \sum_{n=0}^{\infty} \|z_n\| < \infty.$$

Recently, several strong convergence results of the Mann and the Ishikawa iteration processes in general Banach spaces have been established for approximating either fixed points of strong pseudo-contractions acted from a nonempty convex subset K into itself or solutions of nonlinear equations with accretive operators acted from a Banach space X into itself, see [**7**–**9**, **18, 26, 31, 36, 46, 49**]. However, we find that some results mentioned above can be deduced directly from BR1 while almost all other results can be deduced from a version of a convergence theorem due to Bruck and Reich [**6**].

In 1987, Chidume [**10**] proved the following result:

Theorem C1. *Suppose that* K *is a nonempty closed convex and bounded subset of* L_p*,* $p \geq 2$*, and* $T : K \to K$ *is a Lipschitzian strongly pseudo-contractive mapping. Suppose that* $\{c_n\}$ *is a real sequence satisfying the following conditions*:

(i) $0 < c_n < 1$, $n \geq 0$,

(ii) $\sum_{n=0}^{\infty} c_n = \infty$,

(iii) $\sum_{n=0}^{\infty} {c_n}^2 < \infty$.

Then the sequence $\{x_n\}$ *of the Mann iterates defined iteratively by*

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-c_n)x_n + c_nTx_n \quad n \geq 0, \end{cases} \tag{2.8}$$

converges strongly to the unique fixed point of T.

In 1994, Chidume [**11**] extended Theorem C1 from L_p, $p \geq 2$, to uniformly smooth Banach spaces.

Theorem C2. *Let* E *be a real uniformly smooth Banach space,* K *a nonempty closed convex and bounded subset of* E *and* $T : K \to K$ *a continuous strongly pseudo-contractive mapping. Suppose that* $\{c_n\}$ *is a real sequence satisfying the following conditions*:

(i) $0 < c_n < 1$, $n \geq 0$,

(ii) $\sum_{n=0}^{\infty} c_n = \infty$,

(iii) $\sum_{n=0}^{\infty} c_n b(c_n) < \infty$,

where $b : R^+ \to R^+$ *is the function appearing in Reich* [**37**]. *Then the sequence* $\{x_n\}$ *generated by*

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1-c_n)x_n + c_nTx_n \quad n \geq 0, \end{cases} \tag{2.9}$$

converges strongly to the unique fixed point of T.

In 1995, Chidume [**12**] announced another result as follows:

Theorem C3. *Let* E *be a real Banach space with the uniformly convex dual space* E^*. *Suppose that* $T : E \to E$ *is a continuous strongly accretive mapping such that* $(I - T)$ *has bounded range. For a given* $f \in E$, *define* $S : E \to E$ *by* $Sx = f - Tx + x$ *for each* $x \in E$. *Consider the sequence* $\{x_n\}$ *defined iteratively by*

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1-\lambda_n)x_n + \lambda_nSx_n \quad n \geq 0, \end{cases} \tag{2.10}$$

where $\{\lambda_n\}$ *is a real sequence satisfying the following conditions*:

(i) $0 < \lambda_n \leq 1$, $n \geq 0$,

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$,

(iii) $\sum_{n=0}^{\infty} \lambda_n b(\lambda_n) < \infty$, *where* $b : R^+ \to R^+$ *is the function appearing in Reich* [**37**].

Then the sequence $\{x_n\}$ *generated by* (2.10) *converges strongly to the unique solution of the equation* $Tx = f$.

Remark 2.1. By a result of Bogin [**2**], we see that Theorem C3 can be deduced from Theorem C2. Apparently, Theorem C1 is a special case of Theorem C2 while Theorem C2 can be deduced directly from Theorem BR1. The proof of this claim is postponed to Section 3.

In his study of the Ishikawa and Mann iteration methods with errors for strongly accretive operators, Liu [**30**] proved the following theorems:

Theorem L1. *Let* E *be a uniformly smooth Banach space and* $T : E \to E$ *be a Lipschitzian strongly accretive operator with constant* $k \in (0, 1)$ *and Lipschitz constant* $L \geq 1$. *Define a mapping* $S : E \to E$ *by* $Sx = f + x - Tx$. *Let* $\{u_n\}$ *and* $\{v_n\}$ *be two summable sequences in* E *and* $\{\alpha_n\}$, $\{\beta_n\}$ *be two real sequences in* $[0, 1]$ *satisfying the following conditions*:

(i) $\alpha_n \to 0$ *as* $n \to \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iii) $\lim_{n\to\infty} \sup \beta_n < (k/(L^2 - k))$.

Define the sequence $\{x_n\}$ *by*

$$\begin{cases} x_0 \in E, \\ y_n = (1 - \beta_n)x_n + \beta_n S x_n + u_n & n \geq 0, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S y_n + v_n & n \geq 0. \end{cases} \tag{2.11}$$

If $\{Sy_n\}$ *is bounded, then the sequence* $\{x_n\}$ *converges strongly to the solution of the equation* $Tx = f$.

Theorem L2. *Let* E *be a uniformly smooth Banach space and* $T : E \to E$ *a semicontinuous strongly accretive operator with the constant* $k \in (0, 1)$ *and Lipschitz constant* $L \geq 1$. *Define a mapping*

$S : E \to E$ by $Sx = f + x - Tx$. Let $\{u_n\}$ be a summable sequence in E and $\{\alpha_n\}$ be a real sequence in $[0, 1]$ satisfying the following conditions:

(i) $\alpha_n \to 0$ *as* $n \to \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Define the sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S x_n + u_n \quad n \geq 0. \end{cases} \tag{2.12}$$

If $\{Sx_n\}$ is bounded, then the sequence $\{x_n\}$ converges strongly to the solution of the equation $Tx = f$.

In 1996, Osilike [**36**] proved that both the Mann iteration method and the Ishikawa iteration method are applied to approximate either the fixed points of ϕ-hemicontractive mappings or the solutions of ϕ-strongly accretive operators in a real q-uniformly smooth Banach space, where $q > 1$.

Theorem O1. *Let $q > 1$ and E be a real q-uniformly smooth Banach space. Let $T : E \to E$ be a Lipschitzian and ϕ-strongly accretive operator. Suppose that the equation $Tx = f$ has a solution for any given $f \in E$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$ satisfying the following conditions*:

(i) $0 \leq \beta_n \leq {\alpha_n}^{q-1}$, $n \geq 0$,

(ii) $\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n)^{q-1} = \infty$,

(iii) $\sum_{n=0}^{\infty} {\alpha_n}^q < \infty$.

Define a mapping $S : E \to E$ by $Sx = f + x - Tx$ for each $x \in E$. Then the sequence $\{x_n\}$ generated from any $x_0 \in E$ by

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n S x_n & n \geq 0, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S y_n & n \geq 0, \end{cases} \tag{2.13}$$

converges strongly to the unique solution of the equation $Tx = f$.

Theorem O2. *Let $q > 1$ and E be a real q-uniformly smooth Banach space. Let K be a nonempty closed convex subset of E and $T : E \to E$*

be a Lipschitzian ϕ-hemicontractive mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$ satisfying the following conditions:

(i) $0 \le \beta_n \le {\alpha_n}^{q-1}$, $n \ge 0$,

(ii) $\sum_{n=0}^{\infty} \alpha_n (1-\alpha_n)^{q-1} = \infty$,

(iii) $\sum_{n=0}^{\infty} {\alpha_n}^q < \infty$.

Then the sequence $\{x_n\}$ generated from any $x_0 \in E$ by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n & n \ge 0, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n & n \ge 0, \end{cases} \tag{2.14}$$

converges strongly to the unique fixed point of T.

In 1997, Zhou [**46**] extended Theorems C2 and C3 to the Ishikawa iteration process by proving the following results:

Theorem Z1. *Let E be a real uniformly smooth Banach space, K a nonempty closed convex and bounded subset of E and $T: K \to K$ a continuous strongly pseudo-contractive mapping. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$ satisfying the following conditions*:

(i) $\alpha_n,\ \beta_n \to 0$ *as* $n \to \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, for each $x_0 \in K$, the Ishikawa iterative sequence $\{x_n\}$ generated by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n & n \ge 0, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n & n \ge 0, \end{cases} \tag{2.15}$$

converges strongly to the unique fixed point of T.

Theorem Z2. *Let E be a real Banach space with the uniformly convex dual space E^*. Suppose that $T: E \to E$ is a continuous strongly accretive mapping such that $(I-T)$ has bounded range. For a given $f \in E$, define a mapping $S: E \to E$ by $Sx = f - Tx + x$ for each $x \in E$. Consider the sequence $\{x_n\}$ defined iteratively by*

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n & n \ge 0, \\ y_n = (1-\beta_n)x_n + \beta_n S x_n & n \ge 0, \end{cases} \tag{2.16}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences in* $[0, 1]$ *satisfying the following conditions*:

(i) $\alpha_n,\ \beta_n \to 0$ *as* $n \to \infty$,

(ii) $\sum_{n=0}^{\infty} \lambda_n = \infty$.

Then the Ishikawa iterative sequence $\{x_n\}$ *generated by* (2.16) *converges strongly to the unique solution of the equation* $Tx = f$.

In 1997, by using Kato's inequality, Liu [**31**] proved the following result:

Theorem Lu1. *Let* E *be a real Banach space,* K *a nonempty closed convex and bounded subset of* E *and* $T : K \to K$ *a Lipschitzian and strongly pseudo-contractive mapping. Let* $\{c_n\}$ *be a real sequence satisfying the following conditions*:

(i) $0 \le c_n < 1$, $n \ge 0$,

(ii) $c_n \to 0$ *as* $n \to \infty$,

(iii) $\sum_{n=0}^{\infty} c_n = \infty$.

Then the sequence $\{x_n\}$ *defined by*

$$\begin{cases} x_0 \in K, \\ x_{n+1} = (1 - c_n)x_n + c_n T x_n \quad n \ge 0, \end{cases} \tag{2.17}$$

converges strongly to the unique fixed point of T.

By virtue of Liu's idea and technique in [**31**], Chidume and Osilike [**18**] extended Theorem L1 to the Ishikawa iteration process as follows:

Theorem CO. *Let* E *be a real Banach space,* K *a nonempty closed convex subset of* E *and* $T : K \to K$ *a Lipschitzian and strongly pseudo-contractive mapping. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real sequences in* $[0, 1]$ *satisfying the following conditions*:

(i) $\alpha_n,\ \beta_n \to 0$ *as* $n \to \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then the Ishikawa iterative sequence $\{x_n\}$ *generated from an arbitrary* $x_0 \in K$ *by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n & n \geq 0, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n & n \geq 0, \end{cases} \tag{2.18}$$

converges strongly to the unique fixed point of T.

In 1997, Ding [**26**] also proved the following results, which generalized Theorems O1, O2 and CO:

Theorem D1. *Let* E *be an arbitrary Banach space and* $T : D(T) \subset E \to E$ *be a Lipschitzian and* $\phi-$*strongly accretive operator with domain* $D(T)$ *and range* $R(T)$. *Suppose that the equation* $Tx = f$ *has a solution for any* $f \in D(T)$. *Let* $\{u_n\}$ *and* $\{v_n\}$ *be two sequences in* E *and* $\{\alpha_n\}$, $\{\beta_n\}$ *two real sequences in* $[0,1]$ *satisfying the following conditions*:

(i) $\sum_{n=0}^{\infty} \|v_n\| < \infty$, $\sum_{n=0}^{\infty} \|u_n\| < \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iii) $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n)\beta_n < \infty$,

(iv) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$.

Suppose that, for some $x_0 \in D(T)$, *the Ishikawa type iterative sequences* $\{x_n\}$ *and* $\{y_n\}$ *with errors defined by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f + (I-T)y_n) + u_n & n \geq 0, \\ y_n = (1-\beta_n)x_n + \beta_n(f + (I-T)x_n) + v_n & n \geq 0, \end{cases} \tag{2.19}$$

are both contained in $D(T)$. *Then the sequence* $\{x_n\}$ *converges strongly to the unique solution of the equation* $Tx = f$.

Theorem D2. *Let* E *be an arbitrary Banach space and* $A : D(A) \subset E \to E$ *a Lipschitzian and* ϕ*-hemicontractive mapping with domain* $D(A)$ *and range* $R(A)$. *Let* $\{u_n\}$, $\{v_n\}$ *be two sequences in* E *and* $\{\alpha_n\}$, $\{\beta_n\}$ *two real sequences in* $[0,1]$ *such that conditions* (i)–(iv) *in Theorem D1 hold. Suppose that, for some* $x_0 \in D(A)$, *the Ishikawa type iterative sequences* $\{x_n\}$ *and* $\{y_n\}$ *with errors defined by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n A y_n + u_n & n \geq 0, \\ y_n = (1-\beta_n)x_n + \beta_n A x_n + v_n & n \geq 0, \end{cases} \tag{2.20}$$

are both contained in $D(A)$. *Then the sequence* $\{x_n\}$ *converges strongly to the unique solution of* A.

Unlike the approach of Chidume and Osilike [**18**], Chang [**7, 8**] and Chang et al. [**9**] used an inequality which holds in general Banach spaces and proved the following result:

Theorem CSS. *Let* E *be a real Banach space,* K *a nonempty closed convex and bounded subset of* E *and* $T : K \to K$ *a uniformly continuous and strongly pseudo-contractive mapping. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real sequences in* $[0, 1]$ *satisfying the following conditions*:

(i) $\alpha_n, \ \beta_n \to 0$ *as* $n \to \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then the Ishikawa iterative sequence $\{x_n\}$ *generated from an arbitrary* $x_0 \in K$ *by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n & n \geq 0, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n & n \geq 0, \end{cases} \tag{2.21}$$

converges strongly to the unique fixed point of T.

Zhou [**50**] also proved the following results:

Theorem Z3. *Let* E *be an arbitrary Banach space and* $T : E \to E$ *a Lipschitzian and* ϕ*-strongly quasi-accretive operator with the Lipschitz constant* $L \geq 1$. *Set* $L_1 = L + 1$. *Define a mapping* $S : E \to E$ *by* $Sx = x - Tx$ *for each* $x \in E$. *Let* $\{u_n\}$, $\{v_n\}$ *be two sequences in* E *and* $\{\alpha_n\}$, $\{\beta_n\}$ *two real sequences in* $[0, 1]$ *satisfying the following conditions*:

(i) $\|v_n\| \to 0$ *as* $n \to \infty$, $\|u_n\| = o(\alpha_n)$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iii) $\alpha_n, \ \beta_n \to 0$ *as* $n \to \infty$.

Define the Ishikawa iterative sequence $\{x_n\}$ *with errors by*

$$\begin{cases} x_0 \in E, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n + u_n & n \geq 0, \\ y_n = (1-\beta_n)x_n + \beta_n S x_n + v_n & n \geq 0. \end{cases} \tag{2.22}$$

Suppose, furthermore, that $\{Sy_n\}$ *is bounded. Then the sequence* $\{x_n\}$ *converges strongly to the unique solution of the equation* $Tx = 0$.

Theorem Z4. *Let* E *be an arbitrary Banach space,* K *a nonempty convex subset of* E *such that* $K + K \subset K$. *Let* $A : K \to K$ *be a Lipschitzian and* ϕ*-hemicontractive mapping with the Lipschitz constant* $L \geq 1$. *Let* $\{u_n\}$ *and* $\{v_n\}$ *be two sequences in* E *and* $\{\alpha_n\}$, $\{\beta_n\}$ *two real sequences in* $[0, 1]$ *such that conditions* (i)–(iii) *in Theorem* Z3 *hold. Define the Ishikawa iterative sequence* $\{x_n\}$ *iteratively by*

$$
\begin{cases} x_0 \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ay_n + u_n & n \geq 0, \\ y_n = (1 - \beta_n)x_n + \beta_n Ax_n + v_n & n \geq 0. \end{cases} \tag{2.23}
$$

If $\{Ay_n\}$ *is bounded, then the sequence* $\{x_n\}$ *converges strongly to the unique fixed point of* A.

3. A general convergence principle. In this section we establish a general convergence principle from which most of the recent convergence theorems can be deduced easily. We start with a simple theorem, from which Theorems C1, C2 and C3 can be deduced.

Theorem 3.1. *Let* E *be a real uniformly smooth Banach space and* $A : D(A) \subset E \to E$ *a strongly accretive operator with the nonempty kernel* $N(A)$ *and strong accretiveness constant* $k \in (0, 1)$. *Suppose that a sequence* $\{x_n\}$ *can be defined by the explicit scheme*

$$
x_{n+1} = x_n - \lambda_n Ax_n, \quad n \geq 0, \tag{3.1}
$$

where $x_0 \in D(A)$, *and* $\{\lambda_n\}$ *is a positive sequence in* $[0, 1]$ *satisfying* $\sum_{n=0}^{\infty} \lambda_n = \infty$ *and* $\lambda_n \to 0$ *as* $n \to \infty$. *If* $\{Ax_n\}$ *remains bounded, then the sequence* $\{x_n\}$ *converges strongly to the unique zero of* A.

Proof. We observe first that $N(A)$ is a singleton because of the strong accretiveness of A. Since A is strongly accretive, A satisfies the convergence condition and hence it satisfies Bruck-Reich condition.

Now we want to prove that $\{x_n\}$ is bounded and to complete this assertion by using Reich's inequality. Let $M = \sup\{\|Ax_n\| : n \geq 0\} + 1$.

Since $\lambda_n \to 0$ as $n \to \infty$, we see that $b(\lambda_n) \to 0$ as $n \to \infty$. Hence, for the fixed constant $(2k)/M$, there is some fixed positive integer n_0 such that $b(\lambda_n) \le (2k)/M$ for all $n \ge n_0$. Using Reich's inequality and (3.1), we have

$$(3.2)\quad \begin{cases} \|x_{n+1}-x^*\|^2 \le \|x_n-x^*\|^2 - 2\lambda_n\langle Ax_n, j(x_n-x^*)\rangle \\ \qquad\qquad\qquad +M^2\max\{\|x_n-x^*\|,1\}\lambda_n b(\lambda_n) \\ \le (1-2k\lambda_n)\|x_n-x^*\|^2 + M^2\max\{\|x_n-x^*\|,1\}\lambda_n b(\lambda_n). \end{cases}$$

Now we consider two possible cases:

Case 1. $\|x_{n_0}-x^*\| \le 1$. In this case, by (3.2), we have $\|x_{n_0+m}-x^*\| \le 1$ for all $m \ge 0$, which gives $\{x_n\}$ is bounded.

Case 2. $\|x_{n_0}-x^*\| > 1$. In this case, by using (3.2), we have $\|x_{n_0+m}-x^*\| \le \|x_{n_0}-x^*\|$ for all $m \ge 0$. By Theorem BR1, we see that $f(x_n) \to 0$ as $n \to \infty$ or $\sum_{n=0}^{\infty}\|z_n\| < \infty$. In the first case, we know that $x_n \to x^*$ as $n \to \infty$. In the second case, we have $x_n \to y$ as $n \to \infty$. Now it follows from $\liminf_{n\to\infty}\|Ax_n\| = 0$ and the strong accretiveness of A that $y = x^* \in N(A)$. This completes the proof. □

Remark 3.1. Theorem C3 can be deduced directly from our Theorem 3.1 and Theorem C2 can be deduced easily by setting $A = I - T$, where T is as in Theorem C2, while Theorem C1 can be deduced from Theorem C2.

We now turn our attention to the study of convergence for the explicit scheme (1.7). For the sake of simplicity, we only consider the case when A is single-valued.

Theorem 3.2. *Let E be a real uniformly smooth Banach space and $A : D(A) \subset E \to E$ a quasi-accretive operator. Suppose that a sequence $\{x_n\}$ can be defined by the explicit scheme*

$$(3.3)\qquad x_{n+1} = x_n - \lambda_n Ax_n + o(\lambda_n) + e_n, \quad n \ge 0,$$

where $x_0 \in D(A)$, $\{e_n\}$ is a sequence in E such that $\sum_{n=0}^{\infty}\|e_n\| < \infty$ and $\{\lambda_n\}$ is a positive sequence in $[0,1]$ satisfying $\sum_{n=0}^{\infty}\lambda_n = \infty$ and

$\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ and that A satisfies the condition (3.4) *such that*

$$\langle Ax_n, j(x_n - v)\rangle \geq \phi(\|x_n - v\|)\|x_n - v\| \tag{3.4}$$

for any $v \in N(A)$. If $\{Ax_n\}$ remains bounded, then the sequence $\{x_n\}$ converges strongly to v.

Proof. Let $v \in N(A)$ be given and $\|o(\lambda_n)\| = \lambda_n\varepsilon_n$ with $\varepsilon_n \to 0$ as $n \to \infty$. Set

$$M = \sup\{\|Ax_n\| : n \geq 0\} + 1, \quad A_n = x_n - v - \lambda_n Ax_n.$$

By using Reich's inequality and (3.3), we have

(3.5)

$$\begin{aligned}\|x_{n+1} - v\|^2 &\leq \|A_n\|^2 + 2\langle e_n, j(A_n)\rangle + \max\{\|A_n\|, 1\}b(\|e_n\|)\|e_n\| \\ &\leq \|x_n - v\|^2 - 2\lambda_n\langle Ax_n, j(x_n - v)\rangle \\ &\quad + M^2 \max\{\|x_n - v\|, 1\}\lambda_n b(\lambda_n) \\ &\quad + 2\|e_n\|\|A_n\| + \max\{\|A_n\|, 1\}b(\|e_n\|)\|e_n\| \\ &\leq \|x_n - v\|^2 - 2\lambda_n\phi(\|x_n - v\|)\|x_n - v\| \\ &\quad + M^2 \max\{\|x_n - v\|, 1\}\lambda_n b(\lambda_n) \\ &\quad + 2\|e_n\|(\|x_n - v\| + M\lambda_n) \\ &\quad + \max\{\|x_n - v\| + M\lambda_n, 1\}b(\|e_n\|)\|e_n\|.\end{aligned}$$

Now we consider the following possible cases:

Case (I). There is a fixed positive integer n_0 such that $\|x_n - v\| \geq 1$ for all $n \geq n_0$.

Since λ_n, $e_n \to 0$ as $n \to \infty$, we have $b(\lambda_n)$, $b(\|e_n\|) \to 0$ as $n \to \infty$. Hence, there is a fixed positive integer n_1 such that

$$M^2 b(\lambda_n) \leq \phi(1), \quad b(\|e_n\|) \leq 1, \quad 4M\|e_n\| \leq \frac{1}{2}\,\phi(1)$$

for all $n \geq n_1$. Take $n_2 = \max\{n_0, n_1\}$. Then, by using (3.5), we have (3.6)

$$\begin{aligned}\|x_{n+1} - v\|^2 &\leq \|x_n - v\|^2 - 2\lambda_n \phi(1)\|x_n - v\| + M^2\lambda_n b(\lambda_n)\|x_n - v\| \\ &\quad + 2(\|x_n - v\| + M\lambda_n)(1 + b(\|e_n\|)\|e_n\| \\ &\leq \|x_n - v\|^2 - \lambda_n\phi(1) + 4(\|x_n - v\| + M\lambda_n)\|e_n\| \\ &\leq \|x_n - v\|^2 - \frac{1}{2}\,\phi(1)\lambda_n + 4\|x_n - v\|\|e_n\|\end{aligned}$$

for all $n \geq n_2$, which implies that

$$\|x_{n+1} - v\| \leq \|x_n - v\| + 4\|e_n\| \tag{3.7}$$

for all $n \geq n_2$. It follows from (3.7) that $\{\|x_n - v\|\}$ is bounded. This in turn implies that

$$\frac{1}{2}\sum_{n\geq n_2} \lambda_n \leq \|x_{n_2} - v\|^2 + 4C \sum_{n \geq n_2} \|e_n\|$$

for some positive constant C, which is a contradiction. This shows the case (I) is impossible.

Case (II). There is an infinite subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\|x_{n_j} - v\| \leq 1$. In this case, by using induction, we can prove that

$$\|x_{n_j+m} - v\|^2 \leq 1 + 5\sum_{l=0}^{m-1} \|e_{n_j+l}\|,$$

which shows that $\{\|x_n - v\|\}$ is bounded. As in the proof of the corresponding part of Case (I), we can prove that $x_{n_j} \to v$ as $j \to \infty$. As in the proof of boundedness for the sequence $\{x_n\}$, we can prove that $x_n \to v$ as $n \to \infty$. This completes the proof. □

Remark 3.2. Apparently, if A is $\phi-$strongly quasi-accretive, then it satisfies the condition (3.4) for any sequence $\{x_n\}$ in $D(A)$, so that it satisfies the convergence condition [**6**] and hence satisfies the Bruck-Reich condition. Conversely, let A be a quasi-accretive operator which is expending. If A satisfies the Bruck-Reich condition, then it satisfies the convergence condition [**6**]. Furthermore, if both $\{x_n\}$ and $\{Ax_n\}$ are bounded, then A satisfies the condition (3.4). Thus, for any ϕ-strongly quasi-accretive operator A, it satisfies the convergence condition ([**6**]) if and only if it satisfies the Bruck-Reich condition.

Remark 3.3. Actually, Theorem 3.2 can be extended to the following more general setting:

Let E be a real uniformly smooth Banach space and $A : D(A) \subset E \to E$ be a quasi-accretive operator. Assume that $N(A)$ has a nonempty closed convex subset of $N_0(A)$. Let P_0 denote an arbitrary single-valued selection of the nearest point mapping from E onto $N_0(A)$ such that

$$\langle y - P_0x, j(x - P_0x)\rangle \le 0 \tag{3.8}$$

for all $y \in N_0(A)$. Suppose that a sequence $\{x_n\}$ can be defined by the explicit scheme

$$x_{n+1} = x_n - \lambda_n Ax_n + o(\lambda_n) + e_n, \quad n \ge 0, \tag{3.9}$$

where $x_0 \in D(A)$, $\{e_n\}_{n=0}^{\infty}$ is a sequence in E such that $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and $\{\lambda_n\}_{n=0}^{\infty}$ is a positive sequence in $[0, 1]$ satisfying $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\lambda_n \to 0$ as $n \to \infty$. Suppose that there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ and that A satisfies the condition (3.10) such that

$$\langle Ax_n, j(x_n - P_0x_n)\rangle \ge \phi(\|x_n - P_0x_n\|)\|x_n - P_0x_n\|. \tag{3.10}$$

If $\{Ax_n\}$ remains bounded, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a zero of A.

In fact, proceeding as in the proof of Theorem 3.2, we can prove that $\|x_n - P_0x_n\| \to 0$ as $n \to \infty$. By induction, one can prove that $\|x_{n+m} - P_0x_n\| \to 0$ as $n \to \infty$ uniformly for all $m \ge 0$. Consequently, $\|x_{n+m} - x_n\| \to 0$ as $n, m \to 0$. This shows that $\{x_n\}$ is a Cauchy sequence. Assume that $x_n \to z$ as $n \to \infty$. Then $P_0x_n \to z$ as $n \to \infty$. Since $N_0(A)$ is closed, we see that $z \in N_0(A)$.

Thus, taking $N_0(A) = \{v\}$, then we obtain Theorem 3.2.

By virtue of the techniques in the proof of Theorems 3.2 and Remark 3.3, one can prove that the following more general result:

Theorem 3.3. *Let E be a real uniformly smooth Banach space and $A : D(A) \subset E \to E$ be a quasi-accretive operator. Suppose that a sequence $\{x_n\}$ can be defined by the explicit scheme*

$$\begin{cases} x_{n+1} = x_n - \lambda_n A y_n + o(\lambda_n) + e_n & n \geq 0, \\ y_n = x_n - \beta_n A x_n + o(\beta_n) & n \geq 0, \end{cases} \tag{3.11}$$

where $x_0 \in D(A)$, $\{e_n\}_{n=0}^{\infty}$ is a sequence in E such that $\sum_{n=0}^{\infty} \|e_n\| < \infty$, $\{\lambda_n\}_{n=0}^{\infty}$ and$\{\beta_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ satisfying $\sum_{n=0}^{\infty} \lambda_n = \infty$ and λ_n, $\beta_n \to 0$ as $n \to \infty$. Suppose that there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ and that A satisfies the condition (3.13) *such that*

$$\langle A y_n, j(y_n - v)\rangle \geq \phi(\|y_n - v\|)\|y_n - v\| \tag{3.12}$$

for any $v \in N(A)$. If $\{Ax_n\}$ and $\{Ay_n\}$ remain bounded, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to v.

Proof. The proof is similar to that of Theorem 3.2 and so it is omitted. □

Remark 3.4. If we follow the proofs of Theorem 3.2 and Remark 3.3, then Theorem 3.3 can be extended to the following more general setting:

Let E be a real uniformly smooth Banach space and $A : D(A) \subset E \to E$ a quasi-accretive operator. Assume that $N(A)$ has a nonempty closed subset $N_0(A)$. Let P_0 be an arbitrary single-valued selection of the nearest point mapping from E onto $N_0(A)$ such that

$$\langle y - P_0 x, j(x - P_0 x)\rangle \leq 0 \tag{3.13}$$

for all $y \in N_0(A)$. Suppose that a sequence $\{x_n\}$ can be defined by the explicit scheme

$$\begin{cases} x_{n+1} = x_n - \lambda_n A y_n + o(\lambda_n) + e_n & n \geq 0, \\ y_n = x_n - \beta_n A x_n + o(\beta_n) & n \geq 0, \end{cases} \tag{3.14}$$

where $x_0 \in D(A)$ and $\{e_n\}_{n=0}^{\infty}$ is a sequence in E such that $\sum_{n=0}^{\infty} \|e_n\| < \infty$, $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ satisfying $\sum_{n=0}^{\infty} \lambda_n$

$= \infty$ and $\lambda_n,\ \beta_n \to 0$ as $n \to \infty$. Suppose that there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and that A satisfies the condition (3.15) such that

$$\langle Ay_n, j(y_n - P_0 y_n)\rangle \geq \phi(\|y_n - P_0 y_n\|)\|y_n - P_0 y_n\| \tag{3.15}$$

for all $n \geq 0$. If both $\{Ax_n\}$ and $\{Ay_n\}$ remain bounded and $\|P_0 x_n - P_0 y_n\| \to 0$ as $n \to \infty$, then the sequence $\{x_n\}$ converges strongly to a zero of A.

The next theorem deals with convergence for the implicit scheme (1.6). For the sake of simplicity, we only consider the case when A is single-valued.

Theorem 3.4. *Let E be a real Banach space and $A : D(A) \subset E \to E$ be a quasi-accretive operator. Suppose that a sequence $\{x_n\}$ can be defined by the implicit scheme*

$$x_{n+1} + \lambda_n A x_{n+1} = x_n + o(\lambda_n) + e_n, \quad n \geq 0, \tag{3.16}$$

where $x_0 \in D(A)$ and $\{e_n\}_{n=0}^{\infty}$ is a sequence in E such that $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and $\{\lambda_n\}_{n=0}^{\infty}$ is a positive sequence in $[0, 1]$ such that $\sum_{n=0}^{\infty} \lambda_n = \infty$. Suppose that there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and that A satisfies the condition (3.17) such that, for each $x_n \in D(A)$ and $v \in N(A)$, there exists $j(x_n - v) \in J(x_n - v)$ such that

$$\langle Ax_n, j(x_n - v)\rangle \geq \phi(\|x_n - v\|)\|x_n - v\|. \tag{3.17}$$

Then the sequence $\{x_n\}$ converges strongly to v.

Proof. If the conclusion is true with $e_n \equiv 0$ for $n \geq 0$, then it is also true if $\{e_n\}_{n=0}^{\infty}$ has a compact support. Approximating any sequence $\{\|e_n\|\}_{n=0}^{\infty}$ in l^1 by a sequence with a compact support and using the facts that the resolvents J_{λ_n} are contractions for $n \geq 0$, we see that we may assume in the remainder of the proof that $e_n \equiv 0$ for $n \geq 0$. Observe that

$$x_{n+1} - v + \lambda_n A x_{n+1} = x_n - v + o(\lambda_n) \tag{3.18}$$

for any $v \in N(A)$. Evaluating $j(x_{n+1}-v) \in J(x_{n+1}-v)$ on the equality (3.16), we obtain

$$\|x_{n+1}-v\|^2 \le (\|x_n-v\| - \lambda_n(\phi(\|x_{n+1}-v\|)-\varepsilon_n))\|x_{n+1}-v\|. \tag{3.19}$$

Without loss of generality, we may assume that $\|x_{n+1}-v\| > 0$ for all $n \ge 0$. It follows from (3.19) that

$$\|x_{n+1}-v\| \le \|x_n-v\| - \lambda_n[\phi(\|x_{n+1}-v\|)-\varepsilon_n]. \tag{3.20}$$

Set $c_n = \inf\{r \ge 0 : \phi(r) > \varepsilon_n\}$. Then $c_n \to 0$ as $n \to \infty$. If $\varepsilon_n \le \phi(\|x_{n+1}-v\|)$, then we have

$$\|x_{n+1}-v\| \le \|x_n-v\|. \tag{3.21}$$

If $\phi(\|x_{n+1}-v\|) \le \varepsilon_n$, then we have

$$\|x_{n+1}-v\| \le c_n. \tag{3.22}$$

Consequently, we have

$$\|x_{n+1}-v\| \le \max\{\|x_n-v\|, c_n\} \tag{3.23}$$

for all $n \ge 0$. Since $c_n \to 0$ as $n \to \infty$, $\{x_{n+1}-v\}$ is bounded. Let

$$h = \liminf_{n\to\infty} \|x_n-v\|.$$

For any $\varepsilon > 0$, there exists a positive integer such that $\|x_m-v\| < h+\varepsilon$ and $c_n < h+\varepsilon$ for all $n \ge m$. It follows that in fact $\|x_n-v\| < h+\varepsilon$ for all $n \ge m$. Therefore we have

$$\limsup_{n\to\infty} \|x_n-v\| \le h+\varepsilon,$$

so that $h = \lim_{n\to\infty} \|x_n-v\|$. If $h > 0$, then, for sufficiently large n,

$$\phi(\|x_{n+1}-v\|) > \phi\left(\frac{h}{2}\right),$$

which implies that $(h/2)\phi(h/2)\lambda_n \le \|x_n-v\| - \|x_{n+1}-v\|$ and hence $\sum_{n=0}^{\infty} \lambda_n < \infty$. This completes the proof. □

Remark 3.5. Theorem 3.4 can be extended to the following more general setting:

Let E be a real reflexive and smooth Banach space and $A : D(A) \subset E \to E$ be a quasi-accretive operator. Assume that $N(A)$ is proximinal and convex. Let P_0 denote an arbitrary single-valued selection of the nearest point mapping from E onto $N(A)$. Suppose that a sequence $\{x_n\}$ can be defined by the implicit scheme

$$x_{n+1} + \lambda_n A x_{n+1} = x_n + o(\lambda_n) + e_n, \quad n \geq 0, \tag{3.24}$$

where $x_0 \in D(A)$, $\{e_n\}_{n=0}^{\infty}$ is a sequence in E such that $\sum_{n=0}^{\infty} \|e_n\| < \infty$ and $\{\lambda_n\}_{n=0}^{\infty}$ is a positive sequence in $[0,1]$ such that $\sum_{n=0}^{\infty} \lambda_n = \infty$. Suppose that there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ and that A satisfies the condition (3.25) such that, for each $x_n \in D(A)$, there exists $j(x_n - P_0x_n) \in J(x_n - P_0x_n)$ such that

$$\langle Ax_n, j(x_n - P_0x_n)\rangle \geq \phi(\|x_n - P_0x_n\|)\|x_n - P_0x_n\|. \tag{3.25}$$

Then the sequence $\{x_n\}$ converges strongly to a zero of A.

In fact, we observe first that the normalized duality mapping J is single-valued since E is reflexive and smooth. Since $N(A)$ is proximinal and convex, we have

$$\langle y - P_0x, j(x - P_0x)\rangle \leq 0$$

for all $y \in N(A)$. The remainder of the proof is similar to the corresponding part of Theorem 3.4.

Remark 3.6. Our Theorems 3.2$\sim$3.4 and Remarks 3.3$\sim$3.5 are very general results, from which almost all the convergence results on the Ishikawa iterative process (with errors) in arbitrary Banach spaces can be deduced easily.

4. Applications. In this section, we are devoted to present possible applications of the convergence principle established in Section 3. It is expected that almost all the recent convergence results obtained by Chidume [**11**], Weng [42], Osilike [**36**], L.S. Liu [**30**], L.W. Liu [**31**], Chidume and Osilike [**8**], Ding [**26**], Zhou [**48, 53**] and others can be deduced from our general convergence principle.

Corollary 4.1. *Theorem* Z1 *is a corollary of Theorem* 3.3.

Proof. Define a mapping $A : D(A) = K \to E$ by $Ax = x - Tx$ for each $x \in K$. Then A is strongly accretive with $N(A) = \{x^*\}$, the recursive formula (2.15) reduces to

$$\begin{cases} x_{n+1} = x_n - \alpha_n A y_n - \alpha_n \beta_n A x_n \\ \qquad\;\; = x_n - \alpha_n A y_n + o(\alpha_n) & n \geq 0, \\ y_n = x_n - \beta_n A x_n & n \geq 0. \end{cases} \tag{4.2}$$

Moreover, $\{Ax_n\}$ and $\{Ay_n\}$ are all bounded since K is bounded. By Theorem 3.3, we assert that Theorem Z1 holds true. This completes the proof. □

Corollary 4.2. *Theorem* Lu1 *is a corollary of Theorem* 3.4.

Proof. Define a mapping $A : K \to E$ by $Ax = x - Tx$ for each $x \in K$. Then A is strongly accretive with $N(A) = \{x^*\}$. We observe that the recursive formula (2.17) reduces to

$$x_{n+1} = x_n - c_n A x_n, \quad n \geq 0. \tag{4.3}$$

This yields

$$x_{n+1} + c_n A x_{n+1} = x_n + o(c_n), \quad n \geq 0, \tag{4.4}$$

with $\sum_{n=0}^{\infty} c_n = \infty$. Hence the sequence $\{x_n\}_{n=0}^{\infty}$ satisfies (3.9) with $e_n \equiv 0$ for $n \geq 0$. By Theorem 3.4, we see that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Corollary 4.3. *Theorem* CO *is a corollary of Theorem* 3.4.

Proof. Define a mapping $A : K \to E$ by $Ax = x - Tx$ for each $x \in K$. Then A is Lipschitzian and strongly accretive with $N(A) = \{x^*\}$. The recursive formula (2.18) reduces to

$$x_{n+1} + \alpha_n A x_{n+1} = x_n - \alpha_n \beta_n A x_n + \alpha_n (A x_{n+1} - A y_n), \quad n \geq 0. \tag{4.5}$$

Under the assumptions of Theorem CO, we can prove that the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded and $Ax_{n+1} - Ay_n \to 0$ as $n \to \infty$. Thus, the recursive formula (4.4) in fact reduces to

$$x_{n+1} + \alpha_n Ax_{n+1} = x_n + o(\alpha_n), \quad n \geq 0. \tag{4.6}$$

By Theorem 3.4, we know that the conclusion of Theorem CO is true. This completes the proof. □

Corollary 4.4. *Theorem* D1 *is a corollary of Theorem* 3.4.

Proof. Define a mapping $A : D(A) \subset E \to E$ by $Ax = Tx - f$ for each $x \in K$. Then A is Lipschitzian strongly accretive with $N(A) = \{x^*\}$. The recursive formula (2.19) reduces to

$$x_{n+1} + \alpha_n Ax_{n+1} = x_n - \alpha_n\beta_n Ax_n + \alpha_n(Ax_{n+1} - Ay_n), \quad n \geq 0. \tag{4.7}$$

Under the assumptions of Theorem D1, we can prove that the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded and $Ax_{n+1} - Ay_n \to 0$ as $n \to \infty$. Thus, the recursive formula (4.7) in fact reduces to

$$x_{n+1} + \alpha_n Ax_{n+1} = x_n + o(\alpha_n) + e_n, \quad n \geq 0, \tag{4.8}$$

with $\sum_{n=0}^{\infty} \|e_n\| < \infty$. By Theorem 3.4, we know that the conclusion of Theorem D1 is true. This completes the proof. □

Corollary 4.5. *Theorem* Z3 *is a corollary of Theorem* 3.4.

Proof. Under the assumptions of Theorem Z3, the recursive formula (2.22) in fact reduces to

$$x_{n+1} + \alpha_n Tx_{n+1} = x_n + o(\alpha_n), \quad n \geq 0. \tag{4.9}$$

By Theorem 3.4, we assert that the conclusion of Theorem Z3 holds true. This completes the proof. □

5. Open problems. We conclude this paper with the following open problems:

Open problem 1. Can one present a constructive proof for Theorem Lu1?

Open problem 2. Can Theorem Z1 be extended to a real smooth Banach space?

Open problem 3. Can Theorem Z1 be deduced from Theorem 3.2?

Open problem 4. Can one give some sufficient conditions which guarantee the iterative schemes considered in Theorems 3.2 and 3.4 are well-defined?

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