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PHELPS' UNIQUENESS PROPERTY FOR $K(X,Y)$ IN $L(X,Y)$

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ABSTRACT. We study pairs of Banach spaces X and Y with X^* or Y^* having a metric compact approximation of the identity (MCAI) with adjoint operators such that the subspace $K(X,Y)$ of compact operators from X to Y has the Phelps' uniqueness property U in the space of all continuous linear operators $L(X,Y)$, i.e., every functional $f \in K(X,Y)^*$ has a unique norm-preserving extension to $L(X,Y)$.

Our main results are: (1) $K(X,X)$ has property U in $L(X,X)$ whenever X has an MCAI and $K(E,E)$ has property U in $L(E,E)$ for every closed separable subspace E of X having an MCAI; (2) if a Banach space Y has an MCAI, then $K(X,Y)$ has property U in $L(X,Y)$ for all Banach spaces X if and only if $K(l_1,Y)$ has property U in $L(l_1,Y)$. We also show that if a separable dual space X^* has an MCAI with adjoint operators, then property U for $K(X,X)$ in $L(X,X)$ is determined by the properties of the extreme points of the unit ball of $L(X,X)^*$.

0. Introduction. Let X be a (real or complex) Banach space, and let Z be a closed subspace of X. By the Hahn-Banach theorem, every continuous linear functional $g \in Z^*$ has a norm-preserving extension $f \in X^*$. In general, such an extension is highly non-unique. Following Phelps [**16**], we say that Z has *property* U in X if every $g \in Z^*$ has a unique norm-preserving extension $f \in X^*$.

According to the terminology in [**2**], a closed subspace Z of a Banach space X is said to be an *ideal* in X if there exists a contractive projection P on X^* with $\ker P = Z^{\perp}$. It is straightforward to verify that, if Z is an ideal in X, then, for every $f \in X^*$, $Pf \in X^*$ is a norm-preserving extension of the restriction $f|_Z \in Z^*$. Therefore, $\operatorname{ran} P$ is canonically isometric to Z^*. In the sequel, we shall use the (generally non-Hausdorff) weak topology $\sigma(X, \operatorname{ran} P)$. Ideals with property U have been studied e.g. in [**10, 11, 14, 15**].

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In this paper, we study property U for the subspace of compact operators $K(X,Y)$ in the space of all continuous linear operators $L(X,Y)$ between two Banach spaces X and Y. We restrict our attention to the case where X^{**} or Y^* enjoys the Radon-Nikodým property (shortly, RNP) and X or Y has a shrinking metric compact approximation of the identity. Recall that a net (K_α) in $K(X,X)$ with $\|K_\alpha\| \le 1$ is called a *metric compact approximation of the identity* (shortly, MCAI) if $\lim K_\alpha x = x$ for all $x \in X$. If also $\lim {K_\alpha}^* x^* = x^*$ for all $x^* \in X^*$, then (K_α) is called a *shrinking* MCAI. Note that, see [**4**, proof of Lemma 1], if (K_γ) is any weak* convergent (in $K(X,X)^{**}$) shrinking MCAI of X, respectively weak* convergent (in $K(Y,Y)^{**}$) MCAI of Y, then $K(X,Y)$ is an ideal in $L(X,Y)$ with respect to the following *Johnson projection* P on $L(X,Y)^*$ defined by

$$P\phi(T) = \lim_\gamma \phi(TK_\gamma), \quad \phi \in L(X,Y)^*,\ T \in L(X,Y), \tag{0.1}$$

respectively

$$P\phi(T) = \lim_\gamma \phi(K_\gamma T), \quad \phi \in L(X,Y)^*,\ T \in L(X,Y). \tag{0.2}$$

In Section 1, we give a sufficient condition for the convergence of nets with respect to the weak topology induced by the Johnson projection, and establish a criterion of property U for ideals of compact operators in the corresponding space of all continuous linear operators.

In Section 2, we prove that, in certain cases, property U for $K(X,Y)$ in $L(X,Y)$ is separably determined, and study its hereditary properties. In particular, we prove that (1) $K(X,X)$ has property U in $L(X,X)$ whenever X has an MCAI and $K(E,E)$ has property U in $L(E,E)$ for every closed separable subspace E of X having an MCAI; (2) if a Banach space Y has an MCAI, then $K(X,Y)$ has property U in $L(X,Y)$ for all X if and only if $K(l_1,Y)$ has property U in $L(l_1,Y)$. Most of the results of this section are extensions of the corresponding results on M-ideals (for the prototype of (1), see [**12**, Theorem 2]; for the prototype of (2), see [**12**, Corollary 7], see also [**9**, Theorem 2.12] for the separable case), a well-studied subclass of subspaces with property U, see [**3**]. Recall that a closed subspace Z of a Banach space X is said to be an *M-ideal* in X if it is an ideal in X with respect to a projection P on X^* such that, for each $f \in X^*$, one has $\|f\| = \|Pf\| + \|f - Pf\|$.

In Section 3, we prove an extremal test for property U for separable ideals, and, as an application, we show that, under certain assumptions, the uniqueness of the norm-preserving extensions of functionals of the form $x^{**} \otimes y^*$ is sufficient for $K(X,Y)$ to have property U in $L(X,Y)$.

Let us fix some notation. The closed unit ball and the unit sphere of a Banach space X will be denoted, respectively, by B_X and S_X. For a set $A \subset X$, we denote its convex hull by $\operatorname{conv} A$, its linear span by $\operatorname{span} A$, and the set of its extreme points by $\operatorname{ext} A$. The symbol $L(X,Y)$ will stand for the space of continuous linear operators from X to a Banach space Y, and $K(X,Y)$ for its subspace of compact operators. We shall write $L(X)$ and $K(X)$ instead of $L(X,X)$ and $K(X,X)$, respectively. We also denote

$$\mathcal{U}_X = \big\{(x_n)_{n=1}^{\infty} \subset X\colon \|x_1\| \le 1,\ \|x_{n+1} - x_n\| \le 1,\ n \in \mathbf{N}\big\}.$$

1. Auxiliary results.

Our arguments to prove the results in Section 2 are based on

Proposition 1.1 (see [**14**, Theorem 3]). *Let Z be an ideal in a Banach space X with respect to a projection P. The following assertions are equivalent.*

(i) *Z has property U in X.*

(ii) *Whenever $\varepsilon > 0$, $(y_n)_{n=1}^{\infty} \in \mathcal{U}_Z$, K is a convex subset of Z, and x is in the $\sigma(X, \operatorname{ran} P)$-closure of K, then there are $z \in K$ and $n_0 \in \mathbf{N}$ satisfying*

$$\|y_{n_0} + x - z\| \le n_0 + \varepsilon.$$

(iii) *For every $(y_n)_{n=1}^{\infty} \in \mathcal{U}_Z$ and every $x \in B_X$, there is a net (z_α) in B_Z such that $\lim z_\alpha = x$ in the $\sigma(X, \operatorname{ran} P)$-topology and, provided $\varepsilon > 0$, one can find an index α_0 so that, for every $\alpha > \alpha_0$, there is some $n_\alpha \in \mathbf{N}$ satisfying*

$$\|y_{n_\alpha} + x - z_\alpha\| \le n_\alpha + \varepsilon.$$

If $K(X,Y)$ is an ideal in $L(X,Y)$ with respect to some Johnson projection, then, in order to make use of Proposition 1.1 to decide whether $K(X,Y)$ does or does not have property U in $L(X,Y)$, it would be helpful to have some description of the convergence of nets with respect to the weak topology induced by this projection. This will be given in

Lemma 1.2. *Let X and Y be Banach spaces with X^{**} or Y^* enjoying the* RNP *and X, respectively Y, having a shrinking* MCAI, *and let P be a Johnson projection on $L(X,Y)^*$. Then, for any shrinking* MCAI *(K_α) of X, respectively Y, and all $T \in L(X,Y)$, one has $\lim_\alpha TK_\alpha = T$, respectively $\lim_\alpha K_\alpha T = T$, in the $\sigma(L(X,Y), \operatorname{ran} P)$-topology.*

Proof. Let (K_α) be any shrinking MCAI of X (the proof is almost verbatim with some obvious changes if we assume that Y has a shrinking MCAI), and let $T \in L(X,Y)$, $f \in L(X,Y)^*$ and $\varepsilon > 0$. It suffices to show that there is an index α_0 such that $|Pf(T) - Pf(TK_\alpha)| < \varepsilon$ for all $\alpha > \alpha_0$. To this end, first observe that whenever $x^{**} \in X^{**}$ and $y^* \in Y^*$, then, for the functional $x^{**} \otimes y^* \in L(X,Y)^*$ defined by $x^{**} \otimes y^*(V) = x^{**}(V^*y^*)$, $V \in L(X,Y)$, one has $P(x^{**} \otimes y^*) = x^{**} \otimes y^*$. Since X^{**} or Y^* has the RNP, by [**1**, Theorem 1], $\operatorname{span}\{x^{**} \otimes y^*|_{K(X,Y)}\colon x^{**} \in X^{**}, y^* \in Y^*\}$ is a dense subspace of $K(X,Y)^*$, and thus there are $n \in \mathbf{N}$, $x_i^{**} \in X^{**}$, $y_i^* \in Y^*$, $i = 1, \dots, n$, such that, for $g = \sum_{i=1}^n x_i^{**} \otimes y_i^* \in L(X,Y)^*$, one has

$$\begin{aligned}\|Pf - Pg\| &= \|P(f-g)\| = \|(f-g)|_{K(X,Y)}\| \\ &= \|f|_{K(X,Y)} - g|_{K(X,Y)}\| < \frac{\varepsilon}{3\|T\|}.\end{aligned}$$

Choosing an index α_0 so that, for all $\alpha > \alpha_0$, one has

$$\|T^*y_i^* - {K_\alpha}^*T^*y_i^*\| < \frac{\varepsilon}{3n\|x_i^{**}\|}, \quad i = 1, \dots, n,$$

it remains to observe that, for any $\alpha > \alpha_0$,

$$\begin{aligned}|Pf(T) - Pf(TK_\alpha)| &\le \|Pf - Pg\|\,\|T\| + |Pg(T - TK_\alpha)| \\ &\quad + \|Pf - Pg\|\,\|T\|\,\|K_\alpha\| \\ &< \frac{2\varepsilon}{3} + \left|\sum_{i=1}^n x_i^{**}\big((T - TK_\alpha)^* y_i^*\big)\right| < \varepsilon. \qquad \square\end{aligned}$$

The following criterion of property U is now a quick consequence of Proposition 1.1 and Lemma 1.2.

Proposition 1.3. *Let X and Y be Banach spaces with X^{**} or Y^* having the* RNP*, and X, respectively Y, having a shrinking* MCAI*. The following assertions are equivalent.*

(i) *$K(X,Y)$ has property U in $L(X,Y)$.*

(ii) *For every $(S_n)_{n=1}^\infty \in \mathcal{U}_{K(X,Y)}$, $T \in B_{L(X,Y)}$, $\varepsilon > 0$, and every shrinking* MCAI *(K_α) of X, respectively Y, there are $n_0 \in \mathbf{N}$ and $K \in \operatorname{conv}\{K_\alpha\}$ satisfying*

$$\|S_{n_0} + T - TK\| \le n_0 + \varepsilon, \quad \textit{resp.} \quad \|S_{n_0} + T - KT\| \le n_0 + \varepsilon.$$

(iii) *For every $(S_n)_{n=1}^\infty \in \mathcal{U}_{K(X,Y)}$ and $T \in B_{L(X,Y)}$, there exists a shrinking* MCAI *(K_α) of X, respectively Y, such that, provided $\varepsilon > 0$, one can find an index α_0 so that, for every $\alpha > \alpha_0$, there is some $n_\alpha \in \mathbf{N}$ satisfying*

$$(1.1) \quad \|S_{n_\alpha} + T - TK_\alpha\| \le n_\alpha + \varepsilon, \quad \textit{resp.} \quad \|S_{n_\alpha} + T - K_\alpha T\| \le n_\alpha + \varepsilon.$$

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) follow immediately from the corresponding implications of Proposition 1.1 by Lemma 1.2.

(ii) $\Rightarrow$ (iii). Let $(K_\beta)_{\beta \in \mathcal{B}}$ be any shrinking MCAI of X, respectively Y. Consider the set $\mathcal{A} = \{(\beta, \varepsilon): \beta \in \mathcal{B}, \varepsilon \in \mathbf{R}, \varepsilon > 0\}$ directed in the natural way. By (ii), to each $\alpha = (\beta, \varepsilon) \in \mathcal{A}$, one can associate some $K_\alpha \in \operatorname{conv}\{K_\gamma: \gamma \in \mathcal{B}, \gamma > \beta\}$ such that, for some $n_\alpha \in \mathbf{N}$, the condition (1.1) holds. The net $(K_\alpha)_{\alpha \in \mathcal{A}}$ clearly meets all the conditions of (iii). $\square$

2. Hereditary results on property U for $K(X,Y)$ in $L(X,Y)$. Most of the results in this section are extensions of the analogous results for M-ideals.

Throughout this section, we shall repeatedly exploit the following well known lemma (see [**17**, Theorem 15] or [**3**, p. 126], and, e.g., the proofs of [**7**, Theorem 4.2, (c)$\Rightarrow$(a), and Proposition 4.1, (b)$\Rightarrow$(c)]).

Lemma 2.1. *Let a Banach space X have property U in its bidual. Then*

(a) X^* *has the* RNP,

(b) *every* MCAI *of X is shrinking.*

Theorem 2.2. *Let X be a Banach space having an* MCAI*. Suppose that $K(E)$ has property U in $L(E)$ for all closed separable subspaces $E \subset X$ having an* MCAI*. Then $K(X)$ has property U in $L(X)$.*

Remark. 1) The proof of Theorem 2.2 is essentially based on the same effects as its prototype's for M-ideals, see [**12**, Theorem 2], see also [**13**, Corollary 4.3]. However, use of Proposition 1.3 makes our proof somewhat simpler than the proof of the prototype.

2) It is well known that, see e.g., [**3**, p. 301], if X is a Banach space and $K(X)$ is an M-ideal in $L(X)$, then $K(E)$ is an M-ideal in $L(E)$ for all closed subspaces $E \subset X$ having an MCAI. We do not know whether the analogous result is true for property U.

Proof of Theorem 2.2. Since every closed separable subspace of X is contained in some closed separable subspace having an MCAI, and our assumptions yield that every closed separable subspace of X having an MCAI enjoys property U in its bidual, see [**5**, Theorem 4.5] or [**14**, Theorem 7], every closed separable subspace of X has property U in its bidual (because property U in biduals is inherited by closed subspaces). Thus, by [**14**, Corollary 5], X has property U in its bidual. By Lemma 2.1, the latter implies that X^* has the RNP and any MCAI of X is shrinking.

Suppose for contradiction that $K(X)$ does not have property U in $L(X)$. Then, by Proposition 1.3, there are $(S_n)_{n=1}^{\infty} \in \mathcal{U}_{K(X)}$, $T \in B_{L(X)}$, $\varepsilon > 0$, and a shrinking MCAI (K_α) of X such that

$$\|S_n + T - KT\| > n + \varepsilon \quad \text{for all } n \in \mathbf{N} \text{ and all} \quad K \in \operatorname{conv}\{K_\alpha\}.$$

We are going to construct a closed separable subspace $E \subset X$ having an MCAI so that $K(E)$ fails to have property U in $L(E)$.

To this end, first pick any index α_1 and any $z \in B_X$, and put $C_1 = \{z\}$. Now continue as follows. Given indices α_i and finite sets

$C_i \subset X$, $i = 1, \dots, m$ ($m \in \mathbf{N}$) pick an index $\alpha_{m+1} > \alpha_m$ so that

$$\|K_{\alpha_{m+1}} x - x\| < \frac{1}{m+1} \quad \text{for all} \quad x \in C_m.$$

Choose a finite $1/(m+1)$-net A_{m+1} of $\operatorname{conv}\{K_{\alpha_1}, \dots, K_{\alpha_{m+1}}\}$ and a finite set $B_{m+1} \subset B_X$ so that, for all $K \in A_{m+1}$ and $n \in \{1, \dots, m+1\}$, there exists some $x \in B_{m+1}$ satisfying

$$\|(S_n + T - KT)\, x\| > \|S_n + T - KT\| - \frac{1}{m+1}.$$

Then put

$$C_{m+1} = C_m \cup B_{m+1} \cup \left[\bigcup_{\substack{V \in \{T, S_i, K_{\alpha_i} : i=1,\dots,m+1\} \\ x \in C_m \cup B_{m+1}}} \{Vx\} \right].$$

Proceeding as described, we obtain a sequence $(K_{\alpha_m})_{m=1}^\infty \subset B_{K(X)}$ and a sequence $(C_m)_{m=1}^\infty$ of finite subsets of X. Denote $E = \overline{\operatorname{span} \cup_{m=1}^\infty C_m}$. Since $Tx, S_n x, K_{\alpha_m} x \in E$ for all $x \in E$, $n \in \mathbf{N}$, $m \in \mathbf{N}$, denoting $\tilde{T} = T|_E$, $\tilde{S}_n = S_n|_E$, $K_m = K_{\alpha_m}|_E$, $n, m \in \mathbf{N}$, we have $\tilde{T} \in B_{L(E)}$, $(\tilde{S}_n)_{n=1}^\infty \in \mathcal{U}_{K(E)}$, $K_m \in B_{K(E)}$, $m \in \mathbf{N}$. Clearly (K_m) is an MCAI of E. Since property U in biduals is inherited by closed subspaces, the subspace E has property U in E^{**}, and, by Lemma 2.1, (K_m) is shrinking. For all $n \in \mathbf{N}$ and all $K \in \operatorname{conv}\{K_m\}$, one has $\|\tilde{S}_n + \tilde{T} - K\tilde{T}\| > n + \varepsilon$, and, by Proposition 1.3, $K(E)$ fails to have property U in $L(E)$. $\square$

Theorem 2.3. *Let X and Y be Banach spaces with Y having property U in Y^{**} and an* MCAI. *Suppose that $K(E,F)$ has property U in $L(E,F)$ for all closed separable subspaces $E \subset X$ and $F \subset Y$ with F having an* MCAI. *Then $K(X,Y)$ has property U in $L(X,Y)$.*

Proof. Let $(K_\alpha) \subset B_{K(Y)}$ be any MCAI. By Lemma 2.1, (K_α) is shrinking and Y^* has the RNP.

Suppose for contradiction that $K(X,Y)$ does not have property U in $L(X,Y)$. Then, by Proposition 1.3, there are $(S_n)_{n=1}^\infty \in \mathcal{U}_{K(X,Y)}$, $T \in B_{L(X,Y)}$, $\varepsilon > 0$, and a shrinking MCAI (K_α) of Y such that

$$\|S_n + T - KT\| > n + \varepsilon \quad \text{for all} \quad n \in \mathbf{N} \quad \text{and all} \quad K \in \operatorname{conv}\{K_\alpha\}.$$

We are going to construct closed separable subspaces $E \subset X$ and $F \subset Y$ with F having an MCAI so that $K(E,F)$ fails to have property U in $L(E,F)$.

To this end, first pick any index α_1 and $z \in B_X$, and put $C_1 = \{z\}$ and $D_1 = \{S_1 z, Tz\}$. Now continue as follows. Given indices α_i and finite sets $C_i \subset X$, $D_i \subset Y$, $i = 1, \dots, m$ ($m \in \mathbf{N}$) pick an index $\alpha_{m+1} > \alpha_m$ so that

$$\|K_{\alpha_{m+1}} y - y\| < \frac{1}{m+1} \quad \text{for all} \quad y \in D_m.$$

Choose a finite $1/(m+1)$-net A_{m+1} of $\operatorname{conv}\{K_{\alpha_1}, \dots, K_{\alpha_{m+1}}\}$ and a finite set $B_{m+1} \subset B_X$ so that, for all $K \in A_{m+1}$ and $n \in \{1, \dots, m+1\}$, there exists some $x \in B_{m+1}$ satisfying

$$\|(S_n + T - KT)\, x\| > \|S_n + T - KT\| - \frac{1}{m+1}.$$

Then put $C_{m+1} = C_m \cup B_{m+1}$ and

$$D_{m+1} = D_m \cup \Bigg[\bigcup_{\substack{V \in \{S_1, \dots, S_{m+1}, T\} \\ x \in C_{m+1}}} \{Vx\} \Bigg] \bigcup \Bigg[\bigcup_{\substack{i \in \{1, \dots, m+1\} \\ y \in D_m}} \{K_{\alpha_i} y\} \Bigg].$$

Proceeding as described, we obtain a sequence $(K_{\alpha_m})_{m=1}^{\infty} \subset B_{K(Y)}$ and sequences $(C_m)_{m=1}^{\infty}$ and $(D_m)_{m=1}^{\infty}$ of finite subsets of X and Y, respectively. Denote $E = \overline{\operatorname{span} \cup_{m=1}^{\infty} C_m}$ and $F = \overline{\operatorname{span} \cup_{m=1}^{\infty} D_m}$. Since Tx, $S_n x \in F$ for all $x \in E$, $n \in \mathbf{N}$, and $K_{\alpha_m} y \in F$ for all $y \in F$, $m \in \mathbf{N}$, denoting $\widetilde{T} = T|_E$, $\widetilde{S}_n = S_n|_E$, $K_m = K_{\alpha_m}|_F$, $n, m \in \mathbf{N}$, we have $\widetilde{T} \in B_{L(E,F)}$, $(\widetilde{S}_n)_{n=1}^{\infty} \in \mathcal{U}_{K(E,F)}$, $K_m \in B_{K(F)}$, $m \in \mathbf{N}$. Clearly (K_m) is an MCAI of F. Since property U in biduals is inherited by closed subspaces, the subspace F has property U in F^{**}, and, by Lemma 2.1, (K_m) is shrinking. For all $n \in \mathbf{N}$ and all $K \in \operatorname{conv}\{K_m\}$, one has $\|\widetilde{S}_n + \widetilde{T} - K\widetilde{T}\| > n + \varepsilon$, and, by Proposition 1.3, $K(E,F)$ fails to have property U in $L(E,F)$. $\square$

Proposition 2.4. *Let X and Y be Banach spaces with X^{**} or Y^{*} having the* RNP, *and let $K(X,Y)$ have property U in $L(X,Y)$.*

(a) *If* X *has a shrinking* MCAI*, then* $K(X,F)$ *has property* U *in* $L(X,F)$ *for all closed subspaces* $F \subset Y$*.*

(b) *If* Y *has a shrinking* MCAI*, then* $K(X/E,Y)$ *has property* U *in* $L(X/E,Y)$ *for all closed subspaces* $E \subset X$*.*

Remark. For the prototype of Proposition 2.4 for M-ideals, see [**9**, Proposition 2.9].

Proof of Proposition 2.4. (a) Suppose that X has a shrinking MCAI. Let $F \subset Y$ be any closed subspace and let $j\colon F \to Y$ denote the natural embedding. Fix arbitrary $(S_n) \in \mathcal{U}_{K(X,F)}$ and $T \in B_{L(X,F)}$. Clearly $(jS_n) \in \mathcal{U}_{K(X,Y)}$ and $jT \in B_{L(X,Y)}$. Since $K(X,Y)$ has property U in $L(X,Y)$, by Proposition 1.3, there is a shrinking MCAI $(K_\alpha) \subset B_{K(X)}$ such that, provided $\varepsilon > 0$, one can find an index α_0 such that, for every $\alpha > \alpha_0$, there exists some $n_\alpha \in \mathbf{N}$ satisfying

$$\|S_{n_\alpha} + T - TK_\alpha\| = \|jS_{n_\alpha} + jT - jTK_\alpha\| \le n_\alpha + \varepsilon.$$

By Proposition 1.3, $K(X,F)$ has property U in $L(X,F)$.

(b) Suppose that Y has a shrinking MCAI. Let $E \subset X$ be any closed subspace and let $q\colon X \to X/E$ denote the quotient map. Fix arbitrary $(S_n) \in \mathcal{U}_{K(X/E,Y)}$ and $T \in B_{L(X/E,Y)}$. For any $U \in L(X/E,Y)$, one has $\|U\| = \|Uq\|$, thus $(S_nq) \in \mathcal{U}_{K(X,Y)}$ and $Tq \in B_{L(X,Y)}$. Since $K(X,Y)$ has property U in $L(X,Y)$, by Proposition 1.3, there is a shrinking MCAI $(K_\alpha) \subset B_{K(Y)}$ such that, provided $\varepsilon > 0$, one can find an index α_0 such that, for every $\alpha > \alpha_0$, there exists some $n_\alpha \in \mathbf{N}$ satisfying

$$\|S_{n_\alpha} + T - K_\alpha T\| = \|S_{n_\alpha}q + Tq - K_\alpha Tq\| \le n_\alpha + \varepsilon.$$

By Proposition 1.3, $K(X/E,Y)$ has property U in $L(X/E,Y)$. $\square$

Corollary 2.5. *Let* Y *be a Banach space having an* MCAI*. The following assertions are equivalent.*

(i) $K(X,Y)$ *has property* U *in* $L(X,Y)$ *for all Banach spaces* X*.*

(ii) $K(l_1,Y)$ *has property* U *in* $L(l_1,Y)$*.*

Remark. For the prototype of Corollary 2.5 for M-ideals, see [**12**, Corollary 7], see also [**9**, Theorem 2.12] for the separable case. By courtesy of Theorem 2.3, the proof of Corollary 2.5 is more direct than its prototype's one.

Proof of Corollary 2.5. (i) $\Rightarrow$ (ii) is more than obvious.

(ii) $\Rightarrow$ (i). Let X be a Banach space, and let $E \subset X$ and $F \subset Y$ be closed separable subspaces with F having an MCAI. Since $K(l_1, Y)$ has property U in $L(l_1, Y)$, the subspace Y has property U in Y^{**}, see [**14**, Theorem 12], and, by Lemma 2.1, Y^* has the RNP. By Proposition 2.4(a), $K(l_1, F)$ has property U in $L(l_1, F)$. Since property U in biduals is inherited by closed subspaces, the subspace F has property U in its bidual, and thus, by Lemma 2.1, F^* has the RNP and any MCAI of F is shrinking. By Proposition 2.4(b), $K(E, F)$ has property U in $L(E, F)$ (because every closed separable Banach space is isometrically isomorphic to a quotient of ℓ_1). It remains to apply Theorem 2.3. $\square$

3. Extremal test for property U**.** The following theorem is an extremal test for property U for separable ideals.

Theorem 3.1. *Let X be a Banach space, and let Z be a separable ideal in X with respect to a projection $P \in L(X^*)$ such that, for every $x \in B_X$, there is a sequence $(z_m)_{m=1}^{\infty} \subset B_Z$ satisfying $\lim z_m = x$ in the $\sigma(X, \operatorname{ran} P)$-topology. If, for all $\phi \in \operatorname{ext} B_{X^*}$ with $\|\phi\| = \|\phi|_Z\|$, the functional ϕ itself is the only norm-preserving extension to X of its restriction on Z, then Z has property U in X.*

Remark. The assumption for Z to be an ideal in Theorem 3.1 cannot be dropped even if X is finite dimensional, see [**6**, p. 459, Example]. We do not know whether the assertion of Theorem 3.1 remains true if one drops the assumption for B_X to be contained in the sequential $\sigma(X, \operatorname{ran} P)$-closure of B_Z.

Proof of Theorem 3.1. Let $f \in S_{X^*}$ satisfy $\|Pf\| = \|f\|$. It suffices to show that $Pf = f$. Suppose for contradiction that $Pf(x) \neq f(x)$ for some $x \in B_X$. Put $E = \overline{\operatorname{span}(Z \cup \{x\})}$ and $C = \operatorname{ext} B_{E^*}$, and let $Q \in L(E^*)$ be the ideal projection for Z in E induced by P (i.e., for $h \in E^*$, one has $Qh = Pg|_E$ where $g \in X^*$ is any extension of h to X). Let a sequence $(z_m)_{m=1}^{\infty} \subset B_Z$ be such that $\lim z_m = x$ in the $\sigma(X, \operatorname{ran} P)$-topology of X; then also $\lim z_m = x$ in the $\sigma(E, \operatorname{ran} Q)$-topology of E. Denote $h = f|_E$. By Choquet's integral representation theorem, there is a regular Borel probability measure μ on (B_{E^*}, w^*) concentrated on C and representing h. By Lebesgue's bounded convergence theorem, one has

$$\begin{aligned}
Qh(x) &= \lim_{m\to\infty} Qh(z_m)\\
&= \lim_{m\to\infty} h(z_m)\\
&= \lim_{m\to\infty} \int_C \psi(z_m)\, d\mu(\psi)\\
&= \lim_{m\to\infty} \int_C Q\psi(z_m)\, d\mu(\psi)\\
&= \int \lim_{m\to\infty} Q\psi(z_m)\, d\mu(\psi) = \int_C Q\psi(x)\, d\mu(\psi).
\end{aligned}$$

Denote $A = \{\psi \in C \colon Q\psi \neq \psi\}$. Observe that $\|Q\psi\| < 1$ for all $\psi \in A$. (Indeed, let $\psi \in A$. Since $\psi \in \operatorname{ext} B_{E^*}$, then ψ has some norm-preserving extension $\phi \in \operatorname{ext} B_{X^*}$. Since $Q\psi \neq \psi$, also $P\phi \neq \phi$, and thus $\|Q\psi\| = \|P\phi\| < \|\phi\| = 1$ because if $\|P\phi\| = \|\phi\|$, then $P\phi$ and ϕ would be different norm-preserving extensions to X of $\phi|_Z$.) Since

$$\int_C (\psi - Q\psi)(x)\, d\mu(\psi) = h(x) - Qh(x) \neq 0,$$

one has $\mu(A) > 0$, and hence $\int_A \|Q\psi\|\, d\mu(\psi) < \mu(A)$. Thus

$$\begin{aligned}
\|Pf\| = \|Qh\| &= \sup_{y\in B_E} |Qh(y)|\\
&= \sup_{y\in B_E} \left| \int_C Q\psi(y)\, d\mu(\psi)\right|\\
&\leq \int_C \sup_{y\in B_E} |Q\psi(y)|\, d\mu(\psi)\\
&= \int_C \|Q\psi\|\, d\mu(\psi)
\end{aligned}$$

$$= \int_A \|Q\psi\|\, d\mu(\psi) + \int_{C\setminus A} \|Q\psi\|\, d\mu(\psi)$$
$$< \mu(A) + \mu(C \setminus A) = \mu(C) = 1 = \|f\|,$$

a contradiction. □

Recall that if X and Y are Banach spaces, and $x^{**} \in X^{**}$ and $y^* \in Y^*$, then the functional $x^{**} \otimes y^* \in L(X,Y)^*$ is defined by $x^{**} \otimes y^*(T) = x^{**}(T^* y^*)$, $T \in L(X,Y)$.

Corollary 3.2. *Let X and Y be Banach spaces with X^* and Y^* being separable and X or Y having a shrinking* MCAI $(K_m)_{m=1}^{\infty}$. *If, for all $x^{**} \in B_{X^{**}}$ and $y^* \in B_{Y^*}$, the functional $x^{**} \otimes y^* \in L(X,Y)^*$ itself is the only norm-preserving extension to $L(X,Y)$ of its restriction on $K(X,Y)$, then $K(X,Y)$ has property U in $L(X,Y)$.*

Remark. For $x^{**} \in B_{X^{**}}$ and $y^* \in B_{Y^*}$, the functional $x^{**} \otimes y^* \in L(X,Y)^*$ itself is the only norm-preserving extension to $L(X,Y)$ of its restriction on $K(X,Y)$ e.g. whenever x^{**} is a weak*-denting point of $B_{X^{**}}$ or y^* is a weak*-denting point of B_{Y^*}, see [**8**, Lemma 3.1].

Proof of Corollary 3.2. Suppose that $(K_m)_{m=1}^{\infty}$ is a shrinking MCAI of X, respectively Y. Let $\phi \in \mathrm{ext} B_{L(X,Y)^*}$. Since $\mathrm{conv}\,\{x^{**} \otimes y^*\colon x^{**} \in B_{X^{**}},\, y^* \in B_{Y^*}\}$ is weak* dense in $B_{L(X,Y)^*}$, Milman's converse to the Krein-Milman theorem and the weak* compactness of $B_{X^{**}}$ and B_{Y^*} yield that $\phi|_{K(X,Y)} = x^{**} \otimes y^*|_{K(X,Y)}$ for some $x^{**} \in B_{X^{**}}$ and $y^* \in B_{Y^*}$. Observing that $K(X,Y)$ is a separable ideal (because $K(E,F)$ is separable if and only if E^* and F are separable) in $L(X,Y)$ with respect to the projection P defined by (0.1), respectively (0.2), where (K_γ) is a weak* convergent (in $K(X,X)^{**}$, respectively in $K(Y,Y)^{**}$) subnet of (K_m), and, by Lemma 1.2, $\lim TK_m = T$, respectively $\lim K_m T = T$, in the $\sigma(L(X,Y), \mathrm{ran}\, P)$-topology for all $T \in L(X,Y)$, an appeal to Theorem 3.1 finishes the proof. □

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