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LIONS-PEETRE'S INTERPOLATION METHODS ASSOCIATED WITH QUASI-POWER FUNCTIONS AND SOME APPLICATIONS

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ABSTRACT. This paper concerns some properties of Lions-Peetre's interpolation methods of constants and means associated with quasi-power functions, and their applications in harmonic analysis, martingale inequalities, and geometric properties of Banach spaces. We describe Besov-Orlicz spaces and Triebel-Lizorkin-Orlicz spaces in terms of interpolation and wavelet bases. We study the commutators of quasi-logarithmic operators and singular integral operators, Hankel operators in Schatten-Orlicz classes, martingale inequalities for the φ-variation, and the stability of multi-dimensional uniform rotundity under interpolation.

Many problems in analysis can be formulated in terms of the action of operators on function spaces. Interpolation theory is a very powerful tool for obtaining new estimates from old ones. In [**4**], the author investigated Lions-Peetre's interpolation methods of constants and means associated with quasi-power functions, and established its connection with the real interpolation methods in the sense of Brudnyi-Krugljak. These kinds of interpolation methods are a natural generalization of the classical real interpolation methods and may play an important role in other fields of analysis [**4, 5**]. In the present paper, we will study some properties of the above mentioned interpolation methods, and their applications in harmonic analysis, martingale inequalities and geometric properties of Banach spaces. Some classical results can be carried over in a more general context.

The plan of the paper is as follows. Section 1 includes preliminaries about Brudnyi-Krugljak's and Peetre-Gustavsson's interpolation methods. In Section 2, we formulate some useful results for Lions-Peetre's

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interpolation methods with quasi-power functions concerning equivalence, duality, reiteration, interpolation type, compact interpolation and local commutativity. In Section 3, we describe Besov-Orlicz spaces and Triebel-Lizorkin-Orlicz spaces in terms of interpolation, and represent these spaces by using wavelet bases. In Section 4, we deal with the commutators of quasi-logarithmic operators and singular integral operators, as well as Hankel operators in Schatten-Orlicz classes. In Section 5, we obtain some martingale inequalities for the φ-variation. In Section 6, we study the stability of multi-dimensional rotundity under interpolation.

1. Preliminaries. Throughout this paper, we will use the terminology and notation of interpolation theory from [**1, 2**]. The notations $\subseteq$ and $=$ between Banach spaces stand for continuous inclusion and isomorphic equivalence respectively. Let $\overline{X} = (X_0, X_1)$ be a Banach couple with $\Delta\overline{X} = X_0 \cap X_1$ and $\Sigma\overline{X} = X_0 + X_1$, and let X be an intermediate space for Banach couple $\overline{X} = (X_0, X_1)$, we denote by X^0 the regularization for $\overline{X}$, by X' the Banach space dual of X^0, and write the dual couple $\overline{X}' = (X_0', X_1')$. For Banach couples $\overline{X}$ and $\overline{Y}$, we denote $\mathcal{B}(\overline{X}, \overline{Y})$ for the space of all bounded linear operators from $\overline{X}$ to $\overline{Y}$. We simply write $\mathcal{B}(\overline{X}) = \mathcal{B}(\overline{X}, \overline{X})$.

Let us assume that $\overline{X} = (X_0, X_1)$ is a Banach couple. For $t > 0$, the J- and K-functionals on $\Delta\overline{X}$ and $\Sigma\overline{X}$, respectively, are given by

$$J(t, x; \overline{X}) = \|x\|_0 \vee (t\|x\|_1) \quad \text{if} \quad x \in \Delta\overline{X},$$

and

$$K(t, x; \overline{X}) = \inf\left\{ \|x_0\|_0 + t\|x_1\|_1 \,\middle|\, x = x_0 + x_1,\ x_j \in X_j \right\} \quad \text{if} \quad x \in \Sigma\overline{X}.$$

The K- and J-methods of interpolation due to Brudnyi and Krugljak are given as follows. Let Φ be a Banach function space over $(\mathbf{R}^+, dt/t)$ such that $1 \wedge t \in \Phi$ and $\int_0^\infty 1 \wedge (1/t)\,|f(t)|\, dt/t < \infty$ for all $f \in \Phi$. We define

$$K_\Phi(\overline{X}) := \left\{ x \in \Sigma\overline{X} \,\middle|\, \|x\|_{K_\Phi} = \|K(t, x; \overline{X})\|_\Phi < \infty \right\} \quad [\mathbf{2}, (3.3.1)],$$

and define $J_\Phi\big(\overline{X}\big)$ as the space of all $x \in \Sigma\overline{X}$, which permits a canonical representation $x = \int_0^\infty u(t)\, dt/t$ for a strongly measurable function

$u\colon \mathbf{R}^+ \longrightarrow \Delta\overline{X}$, with the norm

$$\|x\|_{J_\Phi} = \inf_u \left\|J(t, u(t); \overline{X})\right\|_\Phi < \infty \qquad [\mathbf{2}, (3.4.3)].$$

The Banach function space Φ is said to be a quasi-power parameter if the Calderón operator S, which is defined by

$$(Sf)(t) = \int_0^\infty 1 \wedge (t/s) f(s) \, \frac{ds}{s}$$

for $f \in L^0(\mathbf{R}^+, dt/t)$, is bounded on Φ. In this case, we have the equivalence

$$J_\Phi(\overline{X}) = K_\Phi(\overline{X}).$$

In a given category, an object B is a retract of the object A, if there are morphisms $\mathcal{J}\colon B \to A$ and $\mathcal{P}\colon A \to B$ in the category, such that $\mathcal{P} \circ \mathcal{J}$ is the identity on B. The Banach couple $\overline{Y}$ is a subcouple of a Banach couple $\overline{X}$ if Y_j is a subspace of X_j ($j = 0, 1$). A subcouple $\overline{Y}$ of $\overline{X}$ is a K-subcouple of $\overline{X}$ if, for some constant C,

$$K(t, y; \overline{Y}) \le C\, K(t, y; \overline{X}) \quad \text{for} \quad t > 0 \quad \text{and} \quad y \in \Sigma\overline{Y}.$$

In the sequel, we need some properties of the K_Φ methods.

Proposition 1.1. *Let $\overline{X}$ and $\overline{Y}$ be Banach couples.*

(i) *If $\overline{X}$ is a retract of $\overline{Y}$, then $J_\Phi(\overline{X})$ is a retract of $J_\Phi(\overline{Y})$ and $K_\Phi(\overline{X})$ is a retract of $K_\Phi(\overline{Y})$.*

(ii) *If $\overline{Y}$ is a K-subcouple of $\overline{X}$, then*

$$K_\Phi(\overline{Y}) = K_\Phi(\overline{X}) \cap \Sigma\overline{Y}.$$

(iii) *Assume that Φ is a reflexive Banach function space and a quasi-power parameter. If X_0 or X_1 is a reflexive space, then $J_\Phi(\overline{X})$ is a reflexive space.*

In fact, part (i) is a natural extension of [**1**, Theorem 6.4.2], part (ii) follows from [**9**, Theorem 2.1] and part (iii) can be induced from [**2**, Proposition 4.6.5 and Corollary 4.6.18].

Let $\rho\colon \mathbf{R}^+ \to \mathbf{R}^+$ be a concave function. We denote

$$\rho^*(t) = 1/\rho(1/t) \quad \text{and} \quad \bar{\rho}(t) = \sup_{s>0} \frac{\rho(st)}{\rho(s)} \quad \text{for} \quad t > 0.$$

A corresponding homogeneous function of two variables, again denoted by ρ, is defined by

$$\rho(t_0, t_1) = t_0 \rho(t_1/t_0) \quad \text{for} \quad t_0,\ t_1 > 0.$$

Let now ρ be quasi-power in the sense that there exist $C > 0$ and $0 < \beta < 1$ for which $\bar{\rho}(t) \le C\,(t^\beta \vee t^{(1-\beta)})$ for all $t > 0$. Throughout the paper, we always assume that

$$\bar{\rho}(t) \le t^\beta \vee t^{(1-\beta)} \quad \text{for all} \quad t > 0.$$

Recall the $\pm$ method G_ρ^0 introduced by Peetre [**15**] as below: $G_\rho^0(\overline{X})$ is the space of all $x \in \Sigma\overline{X}$ such that $x = \sum_{\nu=-\infty}^{\infty} x_\nu$ converges in $\Sigma\overline{X}$ for an admissible sequence $(x_\nu)_\nu$ in $\Delta\overline{X}$, and there is a constant C satisfying

$$\Big\| \sum_{\nu=-\infty}^{\infty} \lambda_\nu 2^{j\nu} x_\nu / \rho(2^\nu) \Big\|_j \le C \sup_\nu |\lambda_\nu|, \quad j = 0, 1$$

for any $(\lambda_\nu)_\nu \in l^\infty$. This space is equipped with the norm $\|x\|_{G_\rho^0} = \inf C$.

2. Lions-Peetre's interpolation methods associated with quasi-power functions. Let $\rho\colon \mathbf{R}^+ \to \mathbf{R}^+$ be a quasi-power function, and let $1 \le p_0, p_1 \le \infty$. We define K_{ρ,p_0,p_1} and J_{ρ,p_0,p_1} as Lions-Peetre's interpolation methods of constants and means associated with the function parameter ρ. More precisely, for a Banach couple $\overline{X} = (X_0, X_1)$, the space $K_{\rho,p_0,p_1}(\overline{X})$ consists of all those $x \in \Sigma\overline{X}$ such that there exist strongly measurable functions $x_j\colon \mathbf{R}^+ \to X_j$ $(j = 0, 1)$ satisfying $x = x_0 + x_1$ and $t^j \|x_j(t)\|_j / \rho(t) \in L^{p_j}(\mathbf{R}^+, dt/t)$ with the norm

$$\|x\|_{K_{\rho,p_0,p_1}} = \inf \Big\{ \max_{j=0,1} \big\| t^j \|x_j(t)\|_j / \rho(t) \big\|_{L^{p_j}(dt/t)} \Big\};$$

and the space $J_{\rho,p_0,p_1}(\overline{X})$ consists of all those $x \in \Sigma\overline{X}$ such that there exists a strongly measurable function $u\colon \mathbf{R}^+ \to \Delta\overline{X}$ satisfying

$x = \int_0^\infty u(t)\,dt/t$ and $t^j \|u(t)\|_j/\rho(t) \in L^{p_j}(\mathbf{R}^+, dt/t)$ $(j = 0, 1)$ with the norm

$$\|x\|_{J_{\rho,p_0,p_1}} = \inf\Big\{\max_{j=0,1}\big\|t^j\|u(t)\|_j/\rho(t)\big\|_{L^{p_j}(dt/t)}\Big\}.$$

Furthermore, we define $\varphi\colon \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ by

$$\varphi^{-1}(t) = t^{1/p_0}\rho(t^{-1/q}), \tag{2.1}$$

where $1/q = 1/p_0 - 1/p_1$. Then φ is a Young function satisfying both Δ_2 and ∇_2 conditions. Let Φ be the weighted Orlicz space of all measurable functions $f\colon \mathbf{R}^+ \longrightarrow \mathbf{C}$ such that $\int_0^\infty \varphi(t^{-q/p_0}|f(t)|)t^q\,dt/t < \infty$, which is equipped with the Luxemberg norm. Observe that Φ is a reflexive Banach function space and a quasi-power parameter.

We now summarize some properties of these interpolation methods obtained by the author in [**4, 5**].

Proposition 2.1. *Let* $\overline{X}$ *be a Banach couple.*

(i) *Equivalence* [**4**, Theorem 1.1]. *If* $p_0 \neq p_1$*, then*

$$K_\Phi(\overline{X}) = K_{\rho,p_0,p_1}(\overline{X}) = J_{\rho,p_0,p_1}(\overline{X}) = J_\Phi(\overline{X}).$$

If $p_0 = p_1 = p$*, then*

$$K_{\rho,p,p}(\overline{X}) = J_{\rho,p,p}(\overline{X}) = J_\rho^p(\overline{X}) = K_\rho^p(\overline{X}),$$

where $J_\rho^p = J_{L_\rho^p}$ *and* $K_\rho^p = K_{L_\rho^p}$*, for which* $f \in L_\rho^p$ *if and only if* $f/\rho \in L^p(\mathbf{R}^+, dt/t)$.

(ii) *Duality* [**4**, Lemma 3.2].

$$J_{\rho,p_0,p_1}(\overline{X})' = K_{\rho^*,p_0',p_1'}(\overline{X}'),$$

where $1/p_j' = 1 - 1/p_j$, $j = 0, 1$.

(iii) *Reiteration* [**5**, Lemma 4.6]. *For* $0 < \theta_0 < 1$ *and* $\theta_1 = 1 - \theta_0$*, let*

$$\frac{1}{r_j} = \frac{1-\theta_j}{p_0} + \frac{\theta_j}{p_1}, \quad j = 0, 1, \quad \text{and} \quad \frac{1}{r} = \frac{1}{r_0} - \frac{1}{r_1}.$$

Then

$$J_{\rho,p_0,p_1} = G^0_\eta\big(J^{r_0}_{\theta_0}, J^{r_1}_{\theta_1}\big),$$

where $\eta(t) = t^{-r\theta_0/q}\rho(t^{r/q})$.

We invoke the following auxiliary result which could be of interest in their own right.

Proposition 2.2. *Let* $\overline{X}$ *and* $\overline{Y}$ *be Banach couples, and let* $X = J_{\rho,p_0,p_1}(\overline{X})$ *and* $Y = J_{\rho,p_0,p_1}(\overline{Y})$.

(i) *Assume that* $x_1, \dots, x_k \in X$ *with the canonical representations*

$$x_i = \int_0^\infty u_i(t)\,dt/t, \quad 1 \le i \le k.$$

If we set $v_i^j(t) = t^j u_i(t)/\rho(t)$ *and*

$$\|v_i^j\|_j = \big\| \|v_i^j(t)\|_j \big\|_{L^{p_j}(dt/t)}, \quad 1 \le i \le k, \quad j = 0, 1,$$

then

$$\|x_1\|_X \cdots \|x_k\|_X \le \bar\rho\left(\big(\|v_1^0\|_0 \cdots \|v_k^0\|_0\big)^{1/k}, \big(\|v_1^1\|_1 \cdots \|v_k^1\|_1\big)^{1/k}\right)^k.$$

(ii) *If* T *is a bounded linear operator from* $\overline{X}$ *to* $\overline{Y}$, *then*

$$\|T\|_{X,Y} \le \bar\rho\,\big(\|T\|_0, \|T\|_1\big),$$

which means that J_{ρ,p_0,p_1} *is an exact interpolation method of type* $\bar\rho$.

Proof. Let

$$M_j = \big(\|v_1^j\|_j \cdots \|v_k^j\|_j\big)^{1/k}, \quad j = 0, 1,$$

and let $a = M_1/M_0$. Then $x_i = \int_0^\infty u_i(at)\,dt/t$ for which

$$\begin{aligned}\left(\int_0^\infty \left(\frac{t^j\|u_i(at)\|_j}{\rho(t)}\right)^{p_j}\frac{dt}{t}\right)^{1/p_j} &= \frac{1}{a^j}\left(\int_0^\infty \left(\frac{t^j\|u_i(t)\|_j}{\rho(t/a)}\right)^{p_j}\frac{dt}{t}\right)^{1/p_j}\\ &\le \frac{\bar\rho(a)}{a^j}\,\|v_i^j\|_j.\end{aligned}$$

It turns out

$$\|x_1\|_X \cdots \|x_k\|_X \leq \bar{\rho}\left(\left(\|v_1^0\|_0 \cdots \|v_k^0\|_0\right)^{1/k}, \left(\|v_1^1\|_1 \cdots \|v_k^1\|_1\right)^{1/k}\right)^k,$$

which gives part (i). Part (ii) is an easy consequence of this inequality when $k = 1$. □

Combing Proposition 2.1 (iii) with [**3**, Theorem 1.1] and [**13**, Theorem 3.6], we obtain

Proposition 2.3. *If T is a bounded linear operator from $\overline{X}$ to $\overline{Y}$, and if*

$$T: X_0 \to Y_0$$

is compact, then $T: J_{\rho,p_0,p_1}(\overline{X}) \to J_{\rho,p_0,p_1}(\overline{Y})$ is compact.

If Φ is a Banach function space over some complete σ-finite measure space (Ω, μ), and if X is a Banach space, we denote by $\Phi[X]$ the space of all X-valued strongly measurable functions f such that $\|f\|_X \in \Phi$ almost everywhere and define the norm $\|f\|_{\Phi[X]} = \big\| \|f\|_X \big\|_\Phi$. For a Young function $\varphi: \mathbf{R}^+ \to \mathbf{R}^+$, let $L^\varphi = L^\varphi(\Omega, \mu)$, and in particular, let $L^p = L^p(\Omega, \mu)$ for $1 \leq p \leq \infty$.

Example. If the function φ is given by (2.1), then we have

$$J_{\rho,p_0,p_1}\left(L^{p_0}, L^{p_1}\right) = L^\varphi \tag{2.2}$$

by Proposition 2.1 (iii) and [**14**, Example 5.3]. Furthermore, we have the following results.

(i) If X is a Banach sequence space possessing the Fatou property, then

$$J_{\rho,p_0,p_1}\left(X[L^{p_0}], X[L^{p_1}]\right) = X\left[L^\varphi\right] \tag{2.3}$$

and

$$J_{\rho,p_0,p_1}\left(L^{p_0}[X], L^{p_1}[X]\right) = L^\varphi[X]. \tag{2.4}$$

(ii) Let $\bar{\varphi}\colon \mathbf{R}^+ \to \mathbf{R}^+$ defined by $\bar{\varphi}^{-1}(t) = t^{1/p_0}\bar{\rho}(t^{-1/q})$, and let $\overline{X}$ be a Banach couple. Then $L^{\bar{\varphi}}\big[J_{\rho,p_0,p_1}(\overline{X})\big]^0$ is an exact interpolation space for the couple $\big(L^{p_0}[X_0], L^{p_1}[X_1]\big)$, and the inclusion

$$J_{\rho,p_0,p_1}\big(L^{p_0}[X_0], L^{p_1}[X_1]\big) \subseteq L^{\bar{\varphi}}\big[J_{\rho,p_0,p_1}(\overline{X})\big]^0 \tag{2.5}$$

holds.

In fact, these results can be obtained by (2.2), together with [**5**, Proposition 3.2] for (2.3), the simple calculation for (2.4), and Proposition 2.2 (i) for (2.5).

We conclude this section by a result on the local commutativity for Lions-Peetre's methods.

Proposition 2.4. *If* $1 < p < \infty$, *then*

$$J_{\rho,p_0,p_1}\big(l^p[X_0], l^p[X_1]\big) = l^p\big[J_{\rho,p_0,p_1}(\overline{X})\big]$$

Proof. Observe first that

$$l^p\big[G^0_\rho(\overline{X})\big] \subseteq G^0_\rho\big(l^p[X_0], l^p[X_1]\big) \tag{2.6}$$

by a simple calculation. If we choose θ_j, r_j, $j = 0, 1$, and η as in Proposition 2.1 (iii), then the inclusion

$$\begin{aligned} l^p\big[J_{\rho,p_0,p_1}(\overline{X})\big] &\subseteq G^0_\eta\big(l^p[J^{r_0}_{\theta_0}(\overline{X})], l^p[J^{r_1}_{\theta_1}(\overline{X})]\big) \\ &= G^0_\eta\big(J^{r_0}_{\theta_0}(l^p[X_0], l^p[X_1]), J^{r_1}_{\theta_1}(l^p[X_0], l^p[X_1])\big) \\ &= J_{\rho,p_0,p_1}\big(l^p[X_0], l^p[X_1]\big) \end{aligned}$$

holds by (2.6) and reiteration. Similarly, we have

$$l^{p'}\big[J_{\rho^*,p_0',p_1'}(\overline{X}')\big] \subseteq J_{\rho^*,p_0',p_1'}\big(l^{p'}[X_0'], l^{p'}[X_1']\big). \tag{2.7}$$

Let X be a Banach space, and let $x = (x_\nu)_\nu \in l^p[X]$. Observe that

$$\|x\|_{l^p[X]} = \sup\Big\{|\langle x', x\rangle| \,\Big|\, x' \in l^{p'}[X'] \text{ with } \|x'\|_{l^{p'}[X']} = 1\Big\}.$$

This, together with (2.7) and Proposition 2.1, gives the converse inclusion

$$J_{\rho,p_0,p_1}\big(l^p[X_0], l^p[X_1]\big) \subseteq l^p\left[J_{\rho,p_0,p_1}(\overline{X})\right],$$

which completes the proof. □

3. Besov-Orlicz and Triebel-Lizorkin-Orlicz spaces, and wavelet base. Let $\mathcal{T}(\mathbf{R}^n)$ be the Schwartz class of test functions on $\mathbf{R}^n$, and let $\mathcal{T}'(\mathbf{R}^n)$ be the space of tempered distributions, which is the dual space of $\mathcal{T}(\mathbf{R}^n)$. According to [**1**, Lemma 6.1.7], we can choose $\phi \in \mathcal{T}(\mathbf{R}^n)$ for which

$$\text{(3.1)} \quad \operatorname{supp}\phi = \left\{\, \xi \,\middle|\, 2^{-1} \le |\xi| \le 2 \,\right\}, \quad \phi(\xi) > 0 \quad \text{for} \quad 2^{-1} < |\xi| < 2$$

and

$$\text{(3.2)} \qquad \sum_{\nu=-\infty}^{\infty} \phi(2^{-\nu}\xi) = 1 \quad \text{for} \quad \xi \neq 0.$$

Now we define functions ϕ_ν in $\mathcal{T}(\mathbf{R}^n)$ by

$$\mathcal{F}\phi_\nu(\xi) = \phi(2^{-\nu}\xi), \quad \nu = 0, \pm 1, \pm 2, \dots .$$

Here $\mathcal{F}$ denotes the Fourier transform.

For $s \in \mathbf{R}$ and $1 \le r \le \infty$, the space $\dot{l}^r_s$ consists of all sequences $\lambda = (\lambda_\nu)_{\nu\in\mathbf{Z}}$ for which

$$\big\|\lambda\big\|_{\dot{l}^r_s} = \left(\sum_{\nu=-\infty}^{\infty} (2^{\nu s}|\lambda_\nu|\,)^r\right)^{1/r} < \infty.$$

Let φ be a Young function satisfying both Δ_2 and ∇_2 conditions, and let $L^\varphi = L^\varphi(\mathbf{R}^n)$ in this section. We define the homogeneous, respectively inhomogeneous, Orlicz-Besove space and Orlicz-Triebel-Lipzorkin space $\dot{B}^s_{\varphi,r}$ and $\dot{F}^s_{\varphi,r}$, respectively $F^s_{\varphi,r}$ and $B^s_{\varphi,r}$, in the following way:

$$\begin{aligned}
\dot{B}^s_{\varphi,r} &= \dot{B}^s_{\varphi,r}(\mathbf{R}^n) = \big\{f \in \mathcal{T}'(\mathbf{R}^n) \,|\, (\phi_\nu * f)_\nu \in \dot{l}^r_s[L^\varphi]\big\},\\
\dot{F}^s_{\varphi,r} &= \dot{F}^s_{\varphi,r}(\mathbf{R}^n) = \big\{f \in \mathcal{T}'(\mathbf{R}^n) \,|\, (\phi_\nu * f)_\nu \in L^\varphi[\dot{l}^r_s]\big\},\\
B^s_{\varphi,r} &= B^s_{\varphi,r}(\mathbf{R}^n) = \dot{B}^s_{\varphi,r}(\mathbf{R}^n) \cap L^\varphi,\\
F^s_{\varphi,r} &= F^s_{\varphi,r}(\mathbf{R}^n) = \dot{F}^s_{\varphi,r}(\mathbf{R}^n) \cap L^\varphi
\end{aligned}$$

with the norms

$$\begin{aligned} \|f\|_{\dot{B}^s_{\varphi,r}} &= \|(\phi_\nu * f)_\nu\|_{\dot{l}^r_s(L^\varphi)}, \\ \|f\|_{\dot{F}^s_{\varphi,r}} &= \|(\phi_\nu * f)_\nu\|_{L^\varphi(\dot{l}^r_s)}, \\ \|f\|_{B^s_{\varphi,r}} &= \|f\|_{\dot{B}^s_{\varphi,r}} + \|f\|_{L^\varphi}, \\ \|f\|_{F^s_{\varphi,r}} &= \|f\|_{\dot{F}^s_{\varphi,r}} + \|f\|_{L^\varphi}. \end{aligned}$$

We make calculus modulo polynomials when dealing with homogeneous spaces. The definition of those spaces does not depend on the choice of the test function ϕ. Some of those spaces are studied in [**1, 7, 8**]. If $p(t) = t^p$ $(1 < p < \infty)$, we write $\dot{B}^s_{p,r} = \dot{B}^s_{\varphi,r}$, $B^s_{p,r} = B^s_{\varphi,r}$, $\dot{F}^s_{p,r} = \dot{F}^s_{\varphi,r}$ and $F^s_{p,r} = F^s_{\varphi,r}$. In fact, those spaces can even be defined for $0 < p \leq 1$ or $p = \infty$. Now we can extend [**1**, Theorem 6.4.5] as below.

Proposition 3.1. *Let $\rho: \mathbf{R}^+ \to \mathbf{R}^+$ be a quasi-power function, let $s \in \mathbf{R}$, let $1 \leq p_0, p_1, r \leq \infty$, and let φ be the function given by* (2.1). *Then we have*

$$\begin{aligned} J_{\rho,p_0,p_1}\big(\dot{B}^s_{p_0,r}, \dot{B}^s_{p_1,r}\big) &= \dot{B}^s_{\varphi,r}, \\ J_{\rho,p_0,p_1}\big(\dot{F}^s_{p_0,r}, \dot{F}^s_{p_1,r}\big) &= \dot{F}^s_{\varphi,r}, \\ J_{\rho,p_0,p_1}\big(B^s_{p_0,r}, B^s_{p_1,r}\big) &= B^s_{\varphi,r}, \\ J_{\rho,p_0,p_1}\big(F^s_{p_0,r}, F^s_{p_1,r}\big) &= F^s_{\varphi,r}. \end{aligned}$$

Proof. Following (2.2)–(2.4), we have

$$\begin{aligned} J_{\rho,p_0,p_1}\big(\dot{l}^r_s[L^{p_0}], \dot{l}^r_s[L^{p_1}]\big) &= \dot{l}^r_s[L^\varphi], \\ J_{\rho,p_0,p_1}\big(L^{p_0}[\dot{l}^r_s], L^{p_1}[\dot{l}^r_s]\big) &= L^\varphi[\dot{l}^r_s]. \end{aligned}$$

Observe that $\dot{B}^s_{\varphi,r}$ is a retract of $\dot{l}^r_s\big[L^\varphi\big]$, and $\dot{F}^s_{\varphi,r}$ is a retract of $L^\varphi\big[\dot{l}^r_s\big]$ by [**7**, Theorem 5.5]. This, together with Proposition 1.1 (i) gives that

$$\begin{aligned} J_{\rho,p_0,p_1}\big(\dot{B}^s_{p_0,r}, \dot{B}^s_{p_1,r}\big) &= \dot{B}^s_{\varphi,r}, \\ J_{\rho,p_0,p_1}\big(\dot{F}^s_{p_0,r}, \dot{F}^s_{p_1,r}\big) &= \dot{F}^s_{\varphi,r}. \end{aligned}$$

It is easy to adapt the proof so that the interpolation result is also valid for the homogeneous spaces. □

Now we define the Orlicz sequence space l^φ to consist of all sequences, and the weighted Orlicz sequence space $\dot{l}_*^\varphi$ as follows:

$$l^\varphi = \left\{ \lambda = (\lambda_n)_{n\geq 1} \,\middle|\, \sum_{n=1}^{\infty} \varphi(|\lambda_n|) < \infty \right\},$$
$$\dot{l}_*^\varphi = \left\{ \lambda = (\lambda_n)_{n\in\mathbf{Z}} \,\middle|\, \sum_{\nu=-\infty}^{\infty} 2^{-\nu} \varphi(|\lambda_\nu|) < \infty \right\}.$$

Both spaces are equipped with the Luxemberg norms. For the test function $\phi \in \mathcal{T}(\mathbf{R^n})$ satisfying (3.1) and (3.2), we define the spaces $\dot{B}_*^\varphi$ and B_*^φ by

$$\dot{B}_*^\varphi = \dot{B}_*^\varphi(\mathbf{R}^n) = \left\{ f \in \mathcal{T}'(\mathbf{R^n}) \mid (\phi_\nu * f)_\nu \in \dot{l}_*^\varphi \right\}$$

and $B_*^\varphi = \dot{B}_*^\varphi \cap L^\varphi$ with the norms

$$\|f\|_{\dot{B}_*^\varphi} = \|(\phi_\nu * f)_\nu\|_{\dot{l}_*^\varphi} \quad \text{and} \quad \|f\|_{B_*^\varphi} = \|(\phi_\nu * f)_\nu\|_{\dot{l}_*^\varphi} + \|f\|_{L^\varphi}.$$

Proposition 3.2. *Let* $\rho: \mathbf{R}^+ \to \mathbf{R}^+$ *be a quasi-power function, let* $1 \leq p_0, p_1 \leq \infty$ *and let* φ *be the function given by* (2.1). *Then we have*

$$J_{\rho,p_0,p_1}\big(\dot{B}^{1/p_0}_{p_0,p_0}, \dot{B}^{1/p_1}_{p_1,p_1}\big) = \dot{B}_*^\varphi,$$
$$J_{\rho,p_0,p_1}\big(B^{1/p_0}_{p_0,p_0}, B^{1/p_1}_{p_1,p_1}\big) = B_*^\varphi.$$

Proof. As in Proposition 3.1, it is enough to show the first identity. If we choose θ_j, r_j $(j = 0, 1)$ and η as in Proposition 2.1 (iii), then

$$\begin{aligned} &J_{\rho,p_0,p_1}\big(\dot{l}^{p_0}_{1/p_0}[L^{p_0}], \dot{l}^{p_1}_{1/p_1}[L^{p_1}]\big) \\ &\qquad = G^0_\eta\Big(J^{r_0}_{\theta_0}\big(\dot{l}^{p_0}_{1/p_0}[L^{p_0}], \dot{l}^{p_1}_{1/p_1}[L^{p_1}]\big), J^{r_1}_{\theta_1}\big(\dot{l}^{p_0}_{1/p_0}[L^{p_0}], \dot{l}^{p_1}_{1/p_1}[L^{p_1}]\big)\Big) \\ &\qquad = G^0_\eta\big(\dot{l}^{r_0}_{1/r_0}[L^{r_0}], \dot{l}^{r_1}_{1/r_1}[L^{r_1}]\big). \end{aligned}$$

Let $\omega_j(t) = 1/t^{1/r_j}$, $j = 0, 1$, and $1/r = 1/r_0 - 1/r_1$. Then

$$\left(\frac{\omega_0(t)}{\omega_1(t)}\right)^r = \frac{1}{t} \quad \text{and} \quad \left(\frac{\omega_1(t)^{1/r_0}}{\omega_0(t)^{1/r_1}}\right)^r = 1$$

and hence

$$G_\eta^0\big(\dot{l}^{r_0}_{1/r_0}[L^{r_0}], \dot{l}^{r_1}_{1/r_1}[L^{r_1}]\big) = \dot{l}^{\varphi}_*[L^\varphi]$$

by [**14**, Example 5.3]. Observe that $\dot{B}^{1/p_j}_{p_j,p_j}$ is a retract of $\dot{l}^{p_j}_{1/p_j}[L^{p_j}]$, $j = 0, 1$, and $\dot{B}^\varphi_*$ is a retract of $\dot{l}^\varphi_*[L^\varphi]$. Thus, the identity

$$J_{\rho,p_0,p_1}\big(\dot{B}^{1/p_0}_{p_0,p_0}, \dot{B}^{1/p_1}_{p_1,p_1}\big) = \dot{B}^\varphi_*$$

holds by Proposition 1.1 (i). □

For a Young function φ, let

$$p_\varphi = \inf_{t>0} \frac{t\varphi'(t)}{\varphi(t)} \quad \text{and} \quad q_\varphi = \sup_{t>0} \frac{t\varphi'(t)}{\varphi(t)}.$$

Recall that φ satisfies both Δ_2- and ∇_2-conditions if and only if $1 < p_\varphi \le q_\varphi < \infty$. We can choose p_0, p_1 such that $1 < p_0 < p_\varphi \le q_\varphi < p_1 < \infty$ and define $\rho: \mathbf{R}^+ \to \mathbf{R}^+$ by

$$\rho(t) = t^{q/p_0}\, \varphi^{-1}(t^{-q}). \tag{3.3}$$

Observe that

$$1 < p_0 < p_\varphi \le \frac{t\varphi'(t)}{\varphi(t)} \le q_\varphi < p_1 < \infty,$$

and

$$\frac{t\rho'(t)}{\rho(t)} = q\left(\frac{1}{p_0} - \frac{t^{-q}(\varphi^{-1})'(t^{-q})}{\varphi^{-1}(t^{-q})}\right) = q\left(\frac{1}{p_0} - \frac{\varphi(s)}{s\varphi'(s)}\right),$$

where $s = \varphi^{-1}(t^{-q})$. This implies that

$$0 < q\left(\frac{1}{p_0} - \frac{1}{p_\varphi}\right) \le \frac{t\rho'(t)}{\rho(t)} \le q\left(\frac{1}{p_0} - \frac{1}{q_\varphi}\right) < 1.$$

Thus, ρ is quasi-power and φ satisfies (2.1).

According to [**8**, Theorem (7.20)], there is a wavelet characterization for the spaces $\dot{F}^s_{p,r} = \dot{F}^s_{p,r}(\mathbf{R})$ and $\dot{B}^s_{p,r} = \dot{B}^s_{p,r}(\mathbf{R})$. Let $w \in \mathcal{T}(\mathbf{R})$ be a wavelet function satisfying

$$\operatorname{supp} \hat{w} \subseteq [-8\pi/3, -2\pi/3] \cup [8\pi/3, 2\pi/3],$$

and hence $\int_{-\infty}^{\infty} x^k w(t)dt = 0$, $k = 0, 1, 2, \dots$, cf. [**8**, Theorem (7.11)], let

$$w_{\nu,k}(t) = 2^{\nu/2}\, w(2^\nu t - k), \quad \nu, k \in \mathbf{N},$$

be the reduced orthonormal wavelet basis of $L^2(\mathbf{R})$ in terms of dilation and translation, and let $\chi_{\nu,k}$ be the characteristic function of $[2^{-\nu}k, 2^{-\nu}(k+1)]$. For $1 \le p, r < \infty$, there are positive constants A and B such that

$$\begin{aligned} A \|f\|_{\dot{F}^s_{p,r}} &\le \left\| \left(\sum_{\nu,k=-\infty}^{\infty} \left(|\langle f, w_{\nu,k}\rangle| 2^{\nu(s+1/2)}\, \chi_{\nu,k}\right)^r \right)^{1/r} \right\|_{L^p} \\ &\le B \|f\|_{\dot{F}^s_{p,r}} \end{aligned} \tag{3.4}$$

for all $f \in \dot{F}^s_{p,r}$, and

$$\begin{aligned} A \|f\|_{\dot{B}^s_{p,r}} &\le \left(\sum_{\nu=-\infty}^{\infty} 2^{r\nu(s+1/2-1/p)} \left(\sum_{k=-\infty}^{\infty} |\langle f, w_{\nu,k}\rangle|^p \right)^{r/p} \right)^{1/r} \\ &\le B \|f\|_{\dot{B}^s_{p,r}} \end{aligned} \tag{3.5}$$

for all $f \in \dot{B}^s_{p,r}$. We extend now (3.4) to spaces $\dot{F}^s_{\varphi,r}$.

Proposition 3.3. *Let $s \in \mathbf{R}$, $1 \le r \le \infty$, and let φ be a Young function satisfying both Δ_2 and ∇_2 conditions. Then there are positive constants A and B such that*

$$A \|f\|_{\dot{F}^s_{\varphi,r}} \le \left\| \left(\sum_{\nu,k=-\infty}^{\infty} \left(|\langle f, w_{\nu,k}\rangle| 2^{\nu(s+1/2)}\, \chi_{\nu,k}\right)^r \right)^{1/r} \right\|_{L^\varphi} \le B \|f\|_{\dot{F}^s_{\varphi,r}},$$

for all $f \in \dot{F}^s_{\varphi,r}$.

Proof. Assume that $1 < p_0 < p_\varphi \le q_\varphi < p_1 < \infty$ and ρ as in (3.3). Let $\dot{f}^s_{\varphi,r}$ be the space of all sequences $\lambda = (\lambda_{\nu,k})_{\nu,k\in\mathbf{Z}}$, for which

$$\|\lambda\|_{\dot{f}^s_{\varphi,r}} = \bigg\|\bigg(\sum_{\nu,k=-\infty}^{\infty}\big(|\lambda_{\nu,k}|2^{\nu(s+1/2)}\,\chi_{\nu,k}\big)^r\bigg)^{1/r}\bigg\|_{L^\varphi};$$

and let U_w and V_w be the operators defined by

$$U_w f = \big(\langle f, w_{\nu,k}\rangle\big)_{\nu,k} \tag{3.6}$$

for $f \in \dot{F}^s_{p_j,r}$, and

$$V_w\lambda = \sum_{\nu,k=-\infty}^{\infty}\lambda_{\nu,k}w_{\nu,k} \tag{3.7}$$

for $\lambda = (\lambda_{\nu,k})_{\nu,k} \in \dot{f}^s_{p_j,r}$. Observe that

$$J_{\rho,p_0,p_1}\big(\dot{f}^s_{p_0,r}, \dot{f}^s_{p_1,r}\big) = \dot{f}^s_{\varphi,r},$$

and U_w is a bounded isomorphism from $\dot{F}^s_{p_j,r}$ to $\dot{f}^s_{p_j,r}$, with the bounded inverse V_w. The results can be obtained by Proposition 3.1 (i) and (3.4). □

Remark. If we denote by $\dot{b}^\varphi_*$ the space of all sequences $\lambda = (\lambda_{\nu,k})_{\nu,k\in\mathbf{Z}}$, for which $\sum_{\nu,k=-\infty}^{\infty}\varphi(2^{\nu/2}|\lambda_{\nu,k}|) < \infty$, equipped with the Luxemberg norm, then

$$J_{\rho,p_0,p_1}\big(\dot{b}^{p_0}_*, \dot{b}^{p_1}_*\big) = \dot{b}^\varphi_*. \tag{3.8}$$

By choosing $r = p = p_j$, $s = 1/p_j$, $j = 0, 1$, in (3.5), and by using Proposition 3.2, we can obtain that

$$A\,\|f\|_{\dot{b}^\varphi_*} \le \big\|\big(\langle f, w_{\nu,k}\rangle\big)\big\|_{\dot{B}^\varphi_*} \le B\,\|f\|_{\dot{b}^\varphi_*}$$

for some constants A, B, and for all $f \in \dot{B}^\varphi_*$.

4. Commutators, Hankel operators and Schatten-Orlicz classes. In this section, we begin with commutator estimates of quasi-logarithmic operators on spaces B^φ_*. Let $\overline{X}$ be a Banach couple, and let

$c > 1$ be a constant. For $x \in \Sigma\overline{X}$, the decomposition $x = x_0(t) + x_1(t)$, $t > 0$, is (c-)*almost optimal* for the K-methods if

$$K(t,x) \le \|x_0(t)\|_0 + t\|x_1(t)\|_1 \le c\,K(t,x).$$

A (c-)*almost optimal projection* for the K-methods is a, usually non-linear, operator $D_K(t)\colon \Sigma\overline{X} \to X_0$ defined by

$$D_K(t)x = D_K(t,\overline{X})x = x_0(t)$$

for some almost optimal decomposition. We can define the corresponding quasi-logarithmic operator $\Omega_{\overline{X}}$ by

$$\Omega_{\overline{X}} = \int_0^\infty \big(I \cdot \chi_{(1,\infty)}(t) - D_K(t)\big)\, x\, \frac{dt}{t}$$

for $x \in \Sigma\overline{X}$. We refer to [**11**] for further details. Let $\overline{X}$ and $\overline{Y}$ be Banach couples. If $\overline{Y}$ is a retract of $\overline{X}$ with the morphisms $\mathcal{J}\colon \overline{Y} \to \overline{X}$ and $\mathcal{P}\colon \overline{X} \to \overline{Y}$, then it is clear that

$$\Omega_{\overline{Y}} = \mathcal{J}\Omega_{\overline{X}}\mathcal{P}. \tag{4.1}$$

Let $\overline{X} = (\dot{b}_*^{p_0}, \dot{b}_*^{p_1})$ and $\overline{Y} = (B^{1/p_0}_{p_0,p_0}, B^{1/p_1}_{p_1,p_1})$. Then $\overline{Y}$ is a retract of $\overline{X}$ with the morphisms $U_w\colon \overline{Y} \to \overline{X}$ and $V_w\colon \overline{X} \to \overline{Y}$, where U_w and V_w are given in (3.6) and (3.7). Observe that $\dot{b}_*^{p_j}$ ($j = 0,1$) is a weighted l^{p_j} space consisting of all sequences $\lambda = (\lambda_{\nu,k})_{\nu,k\in\mathbf{Z}}$ for which

$$\big\|\lambda\big\|_{\dot{b}_*^{p_j}} = \bigg(\sum_{\nu,k=-\infty}^{\infty} \big(2^{\nu/2}\,|\lambda_{\nu,k}|\big)^{p_j}\bigg)^{1/p_j} < \infty.$$

By [**11**, Section 4.3], we have $(\Omega_{\overline{X}}\lambda)_{\nu,k} = \lambda_{\nu,k}\log|r_\lambda(\nu,k)|^{1/q}$ for $\lambda \in \Sigma\overline{X}$, where $r_\lambda(\nu,k) = |\{(\nu',k') \,|\, |\lambda_{\nu,k}| > |\lambda_{\nu',k'}|\}|$. Consequently,

$$\Omega f = \Omega_{\overline{Y}} f = \sum_{\nu,k=-\infty}^{\infty} \nu\langle f, w_{\nu,k}\rangle \big(\log|r_\lambda(\nu,k)|^{1/q}\big) w_{\nu,k}$$

for all $f \in \Sigma\overline{Y}$ by (4.1). Combining Proposition 3.2, (3.8) and [**6**, Theorem 4.3], we obtain

Proposition 4.1. *Assume* $1 < p_0,\, p_1 < \infty$. *For* $f \in B^{1/p_0}_{p_0,p_0} + B^{1/p_1}_{p_1,p_1}$, *let* $\lambda = U_w(f)$, *and let*

$$\Omega f = \sum_{\nu,k=-\infty}^{\infty} \nu \langle f, w_{\nu,k}\rangle \big(\log |r_\lambda(\nu,k)|^{1/q}\big) w_{\nu,k}.$$

If T *is a bounded linear operator on* $B^{1/p_j}_{p_j,p_j}$, $j = 0,1$, *then* $\Omega T - T\Omega$ *is a bounded nonlinear operator on* B^{φ}_{*}.

Next we deal with commutators of singular integral operators in Schatten-Orlicz classes. Consider a Hilbert space H and a Young function $\varphi: \mathbf{R}^+ \to \mathbf{R}^+$. For a compact operator T on H, let $(s_n(T))_{n\geq 1}$ be the sequence of eigenvalues of $|T|$, counted according to multiplicity. We say that $T \in S^{\varphi}$, the Schatten-Orlicz φ-class if $(s_n(T))_n \in l^{\varphi}$. The norm on S^{φ} is given by $\|T\|_{S^{\varphi}} = \|(s_n(T))_n\|_{l^{\varphi}} < \infty$. In particular, we denote S^p $(1 \leq p < \infty)$ the Schatten p-class, $S^{\infty} = \mathcal{B}(H)$ and S^0 the space of all finite rank operators on H with norm $\|T\|_{S^0} = \operatorname{rank}(T)$.

Let T be a Calderón-Zygmund transform, a singular integral operator with kernel $K(t-s)$, where K is a nonzero C^{∞} function on $\mathbf{R}^n$ except at the origin, and is homogeneous of degree $-n$ with mean value zero on spheres centered at the origin. For $f \in L^2(\mathbf{R}^n)$, let M_f be the pointwise multiplication by f, and let $C_f = M_f T - T M_f$ be the commutator for the operator T on $L^2(\mathbf{R}^n)$. In fact,

$$C_f x(t) = \int_{-\infty}^{\infty} K(t-s)\big(f(t) - f(s)\big) x(s)\, ds.$$

According to [**17**] and [**10**], if $f \in B^*_p(\mathbf{R})$, or $f \in B^*_p(\mathbf{R}^n)$ for $n \geq 2$ and $p_\varphi > n$, then $C_f \in S^p$. In terms of Proposition 3.2, we can obtain the following result:

Proposition 4.2. *Let* $\varphi: \mathbf{R}^+ \to \mathbf{R}^+$ *be a Young function satisfying both* Δ_2 *and* ∇_2 *conditions.*

(i) *If* $f \in B^*_{\varphi}(\mathbf{R})$, *then* $C_f \in S^{\varphi}$.

(ii) *If* $f \in B^*_{\varphi}(\mathbf{R}^n)$ *for* $n \geq 2$ *and* $p_\varphi > n$, *then* $C_f \in S^{\varphi}$.

Let us now turn our attention to Hankel operators in Schatten-Orlicz classes. Consider the operator $U\colon K_{\rho,p_0,p_1}(S^{p_0},S^{p_1}) \to l^{\varphi}$, defined by

$$U(T) = (s_n(T))_n.$$

Thus we have $K_{\rho,p_0,p_1}(S^{p_0},S^{p_1}) \subseteq S^{\varphi}$ by (2.2) and interpolation. By using the duality argument, we obtain

$$K_{\rho,p_0,p_1}(S^{p_0},S^{p_1}) = S^{\varphi}. \tag{4.2}$$

If $H = l^2$, then each bounded operator on l^2 can be presented by a matrix $(a_{i,j})_{i,j=1}^{\infty}$. Let Γ^{φ} denote the subspace of S^{φ} consisting of Hankel matrices, i.e., matrices of the form $(a_{i+j})_{i,j=1}^{\infty}$.

Proposition 4.3. *Let $\varphi_j\colon \mathbf{R}^+ \to \mathbf{R}^+$ $(j = 0,1)$ be Young functions satisfying both Δ_2 and ∇_2 conditions. Then $(\Gamma_{\varphi_0},\Gamma_{\varphi_1})$ is a K-subcouple of $(S_{\varphi_0},S_{\varphi_1})$.*

Proof. Let us choose p_0, p_1 satisfying

$$1 < p_0 < p_{\varphi_0} \wedge p_{\varphi_1} \le q_{\varphi_0} \vee q_{\varphi_1} < p_1 < \infty,$$

and define $\rho_j\colon \mathbf{R}^+ \to \mathbf{R}^+$ as $\rho_j(t) = t^{-q/p_0}\varphi_j^{-1}(t^{-q})$, $j = 0,1$. Let Φ_j be the weighted Orlicz space corresponding to the indices p_0, p_1 and the function φ_j. Observe that

$$S^{\varphi_j} = K_{\rho_j,p_0,p_1}(S^{p_0},S^{p_1}) = K_{\Phi_j}(S^{p_0},S^{p_1})$$

by (4.2), and $(\Gamma^{p_0},\Gamma^{p_1})$ is a K-couple of (S^{p_0},S^{p_1}) by [**9**, Theorem 8.2]. Thus,

$$\Gamma^{\varphi_j} = K_{\Phi_j}(\Gamma^{p_0},\Gamma^{p_1})$$

by [**9**, Theorem 2.1] and hence $(\Gamma^{\varphi_0},\Gamma^{\varphi_1})$ is a K-couple of $(S^{\varphi_0},S^{\varphi_1})$ by [**9**, Theorem 2.2 (i)]. $\square$

Remark. As a consequence of Proposition 4.3, each bounded operator on l^2 has a simultaneous good Hankel approximation with respect to all Schatten-Orlicz classes S^{φ}. That is, if $T \in \mathcal{B}(l^2)$, then there is a Hankel operator K on l^2 such that

$$\|T-K\|_{S^{\varphi}} \le C_{\varphi} \inf\left\{\, \|T-R\|_{S^{\varphi}} \,\middle|\, R \in S^{\varphi} \,\right\},$$

where the constant C_{φ} depends only on φ.

5. Martingale inequalities for the strong φ-variation. Let $\varphi: \mathbf{R}^+ \to \mathbf{R}^+$ be a Young function. The strong φ-variation of a sequence $\lambda = (\lambda_n)_{n\geq 0}$, denoted by $W_\varphi(x)$, is defined as follows

$$W_\varphi(\lambda) = \sup \left\{ \|(\lambda_{n_k} - \lambda_{n_{k-1}})_k\|_{l^\varphi} \mid 0 \leq n_0 \leq n_1 \leq \cdots \right\}.$$

The corresponding Banach space V_φ is defined by

$$V_\varphi = \left\{ \lambda \in \mathbf{R}^N \;\middle|\; \|\lambda\|_{V_\varphi} = W_\varphi(\lambda) < \infty \right\}.$$

We refer to [**12**, Section 2.3] for more information about this space.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with the filtration $\{\mathcal{F}_n\}_{n\geq 0}$ for which $\mathcal{F} = \vee_{n\geq 0}\mathcal{F}_n$. The conditional expectation operators relative to $\mathcal{F}_n$ are denoted by $E = E_0$ and E_n for $n \geq 1$. For each random variable $f \in L^1(\Omega, \mathcal{F}, P)$ with $Ef = 0$, we consider the corresponding martingale $f = (f_n)_{n\geq 0}$, where $f_n = E_n f$. Moreover, we define $f^* = \sup_{n\geq 0} |f_n|$, and define the martingale differences of f by

$$d_0 = d_0(f) = 0 \quad \text{and} \quad d_n = d_n(f) = f_n - f_{n-1}, \quad n \geq 1.$$

By using the classical real interpolation method, Pisier and Xu [**16**] proved the following inequalities concerning the strong p-variation of martingales: There exists a constant C_p depending on p such that

$$\|W_p(f)\|_{l^p} \leq C_p \, \|(d_n)_n\|_{l^p}, \quad 1 \leq p < 2, \tag{5.1}$$

and

$$\|W_p(f)\|_{l^p} \leq C_p \, \|f^*\|_{l^p}, \quad 2 < p < \infty, \tag{5.2}$$

for all martingales f. In this section, we extend (5.1)–(5.2) to the strong φ-variation.

In terms of the reiteration in Proposition 2.1 (iii) and the similar arguments as in the proof of [**16**, (1.9) and Theorem 2.1], we have

Lemma 5.1. *Let $\rho: \mathbf{R}^+ \to \mathbf{R}^+$ be quasi-power, and let φ given in* (2.1)*.*

(i) $J_{\rho,p_0,p_1}(V_{p_0}, V_{p_1}) \subseteq V_\varphi$.

(ii) *If* D_φ *is the subset of* $L^\varphi(\Omega \times \mathbf{N})$ *formed of all sequences* $u = (u_n)_{n\geq 0}$ *such that* u_n *is* $\mathcal{F}_n$*-measurable for all* $n \geq 0$ *and* $E_{n-1}(u_n) = 0$ *for all* $n \geq 1$*, then*

$$D_\varphi = J_{\rho,p_0,p_1}(D_{p_0}, D_{p_1}).$$

Proposition 5.1. *Assume that* $\varphi: \mathbf{R}^+ \to \mathbf{R}^+$ *is a super-multiplicative Young function for which* $1 < p_\varphi \leq q_\varphi < 2$ *or* $2 < p_\varphi \leq q_\varphi < \infty$*. Then there is a constant* C_φ *depending only on* φ *such that the inequalities*

$$\|W_\varphi(f)\|_{l^\varphi} \leq C_\varphi \|(d_n)_n\|_{l^\varphi}, \quad 1 < p_\varphi \leq q_\varphi < 2,$$

and

$$\|W_\varphi(f)\|_{l^\varphi} \leq C_\varphi \|f^*\|_{l^\varphi}, \quad 2 < p_\varphi \leq q_\varphi < \infty,$$

hold for all martingales f*.*

Proof. Assume that $1 < p_0 < p_\varphi \leq q_\varphi < p_1 < 2$ and ρ as in (3.3). Then ρ is quasi-power with $\bar{\rho} = \rho$, and φ satisfies (2.1). As in the proof of [**16**, Theorem 2.1], let T be the operator which maps any u in D_1 to the martingale $f = (f_n)_{n\geq 0}$ defined by $f_n = \sum_{k=0}^n u_k$. It is known that T is bounded from D_p to $l^p[V_p]$ for $1 \leq p < 2$. This, combined with Lemma 5.1, (2.2) and (2.5), implies that

$$T: D_\varphi = J_{\rho,p_0,p_1}(D_{p_0}, D_{p_1}) \longrightarrow J_{\rho,p_0,p_1}\left(l^{p_0}[V_{p_0}], l^{p_1}[V_{p_1}]\right),$$

and

$$J_{\rho,p_0,p_1}\left(l^{p_0}[V_{p_0}], l^{p_1}[V_{p_1}]\right) \subseteq l^\varphi\left[J_{\rho,p_0,p_1}(V_{p_0}, V_{p_1})\right] \subseteq l^\varphi\,[V_\varphi].$$

Thus, $\|T(u)\|_{l^\varphi(V_\varphi)} \leq C_\varphi \|u\|_{D_\varphi}$. This gives the inequality

$$\|W_\varphi(f)\|_{l^\varphi} \leq C_\varphi \|(d_n)_n\|_{l^\varphi}.$$

For $2 < p_0 < p_\varphi \leq q_\varphi < \infty$, the inequality $\|W_\varphi(f)\|_{l^\varphi} \leq C_\varphi \|f^*\|_{l^\varphi}$ can be obtained by a similar argument as above and in the proof of [**16**, Theorem 2.4]. ◻

6. On multi-dimensional uniform rotundity. Like the classical real methods, many important geometric properties of Banach spaces are stable under Lions-Peetre's methods associated with quasi-power functions. For instance, if X_0 or X_1 is a reflexive space, then $J_{\rho,p_0,p_1}(\overline{X})$ is also a reflexive space. This is an immediate consequence of Proposition 1.1 (iii) and Proposition 2.1 (i). Another example is the uniform convexity [**5**, Proposition 4.5]. In this final section, we consider the multi-dimensional uniform rotundity.

Let X be a Banach space, and let $x_i \in X$ for $0 \le i \le k$. The k-dimensional volume enclosed by $x_0, x_1, \dots, x_k$ is defined by

$$\mathcal{A}_X(\{x_i\}_i) = \sup_{x_l^* \in X', \|x_l^*\|_{X'} \le 1} \left\{ \begin{vmatrix} 1 & \cdots & 1 \\ \langle x_1^*, x_0\rangle & \cdots & \langle x_1^*, x_k\rangle \\ \vdots & \ddots & \vdots \\ \langle x_k^*, x_0\rangle & \cdots & \langle x_k^*, x_k\rangle \end{vmatrix} \right\}.$$

The modulus of k-rotundity of X is defined by

$$\delta_X^{(k)}(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x_0 + x_1 + \cdots + x_k}{k+1} \right\|_X \;\middle|\; \|x_i\|_X \le 1,\, \mathcal{A}\big(\{x_i\}\big) \ge \varepsilon \right\}$$

for $0 \le \varepsilon \le (k+1)^{(k+1)/2}$. The space X is k-uniformly rotund (k-UR in short)), or equivalently k-uniformly convex, if $\delta_X^{(k)}(\varepsilon) > 0$ for $\varepsilon > 0$. If X is a k-UR Banach space and if $1 < p < \infty$, then $L^p[X]$ is also k-UR [**18**]. Let us begin with an extension of [**18**, Proposition 8].

Proposition 6.1. *Suppose that $1 < p_0, p_1 < \infty$, and $\rho\colon \mathbf{R}^+ \to \mathbf{R}^+$ is quasi-power. For a Banach couple $\overline{X}$, let*

$$X = J_{\rho,p_0,p_1}(\overline{X}).$$

If $x_i \in X$, $0 \le i \le k$, for which $x_i = \int_0^\infty u_i(t)\,dt/t$, then

$$\mathcal{A}_X(\{x_i\}_i) \le k^{k/2}\,\bar{\rho}\left(\mathcal{A}_{L^{p_0}[X_0]}\left(\left\{\frac{u_i(t)}{\rho(t)}\right\}_i\right)^{1/k}, \right.$$
$$\left. \mathcal{A}_{L^{p_1}[X_1]}\left(\left\{\frac{u_i(t)}{\rho(t)}\right\}_i\right)^{1/k}\right)^k.$$

Proof. For $0 \le i \le k-1$, let d_i be the distance between x_i and the affine $[x_{i+1}, \dots, x_k]$ span of $x_{i+1}, \dots, x_k$. Then

$$d_0 \cdot d_1 \cdots d_{k-1} \le \mathcal{A}_X(\{x_i\}_i) \le k^{k/2} d_0 \cdot d_1 \cdots d_{k-1}.$$

Without loss of generality, we may assume that $k = 2$. For a strongly measurable function $u: \mathbf{R}^+ \to \Delta\overline{X}$, we denote

$$v^j(t) = t^j u(t)/\rho(t) \quad \text{and} \quad \|v^j\|_j = \big\|\|v^j(t)\|_j\big\|_{L^{p_j}(dt/t)}, \quad j = 0, 1.$$

Now we have

$$\begin{aligned}\mathcal{A}_X&(x_0, x_1, x_2)\\ &\le 2 \inf_{z\in[x_1,x_2]} \big(\|x_1 - x_2\|_X \|z - x_0\|_X\big)\\ &\le 2 \inf_{u\in[u_1,u_2]} \bar{\rho}\Big(\sqrt{\|v_2^0 - v_1^0\|_0 \|v^0 - v_0^0\|_0}, \sqrt{\|v_2^1 - v_1^1\|_1 \|v^1 - v_0^1\|_1}\Big)^2\end{aligned}$$

by Proposition 2.2 (i). This implies that

$$\begin{aligned}&\mathcal{A}_X(x_0, x_1, x_2)\\ &\le 2 \inf_{u_0,u_1,u_2} \inf_{u\in[u_1,u_2]} \bar{\rho}\Big(\sqrt{\|v_2^0 - v_1^0\|_0 \|v^0 - v_0^0\|_0}, \sqrt{\|v_2^1 - v_1^1\|_1 \|v^1 - v_0^1\|_1}\Big)^2.\end{aligned}$$

Since

$$\mathcal{A}_{L^{p_j}[X_j]}(v_0^j, v_1^j, v_2^j) \ge \|v_2^j - v_1^j\|_j \inf_{u^j} \|v^j - v_0^j\|_j, \quad j = 0, 1,$$

it follows that

$$\mathcal{A}_X(x_0, x_1, x_2) \le 2\bar{\rho}\big(\mathcal{A}_{L^{p_0}[X_0]}(v_0^0, v_1^0, v_2^0)^{1/2},\, \mathcal{A}_{L^{p_1}[X_1]}(v_0^1, v_1^1, v_2^1)^{1/2}\big)^2,$$

which completes the proof. □

The following result is a generalization of [**18**, Theorem 10] and [**5**, Proposition 4.5].

Proposition 6.2. *If X_0 or X_1 is k-*UR*, then $X = J_{\rho,p_0,p_1}(\overline{X})$ is also k-*UR*. Moreover, if we denote $\delta = \delta_X^{(k)}$, $\delta_{p_j,j} = \delta_{L^{p_j}(X_j)}^{(k)}$ $(j = 0, 1)$, then*

$$\delta(\varepsilon) \ge 1 - \bar{\rho}\big(1 - \delta_{p_0,0}(C\,\varepsilon^{1-\beta} \wedge \varepsilon^{\beta}), 1 - \delta_{p_1,1}(C\,\varepsilon^{1-\beta} \wedge \varepsilon^{\beta})\big)$$

for a positive constant C and for $\varepsilon > 0$ small enough.

Proof. Let $x_i \in X$ with $\|x_i\|_X \le 1$ $(0 \le i \le k)$ and $\mathcal{A}(\{x_i\}_i) \ge \varepsilon > 0$. For $\eta > 0$, by Proposition 6.1, we have

$$\begin{aligned}\frac{\varepsilon}{(1+\eta)^k} &\le \mathcal{A}_X\left(\left\{\frac{x_i}{1+\eta}\right\}_i\right)\\ &\le k^{k/2}\,\bar{\rho}\big(\mathcal{A}_{L^{p_0}(X_0)}(\{v_i^0\}_i)^{1/k},\, \mathcal{A}_{L^{p_1}(X_1)}(\{v_i^1\}_i)^{1/k}\big)^k\\ &\le C\max_{j=0,1}\big\{\mathcal{A}_{L^{p_j}(X_j)}(\{v_i^j\}_i)^{1-\beta}\mathcal{A}_{L^{p_{1-j}}(X_{1-j})}(\{v_i^{1-j}\}_i)^{\beta}\big\}.\end{aligned}$$

This, together with the inequality $\mathcal{A}_{L^{p_j}(X_j)}(\{v_i^j\}_i) \le (k+1)^{(k+1)/2}$, implies that

$$\mathcal{A}_{L^{p_j}(X_j)}\big(\{v_i^j\}_i\big) \ge C\left(\frac{\varepsilon}{(1+\eta)^k}\right)^{1/(1-\beta)} \wedge \left(\frac{\varepsilon}{(1+\eta)^k}\right)^{1/\beta},$$

and hence

$$\left\|\frac{\sum_{i=1}^k v_i^j}{k+1}\right\|_{L^{p_j}(X_j)} \le 1-\delta_{p_j,j}\big(C((1+\eta)^{-k}\varepsilon)^{1/(1-\beta)} \wedge ((1+\eta)^{-k}\varepsilon)^{1/\beta}\big).$$

Consequently,

$$\begin{aligned}\frac{1}{(1+\eta)}\left\|\frac{\sum_{i=1}^k x_i}{k+1}\right\|_X &\le \bar{\rho}\left(\left\|\frac{\sum_{i=1}^k v_i^0}{k+1}\right\|_{L^{p_0}[X_0]}, \left\|\frac{\sum_{i=1}^k v_i^1}{k+1}\right\|_{L^{p_1}[X_1]}\right)\\ &\le \bar{\rho}\left(1-\delta_{p_0,0}\left(C\left(\frac{\varepsilon}{(1+\eta)^k}\right)^{1/(1-\beta)} \wedge \left(\frac{\varepsilon}{(1+\eta)^k}\right)^{1/\beta}\right),\right.\\ &\qquad \left. 1-\delta_{p_1,1}\left(C\left(\frac{\varepsilon}{(1+\eta)^k}\right)^{1/(1-\beta)} \wedge \left(\frac{\varepsilon}{(1+\eta)^k}\right)^{1/\beta}\right)\right).\end{aligned}$$

Therefore, X is also k-UR, and the estimate for δ is obtained by letting $\eta \to 0$. □

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