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EXPLICIT EQUATIONS OF SOME ELLIPTIC MODULAR SURFACES

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ABSTRACT. We present explicit equations of semi-stable elliptic surfaces (i.e., having only type I_n singular fibers) which are associated to the torsion-free genus zero congruence subgroups of the modular group as classified by A. Sebbar.

1. Introduction. The purpose of this paper is to give explicit equations for the elliptic modular surfaces associated to torsion-free genus zero congruence subgroups Γ of $PSL_2(\mathbf{Z})$. We will call them *elliptic modular surfaces of genus zero* for short. By the Noether formula, the Euler number of any elliptic surface is a (nonnegative) multiple of 12. It turns out that an elliptic modular surface of genus zero has Euler number one of $12, 24, 36, 48$ and 60. Indeed, if it has an Euler number bigger than 12, then it is semi-stable and the Euler number equals the index of the corresponding group Γ in $PSL_2(\mathbf{Z})$. Sebbar's classification [**17**] of all such torsion free congruence subgroups Γ of genus zero implies that they have index ≤ 60 in $PSL_2(\mathbf{Z})$.

This paper can be regarded as a natural sequel to the article of Beauville [**2**] which deals with elliptic modular surfaces (of genus zero) having Euler number 12, and the article of Livné and Yui [**12**] which considered the case of elliptic modular surfaces of genus zero with Euler number 24.

We will recall the results of Beauville [**2**] and of Livné and Yui [**12**] (presenting a different approach in the latter case). Furthermore, we give explicit defining equations for the elliptic modular surfaces with Euler number one of $36, 48$ and 60.

Although our approach is different, these results could also be found using McKay and Sebbar's tables in [**13**], which provide one with the

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j-invariant of each of the elliptic modular surfaces of genus zero. We now illustrate how this works for a typical example (with Euler number 36).

Example 1.1. Consider the group $\Gamma = \Gamma_0(2) \cap \Gamma(3)$. According to [**13**, Table 5], the j-invariant $j(t)$ of the corresponding elliptic surface equals

$$\frac{(t^3+4)^3(t^3+6t^2+4)^3(t^6-6t^5+36t^4+8t^3-24t^2+16)^3}{t^6(t+1)^3(t^2-t+1)^3(t-2)^6(t^2+2t+4)^6}.$$

To obtain a semi-stable elliptic surface with this j-invariant, start from any elliptic surface with j-invariant j, such as the one given by

$$y^2 + xy = x^3 - \frac{36}{j-1728}\,x - \frac{1}{j-1728}.$$

The corresponding elliptic surface over the j-line has only three singular fibers, namely an I_1-fiber over $j=\infty$, a II-fiber over $j=0$ and a III^*-fiber over $j=1728$. Now substitute $j=j(t)$ in the equation. One obtains in this way an elliptic surface which has I_0^*-fibers over all t such that $j(t)=0$ (this can be read off from the ramification of the function $j(t)$, together with the fact that the surface given above has a fiber of type II over $j=0$; compare [**3**, p. 46]). Hence, we have to take a quadratic twist over an extension ramified at all these t in order to make these fibers smooth. The equation $j(t)=\infty$ has four triple (one at $t=\infty$) and four six-fold roots; hence, the resulting surface has four fibers of type I_3 and four fibers of type I_6. Finally, the equation $j(t)=1728$ has 18 roots, each with multiplicity 2. This implies that our new surface has fibers of type I_0^* over these 18 values of t. Therefore, after the quadratic twist over an extension ramified only at these 18 t's, one obtains the semi-stable elliptic surface we were looking for.

It should be clear from the above example that this method works in general; however, it is rather elaborate and the resulting equations (even for a rather small example such as $j(t)$ above) are quite cumbersome. Therefore, we offer an alternative approach.

Our method consists of describing a different base change resulting in the desired surface. Instead of starting from a surface with j-invariant j, we start from a surface corresponding to a torsion free group $\widetilde{\Gamma}$ with $\Gamma \subset \widetilde{\Gamma} \subset PSL_2(\mathbf{Z})$ and $\widetilde{\Gamma}/\Gamma$ cyclic of order > 1. The surprising observation which makes our method work, is that such a group exists for most cases in Sebbar's list. We then start from the surface corresponding to $\widetilde{\Gamma}$ and construct the one for Γ as a cyclic base change of the former.

We now illustrate this method on the same example as used above.

Example 1.2. The group $\Gamma := \Gamma_0(2) \cap \Gamma(3)$ has index two in $\widetilde{\Gamma} := \Gamma_0(6)$. The latter group already appears in the paper of Beauville [**2**]; in fact, it corresponds to the surface with affine equation

$$(x+y)(y+1)(x+1) + txy = 0.$$

This surface has as singular fibers an I_6 (over $t = \infty$), an I_3 (over $t = 0$), an I_2 (over $t = 1$) and an I_1 (over $t = -8$). Now take the cyclic triple cover of the t-line which ramifies over $t = 1$ and $t = -8$ only. Explicitly, this cover is described as $s \mapsto t = (1 + 8s^3)/(1 - s^3)$, for a coordinate s on the cover such that $s = 0$ and $s = \infty$ are the branch points. The resulting base changed surface has equation

$$(s^3 - 1)(x+y)(y+1)(x+1) = (8s^3 + 1)xy,$$

and its singular fibers are of type I_6 (over $s^3 = 1$ and $s = 0$) and of type I_3 (over $8s^3 = -1$ and $s = \infty$).

We mention here that the problem of finding explicit equations for elliptic surfaces (not necessarily modular) has been getting considerable attention. Miranda and Persson [**15**] determined all possible configurations of bad fibers of semi-stable elliptic $K3$ surfaces; for instance, it turns out that such a surface must have at least 6 (and at most 24) singular fibers, and there exist precisely 112 possible configurations of bad fibers for such surfaces with six singular fibers. These surfaces have Picard number $2 + 24 - 6 = 20$, hence they are extremal elliptic $K3$ surfaces. Such extremal surfaces have been studied by Nori [**16**]. Quite recently a short proof of a characterization of all such surfaces was obtained by Kloosterman [**10**]. Lang [**8, 9**] studied rational extremal elliptic surfaces in positive characteristic. The result of Miranda and Persson uses the construction of the j-function of the surfaces involved.

This relies on the Riemann existence theorem, hence it does not explicitly construct this j-function. In particular, no equations of the surfaces are given.

The first semi-stable $K3$ surface in the list of Miranda and Persson corresponds to the configuration of singular fibers $I_1, I_1, I_1, I_1, I_1, I_{19}$. Shioda [**19**] and Iron [**7**] have independently and with different methods obtained explicit equations realizing such a surface. In fact, over the complex numbers it is unique: this follows from an observation of Kloosterman [**10**, Theorem 8.2], and also from the following argument due to Shioda [**19**]. One may present any such surface by an equation $y^2 = x^3 - 3f(t)x - 2g(t)$, with f and g polynomials of degree 8, respectively 12, and 'discriminant' $d(t) := g^2 - f^3$ of degree 5 (here the surface is given in such a way that the I_{19}-fiber is over $t = \infty$). A special case of a result of Stothers [**20**] shows that, up to obvious scalings, the equation $d = g^2 - f^3$ has a unique solution in polynomials of the given degrees. This solution can already be found in a paper by Marshall Hall, Jr., see [**5**, p. 185].

The paper of Livné and Yui [**12**] presents equations of nine other extremal semi-stable elliptic $K3$ surfaces, namely all modular ones. In subsection 2.3 below, it is explained how to obtain such equations by the method of our paper. In fact, as we will discuss in Section 5, equations for at least 18 of the 112 entries in the list of Miranda and Persson may be obtained from Beauville's list mentioned above, by using quadratic base changes.

2. Preliminaries.

2.1 Congruence subgroups. Throughout, we use the notion *congruence subgroup* and the notations $\Gamma(m)$, $\Gamma_1(m)$ and $\Gamma_0(m)$ in the context of subgroups of $PSL_2(\mathbf{Z}) = SL_2(\mathbf{Z})/\pm 1$. Note that this differs slightly from a lot of literature where the same notions and notations are used in the context of $SL_2(\mathbf{Z})$; we take the image under the natural quotient map of these.

Sebbar [**17**] gave the complete classification of all torsion-free genus zero congruence subgroups of the modular group $PSL_2(\mathbf{Z})$. Since the groups are assumed to be torsion free, the natural morphism from the corresponding compact Riemann surface $\Gamma\backslash\mathfrak{H}^*$ (of genus zero) to

$PSL_2(\mathbf{Z})\backslash\mathfrak{H}^*$ is ramified only in the cusps. The ramification index is classically called the *cusp width*. The Zeuthen-Hurwitz formula in this case gives $\mu = 6(k-2)$ where $\mu = [PSL_2(\mathbf{Z}) : \Gamma]$ is the index and k is the number of cusps for Γ. In particular, the index is a multiple of 6 for the type of subgroups considered.

In the table, the notation $\Gamma(m; m/d, \varepsilon, \chi)$ is used for a Larcher congruence subgroup. It is defined as follows. Let $m > 0$ be an integer, and let $d > 0$ be a divisor of m. Write $m/d = h^2 n$ with n square-free. Take integers $\varepsilon > 0$ a divisor of h, and $\chi > 0$ a divisor of $\gcd(d\varepsilon, m/(d\varepsilon^2))$. Then

$$\begin{aligned} &\Gamma(m; m/d, \varepsilon, \chi) \\ &\quad := \left\{ \pm \begin{pmatrix} 1 + (m/\varepsilon\chi)\,\alpha & d\,\beta \\ (m/\chi)\,\gamma & 1 + (m/\varepsilon\chi)\,\delta \end{pmatrix};\ \gamma \equiv \alpha \bmod \chi \right\} \Big/ \pm 1, \end{aligned}$$

where α, β, γ and δ are integers and the matrices are supposed to be in $SL_2(\mathbf{Z})$.

The following table summarizes Sebbar's classification results.

TABLE 1. All torsion-free genus zero congruence subgroups.

Index	t	Group	Cusp widths
6	3	$\Gamma(2)$	2,2,2
		$\Gamma_0(4)$	4,1,1
12	4	$\Gamma(3)$	3,3,3,3
		$\Gamma_0(4) \cap \Gamma(2)$	4,4,2,2
		$\Gamma_1(5)$	5,5,1,1
		$\Gamma_0(6)$	6,3,2,1
		$\Gamma_0(8)$	8,2,1,1
		$\Gamma_0(9)$	9,1,1,1

TABLE 1. (Continued).

Index	t	Group	Cusp widths
24	6	$\Gamma(4)$	4,4,4,4,4,4
		$\Gamma_0(3) \cap \Gamma(2)$	6,6,6,2,2,2
		$\Gamma_1(7)$	7,7,7,1,1,1
		$\Gamma_1(8)$	8,8,4,2,1,1
		$\Gamma_0(8) \cap \Gamma(2)$	8,8,2,2,2,2
		$\Gamma(8;4,1,2)$	8,4,4,4,2,2
		$\Gamma_0(12)$	12,4,3,3,1,1
		$\Gamma_0(16)$	16,4,1,1,1,1
		$\Gamma(16;16,2,2)$	16,2,2,2,1,1
36	8	$\Gamma_0(2) \cap \Gamma(3)$	6,6,6,6,3,3,3,3
		$\Gamma_1(9)$	9,9,9,3,3,1,1,1
		$\Gamma(9;3,1,3)$	9,9,3,3,3,3,3,3
		$\Gamma_1(10)$	10,10,5,5,2,2,1,1
		$\Gamma_0(18)$	18,9,2,2,2,1,1,1
		$\Gamma(27;27,3,3)$	27,3,1,1,1,1,1,1
48	10	$\Gamma_1(8) \cap \Gamma(2)$	8,8,8,8,4,4,2,2,2,2
		$\Gamma(8;2,1,2)$	8,8,4,4,4,4,4,4,4,4
		$\Gamma_1(12)$	12,12,6,4,4,3,3,2,1,1
		$\Gamma(12;6,1,2)$	12,6,6,6,6,4,2,2,2,2
		$\Gamma_0(16) \cap \Gamma_1(8)$	16,16,4,4,2,2,1,1,1,1
		$\Gamma(16;8,2,2)$	16,16,2,2,2,2,2,2,2,2
		$\Gamma(24;24,2,2)$	24,8,3,3,3,3,1,1,1,1
		$\Gamma(32;32,4,2)$	32,8,1,1,1,1,1,1,1,1
60	12	$\Gamma(5)$	5,5,5,5,5,5,5,5,5,5,5,5
		$\Gamma_0(25) \cap \Gamma_1(5)$	25,25,1,1,1,1,1,1,1,1,1,1

2.2 Elliptic modular surfaces. Here we recall some basic facts about a special class of elliptic surfaces, called *elliptic modular surfaces*, associated to congruence subgroups $\Gamma \subset PSL_2(\mathbf{Z})$ as discussed in subsection 2.1. The concept of elliptic modular surfaces was first introduced by Shioda in [**18**].

The quotient of $\mathfrak{H} \times \mathbf{C}$ by all automorphisms of the form

$$(\tau, z) \longmapsto (\gamma\tau, (c\tau + d)^{-1}(z + m\tau + n))$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $(m, n) \in \mathbf{Z}^2$ defines a surface equipped with a morphism to the modular curve X_Γ attached to Γ. The fiber over the image in X_Γ of a general point $\tau \in \mathfrak{H}$ is the elliptic curve corresponding to the lattice $\mathbf{Z} \oplus \mathbf{Z}\tau$. The surface obtained in this way can be extended to an elliptic surface over the modular curve X_Γ. This surface is called the *elliptic modular surface* associated to Γ.

McKay and Sebbar [**13**] observed that all groups Γ in the list given in subsection 2.1 (except the two which have index 6 in $PSL_2(\mathbf{Z})$) have an explicit lift $\overline{\Gamma}$ to $SL_2(\mathbf{Z})$ with the following properties:

(a) $\overline{\Gamma}$ has no elliptic elements,

(b) $\overline{\Gamma}$ contains no element of trace equal to -2.

Hence following a classical method of Kodaira, they obtained the following result.

Theorem 2.1. *Let Γ be one of the subgroups in Table* 1 *with index $\mu > 6$, and let t be the number of inequivalent cusps for Γ. Then the elliptic modular surfaces attached to Γ is semi-stable, and has t singular fibers of type I_n corresponding to these cusps. Moreover, the fiber over such a cusp is of type I_n where n equals the cusp width.*

Moreover, McKay and Sebbar explicitly give the relation between the Hauptmodul t for Γ (which is by definition a generator of the function field of the rational curve X_Γ) and the elliptic modular function j. In other words, they present the rational function $j(t)$ which gives the j-invariant of the corresponding elliptic modular surface. Let $I_{n_1}, \cdots, I_{n_k}$ be the singular fibers of such a surface, then Kodaira's results imply $\sum n_i = \mu$. In particular, the index μ gives the Euler number of the surface, which in turn gives its geometric genus $p_g = \mu/12 - 1$. Therefore, the index 12 subgroups give rise to rational elliptic surfaces, the index 24 to elliptic $K3$ surfaces, and the remaining indices $36, 48$ and 60 to elliptic surfaces of general type with $p_g = 2, 3$ and 4, respectively.

These elliptic surfaces are *extremal*, which means that their Néron-Severi group has rank equal to their Hodge number $h^{1,1}$ and is generated by a section and all components of singular fibers. To see this, note that the Euler number equals the sum of Hodge numbers $h^0 + h^{0,2} + h^{1,1} + h^{2,0} + h^4 = 2 + 2p_g + h^{1,1}$, hence $h^{1,1} = 5\mu/6$. The components of singular fibers plus a section generate an indivisible subgroup of the Néron-Severi group of rank $2 + \sum_{i=1}^{k}(n_i - 1) = 2 + \mu - k$. In subsection 2.1 it was remarked that $\mu = 6(k-2)$. Therefore, it follows that $2 + \mu - k = 5\mu/6 = h^{1,1}$, which shows that these surfaces are extremal.

2.3 Defining equations corresponding to index 12, respectively 24. The defining equations for the rational elliptic surfaces were given by Beauville [**2**] (not in a Weierstrass form; however, it is easy to derive Weierstrass equations from the given ones as is explained, e.g., in [**4**, Chapter 8]). Equations for the elliptic $K3$ surfaces were obtained by Livné and Yui [**12**] (in a Weierstrass form).

For the sake of completeness we list these known cases. First we list Beauville's defining equations for the rational elliptic modular surfaces in $\mathbf{P}^2 \times \mathbf{P}^1$ where $[x : y : z]$ denotes the projective coordinate for $\mathbf{P}^2$ and $t \in \mathbf{P}^1$ is the affine coordinate for $\mathbf{P}^1$.

TABLE 2. Defining equations corresponding to index 12 groups.

modular curve	group	defining equation
$X(3)$	$\Gamma(3)$	$x^3 + y^3 + z^3 + t\,xyz = 0$
	$\Gamma_0(4) \cap \Gamma(2)$	$x(x^2 + z^2 + 2zy) + t\,z(x^2 - y^2) = 0$
$X_1(5)$	$\Gamma_1(5)$	$x(x-z)(y-z) + t\,zy(x-y) = 0$
$X_0(6)$	$\Gamma_0(6)$	$(x+y)(y+z)(z+x) + t\,xyz = 0$
$X_0(8)$	$\Gamma_0(8)$	$(x+y)(xy - z^2) + t\,xyz = 0$
$X_0(9)$	$\Gamma_0(9)$	$x^2y + y^2z + z^2x + t\,xyz = 0$

Now we list defining equations for the elliptic modular $K3$ surfaces obtained in [**12**]. As Fuchsian groups in $PSL_2(\mathbf{R})$, the nine groups of index 24 are further classified into four sets of mutually conjugate groups:

$$\Gamma(4) \quad \text{and} \quad \Gamma_0(8)\cap\Gamma(2) \quad \text{and} \quad \Gamma_0(16),$$
$$\Gamma_0(3)\cap\Gamma(2) \quad \text{and} \quad \Gamma_0(12),$$
$$\Gamma_1(7),$$
$$\Gamma_1(8), \quad \text{and} \quad \Gamma_1(8;4,1,2) \quad \text{and} \quad \Gamma_1(16;16,2,2).$$

Conjugation of these groups translates into isogenies between the corresponding elliptic surfaces. In fact, all isogenies which show up here are composed of isogenies of degree 2. Since formulas for the image under such isogenies are well known [**4**, pp. 58–59], we only present one equation from each isogeny class.

TABLE 3. Defining equations corresponding to index 24 groups.

group	defining equation
$\Gamma(4)$	$y^2 = x(x-1)(x-(t+t^{-1})^2/4)$
$\Gamma_1(7)$	$y^2+(1+t-t^2)xy+(t^2-t^3)y = x^3+(t^2-t^3)x$
$\Gamma(8;4,1,2)$	$y^2 = x^3-2(8t^4-16t^3+16t^2-8t+1)x^2+(8t^2-8t+1)(2t-1)^4x$
$\Gamma_0(12)$	$y^2+(t^2+1)xy-t^2(t^2-1)y = x^3-t^2(t^2-1)x^2$

We should remark that the equation for level 4 was found in [**18**].

We now describe a different method to obtain defining equations corresponding to all nine index 24 groups. In fact, this geometric method works not only for index 24 groups but for other groups as well. This will be done by realizing all but one of these elliptic modular surfaces as double covers of Beauville's rational elliptic modular surfaces and then by deriving defining equations from those of Beauville. The remaining ninth surface is the one corresponding to $\Gamma_1(7)$. An explicit equation for this case was already obtained by Kubert [**11**] in 1976; see also [**6**, Table 3]. It is given by

$$y^2+(1-c)xy-by = x^3-bx^2$$

where $b = t^3-t^2$ and $c = t^2-t$.

2.3.1 *Two subgroups of* $\Gamma_0(4) \cap \Gamma(2)$. The first example we consider is the index 12 subgroup $\Gamma_0(4) \cap \Gamma(2)$. According to Beauville, it corresponds to the surface with affine equation

$$x(x^2 + 2y + 1) + t(x^2 - y^2) = 0.$$

A Weierstrass equation for the same surface is obtained by taking t and $\xi := tx$ and $\eta := t^2y - tx$ as new coordinates. The resulting equation is

$$\eta^2 = \xi^3 + (t^2 + 1)\xi^2 + t^2\xi.$$

The configuration of singular fibers of this surface is given in the following table.

$t =$	0	1	-1	∞
fiber:	I_4	I_2	I_2	I_4

The group $\Gamma_0(4) \cap \Gamma(2)$ contains $\Gamma(4)$ as well as $\Gamma(8; 4, 1, 2)$ as index 2 subgroups. The first one corresponds to a double cover of modular curves ramified over the cusps of width 2. The second one yields a double cover ramified over one cusp of width 2 and one cusp of width 4.

An explicit rational map which realizes the first double cover, is

$$s \longmapsto t := (s^2 - 1)/(s^2 + 1),$$

with $s = 0$ and $s = \infty$ as ramification points. Hence the base changed surface with equation

$$(s^2 + 1)x(x^2 + 2y + 1) + (s^2 - 1)(x^2 - y^2) = 0$$

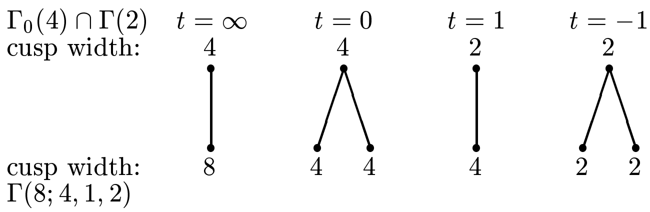
is a semi-stable surface with 6 singular fibers of type I_4, corresponding to $\Gamma(4)$. We express the data of the associated double cover of modular curves by the following diagram.

$\Gamma_0(4) \cap \Gamma(2)$	$t = \infty$	$t = 0$	$t = 1$	$t = -1$
cusp width:	4	4	2	2
cusp width:	4 4	4 4	4	4
$\Gamma(4)$				

Similarly, an explicit rational map realizing the second double cover is

$$s \longmapsto t := s^2 + 1.$$

This corresponds to the diagram



Hence $\Gamma(8; 4, 1, 2)$ is realized by

$$x(x^2 + 2y + 1) + (s^2 + 1)(x^2 - y^2) = 0.$$

2.3.2 *Two subgroups of* $\Gamma_0(6)$. The Beauville surface corresponding to $\Gamma_0(6)$ was already described in Example 1.2. Its equation is

$$(x + y)(x + 1)(y + 1) + txy = 0$$

and its singular fiber configuration is given by

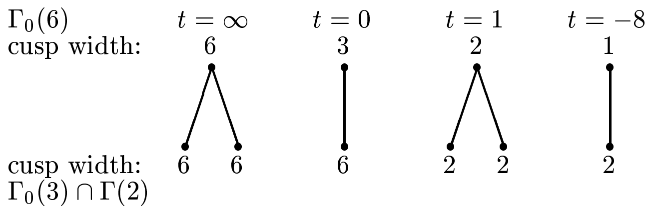
$t =$	-8	1	0	∞
fiber:	I_1	I_2	I_3	I_6

Two subgroups of index 2 of $\Gamma_0(6)$ are $\Gamma_0(3) \cap \Gamma(2)$ and $\Gamma_0(12)$. The corresponding double cover of modular curves ramifies over the cusps of width 1 and 3 in the first case. In the second case, it ramifies over the cusps of width 2 and 6.

So, to realize the surface corresponding to $\Gamma_0(3) \cap \Gamma(2)$ one can use the base change obtained from

$$s \longmapsto t := s^2/(1 - 8s^2).$$

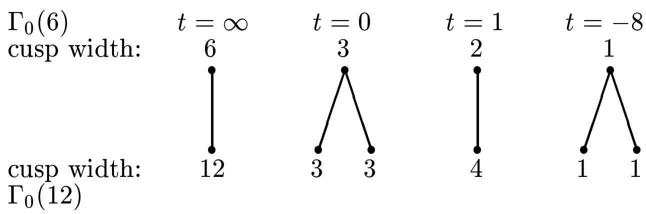
This yields the diagram



For the group $\Gamma_0(12)$ the map

$$s \longmapsto t := 1 - s^2$$

gives the desired covering. In a diagram:



Hence, one obtains the surfaces

$$\text{for} \quad \Gamma_0(3)\cap\Gamma(2): \quad (8s^2-1)(x+y)(x+1)(y+1) = s^2xy$$

and

$$\text{for} \quad \Gamma_0(12): \quad (x+y)(x+1)(y+1) + (1-s^2)xy = 0.$$

2.3.3 *Four subgroups of* $\Gamma_0(8)$. The Beauville surface corresponding to $\Gamma_0(8)$ was given by Beauville as

$$(x+y)(xy-1) + \tilde{t}xy = 0.$$

Alternatively, it is defined by the Weierstrass equation

$$\eta^2 = \xi^3 + (2-t^2)\xi^2 + \xi,$$

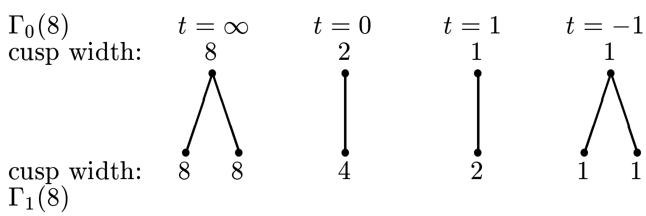
in which $\eta = (x+y-(txy/2))y^{-2}$ and $\xi = x/y$ and $t = i\tilde{t}/2$. The configuration of singular fibers of this surface is as follows.

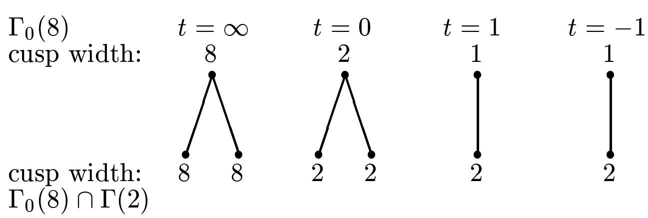
$t =$	∞	0	1	-1
fiber:	I_8	I_2	I_1	I_1

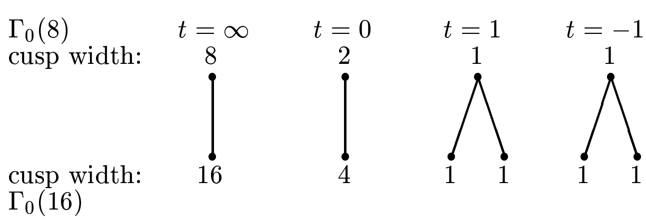
The reason for rescaling Beauville's parameter, written as $\tilde{t}$ here, is precisely the fact that in the new parameter t, all singular fibers occur over rational points. The group $\Gamma_0(8)$ turns out to contain four index 2 genus zero congruence subgroups, namely

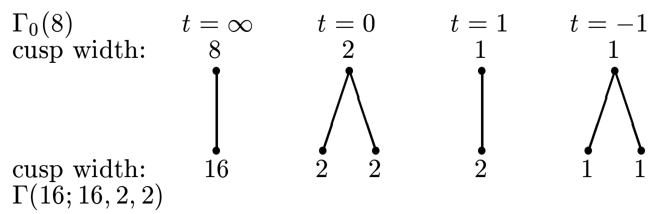
$$\Gamma_1(8), \quad \Gamma_0(8) \cap \Gamma(2), \quad \Gamma_0(16), \quad \text{and} \quad \Gamma(16; 16, 2, 2).$$

The corresponding double covers of modular curves ramify over two cusps, which are of width 1 and 2 in the first case, both of width 1 in the second case, of width 8 and 2 in the third case, and of width 8 and 1 in the remaining case. This is expressed in the next four diagrams.









Explicit covering maps, which can be used for this, are presented in the following table.

$$\begin{aligned} \text{for} \quad \Gamma_1(8): &\quad s \mapsto t := s^2/(s^2+1); \\ \text{for} \quad \Gamma_0(8) \cap \Gamma(2): &\quad s \mapsto t := (s^2-1)/(s^2+1); \\ \text{for} \quad \Gamma_0(16): &\quad s \mapsto t := s^2; \\ \text{for} \quad \Gamma(16;16,2,2): &\quad s \mapsto t := s^2+1. \end{aligned}$$

It should be evident how to obtain explicit equations for the base changed surfaces from this.

3. The main result. In this section we will give explicit equations for the elliptic surfaces of general type corresponding to the remaining genus zero congruence subgroups of index 36, 48 and 60 in Table 1. The observation which makes this a rather easy task is the following.

Theorem 3.1. *With the exception of the two groups $\Gamma_1(7)$ and $\Gamma_1(10)$, all torsion-free genus zero congruence subgroups of index at least 24 in $PSL_2(\mathbf{Z})$ correspond to modular curves which are obtained by a composition of cyclic covers from the modular curves in Beauville's list, see Table 2.*

Here, a composition of cyclic covers is only needed for the groups $\Gamma_1(12)$ and $\Gamma_0(16) \cap \Gamma_1(8)$; in all other cases, a single cyclic map suffices.

For the case of index 24, Theorem 3.1 was proven in subsections 2.3.1, 2.3.2 and 2.3.3 above. We will consider the remaining cases in Section 4 below.

Note that the exceptional case $\Gamma_1(7)$ was already discussed in subsection 2.3. For the remaining exceptional case $\Gamma_1(10)$, we quote

[**6**, Table 6], which provides the equation

$$y^2 = x(x^2 + ax + b)$$

with

$$a = -(2t^2 - 2t + 1)(4t^4 - 12t^3 + 6t^2 + 2t - 1)$$

and

$$b = 16(t^2 - 3t + 1)(t - 1)^5 t^5$$

for the corresponding elliptic modular surface. This is a surface with two I_5-fibers at $t = \infty$ and at $t = 1/2$, two I_{10}-fibers at $t = 0$ and at $t = 1$, two I_2-fibers at the roots of $t^2 - 3t + 1 = 0$, and finally, two I_1-fibers at the roots of $4t^2 - 2t - 1 = 0$.

Note that a cyclic cover $\varphi : \mathbf{P}^1 \to \mathbf{P}^1$ of degree d ramifies over exactly two points. Write these ramification points as $t = a$ and $t = b$ for some coordinate t on the second $\mathbf{P}^1$ (which, for convenience, we take in such a way that $a \neq \infty$). Choose a coordinate s on the first $\mathbf{P}^1$ such that $s = 0$ is mapped to $t = a$ and $s = \infty$ is mapped to $t = b$. With such choices, φ can be given as

$$\varphi : \quad s \longmapsto t := \frac{bs^d + \lambda a}{s^d + \lambda}$$

for an arbitrary choice of $\lambda \neq 0$. In case $b = \infty$ one can take $\varphi(s) = \lambda s^d + a$. Hence, once we have determined the degrees and the ramification points for the maps needed in Theorem 3.1, it is evident how to obtain explicit equations for the corresponding elliptic modular surfaces starting from Beauville's equations.

4. Proof of the main result.

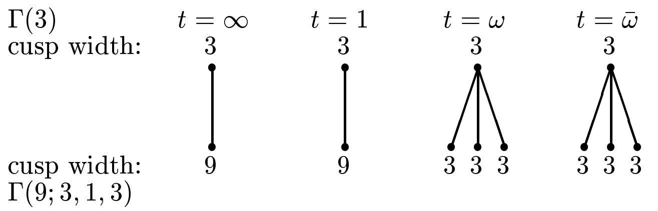
Here we present a proof of Theorem 3.1 by considering the necessary groups in Beauville's list one by one.

4.1 An index 3 subgroup of $\Gamma(3)$.

The elliptic modular surface associated with $\Gamma(3)$ is the famous Hessian family given in affine form as

$$x^3 + y^3 + 1 - 3txy = 0.$$

Note that we have slightly rescaled the parameter used by Beauville. The family has four fibers of type I_3, corresponding to $t = \infty$, 1, ω, $\overline{\omega}$. Here ω denotes a primitive cube root of unity (so $\omega^2 + \omega + 1 = 0$). The subgroup $\Gamma(9; 3, 1, 3)$ yields a cyclic covering of degree 3 of the modular curve $X(3)$. It ramifies over two of the cusps. This is illustrated in the following diagram.



4.2 Two index 3 subgroups of $\Gamma_0(9)$. The elliptic modular surface associated with $\Gamma_0(9)$ is presented by Beauville as

$$x^2y + y^2 + x + txy = 0.$$

Using new coordinates $\xi := -4/x$ and $\eta := (8y + 4x^2 + 4tx)/x^2$, this is transformed into the Weierstrass equation

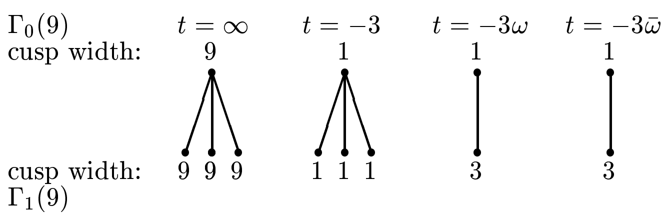
$$\eta^2 = \xi^3 + t^2\xi^2 - 8t\xi + 16.$$

This surface has three singular fibers of type I_1 at $t = -3, -3\omega, -3\bar{\omega}$ (with $\omega^2 + \omega + 1 = 0$) and a fiber of type I_9 at $t = \infty$.

The subgroup $\Gamma_1(9)$ corresponds to a cyclic triple covering of $X_0(9)$ ramified over two cusps of width 1. With the parameter t used here, this covering can be expressed by

$$s \longmapsto t := \frac{s^3 - 3s + 1}{s^2 - s}.$$

It corresponds to the following diagram.



The subgroup $\Gamma(27;27,3,3)$ yields the cyclic triple covering of $X_0(9)$ ramifying over the other two cusps (of width 9 and 1, respectively). Hence, in a diagram:

$\Gamma_0(9)$	$t=\infty$	$t=-3$	$t=-3\omega$	$t=-3\bar{\omega}$
cusp width:	9	1	1	1
cusp width:	27	3	1 1 1	1 1 1
$\Gamma(27;27,3,3)$				

4.3 Two index 3 and two index 4 subgroups of $\Gamma_0(6)$. The group $\Gamma_0(6)$ we already discussed in subsection 2.3.2. A corresponding modular elliptic surface is

$$(x+y)(x+1)(y+1)+txy=0,$$

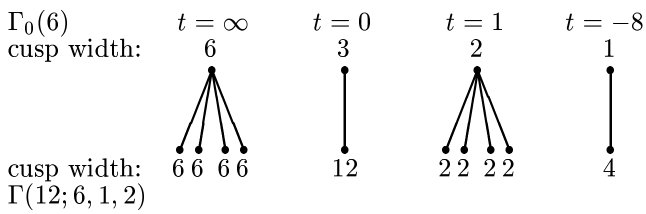
with at $t=-8$ an I_1-fiber, at $t=1$ an I_2-fiber, at $t=0$ an I_3-fiber, and at $t=\infty$ an I_6-fiber. The degree 3 cyclic covering of $X_0(6)$ ramified over the cusps of width 1 and 2 corresponds to $\Gamma_0(2)\cap\Gamma(3)$.

$\Gamma_0(6)$	$t=\infty$	$t=0$	$t=1$	$t=-8$
cusp width:	6	3	2	1
cusp width:	6 6 6	3 3 3	6	3
$\Gamma_0(2)\cap\Gamma(3)$				

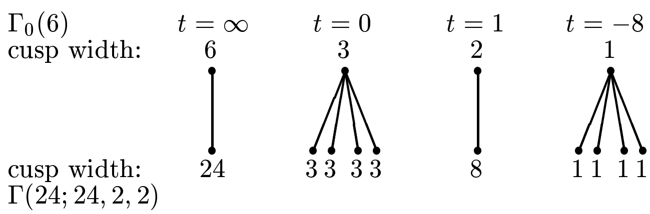
Similarly, the degree 3 cyclic cover ramified over the cusps of width 3 and 6 yields $\Gamma_0(18)$.

$\Gamma_0(6)$	$t=\infty$	$t=0$	$t=1$	$t=-8$
cusp width:	6	3	2	1
cusp width:	18	9	2 2 2	1 1 1
$\Gamma_0(18)$				

Two more subgroups of $\Gamma_0(6)$ appearing in Table 1 have index 4. The degree 4 cyclic covering of $X_0(6)$ ramifying over the cusps of width 3 and 1 corresponds to $\Gamma(12;6,1,2)$.



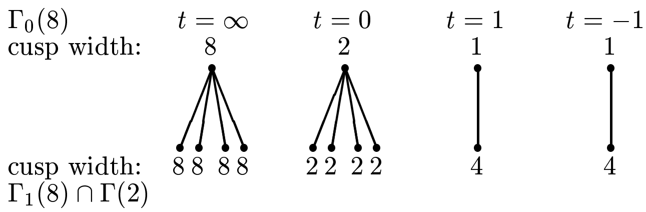
Finally, the degree 4 cyclic covering which ramifies over the cusps of width 6 and 2 comes from the group $\Gamma(24;24,2,2)$.



4.4 Two index 4 subgroups of $\Gamma_0(8)$. From subsection 2.3.3 one sees that $\Gamma_0(8)$ gives the elliptic modular surface

$$\eta^2 = \xi^3 + (2-t^2)\xi^2 + \xi,$$

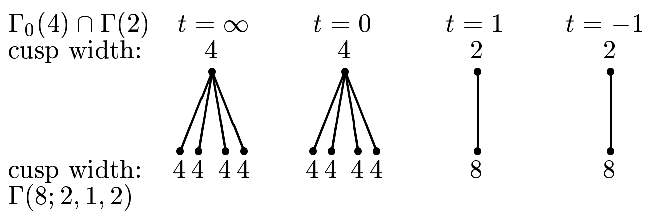
with two I_1-fibers at $t=\pm1$, an I_2-fiber at $t=0$ and an I_8-fiber at $t=\infty$. The groups $\Gamma_1(8)\cap\Gamma(2)$ and $\Gamma(32;32,4,2)$ both have index 4 in $\Gamma_0(8)$. They correspond to cyclic coverings of degree 4 which are ramified over the cusps of width 1 for the first group, and over the cusps of width 2 and 8 for the second. This is summarized in the next two diagrams.



4.5 Two index 4 subgroups of $\Gamma_0(4)\cap\Gamma(2)$. As we saw in subsection 2.3.1, an equation for the elliptic modular surface related with $\Gamma_0(4)\cap\Gamma(2)$ is

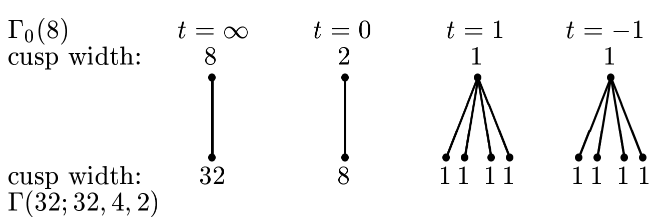
$$\eta^2 = \xi^3 + (t^2+1)\xi^2 + t^2\xi,$$

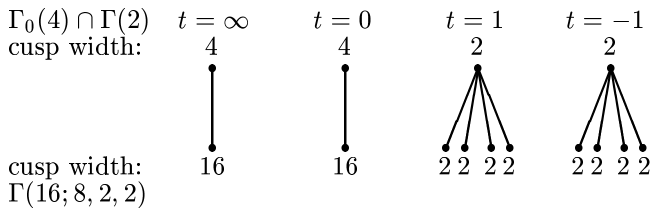
having I_2-fibers at $t=\pm 1$ and I_4-fibers at $t=0,\infty$. The group $\Gamma(8;2,1,2)$ comes from a covering ramified over the cusps of width 2, and the group $\Gamma(16;8,2,2)$ comes from a covering ramified over the cusps of width 4. We express this as follows.



4.6 Two index 5 subgroups of $\Gamma_1(5)$. The Beauville elliptic modular surface for $\Gamma_1(5)$ is given by

$$x(x-1)(y-1)+ty(x-y)=0.$$





A corresponding Weierstrass equation is obtained using the coordinate change $\xi := 4t/x$ and $\eta := 4t(2ty - tx - x^2 + x)/x^2$; the resulting equation is

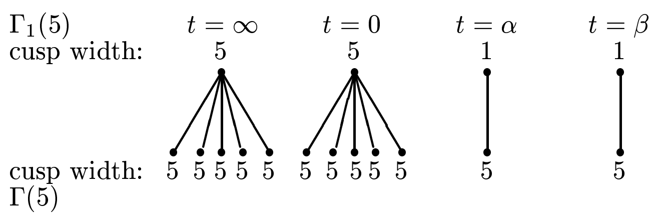
$$\eta^2 = \xi^3 + (t^2 - 6t + 1)\xi^2 + 8t(t-1)\xi + 16t^2.$$

This defines a surface with singular fibers at $t = 0, \infty$ (of type I_5) and two fibers of type I_1 at $t = \alpha := (11/2) + (5/2)\sqrt{5}$ and at $t = \beta := (11/2) - (5/2)\sqrt{5}$.

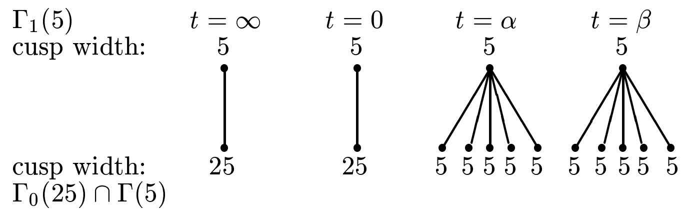
The subgroup $\Gamma(5)$ corresponds to the cyclic quintic covering of $X_1(5)$ which ramifies over the two cusps of width 1. In terms of the coordinate t which we use on $X_1(5)$, this covering can be expressed as

$$s \longmapsto t := \frac{(s+5)(11s^4 + 70s^3 + 200s^2 + 250s + 125)}{2s(s^4 + 50s^2 + 125)}.$$

In a diagram, the covering yields the following.



The remaining subgroup $\Gamma_0(25) \cap \Gamma_1(5)$ of $\Gamma_1(5)$ corresponds to the cyclic quintic covering of $X_1(5)$ which ramifies over the two cusps of width 5.



4.7 The remaining groups: $\Gamma_1(12)$ and $\Gamma_0(16)\cap\Gamma_1(8)$. With respect to the natural morphism attached to inclusion of groups, the modular curves associated with $\Gamma_1(12)$ and with $\Gamma_0(16)\cap\Gamma_1(8)$ are not cyclic coverings of any of the modular curves appearing in Beauville's list: such a cyclic morphism would have degree 4, and then two of the cusps should have a width divisible by 4 and the others should come in 4-tuples having the same width. As can be read off from Table 1, this is not the case.

To find explicit equations for the corresponding elliptic modular surfaces, we use the inclusions

$$\Gamma_1(12)\subset\Gamma_0(12)\subset\Gamma_0(6)$$

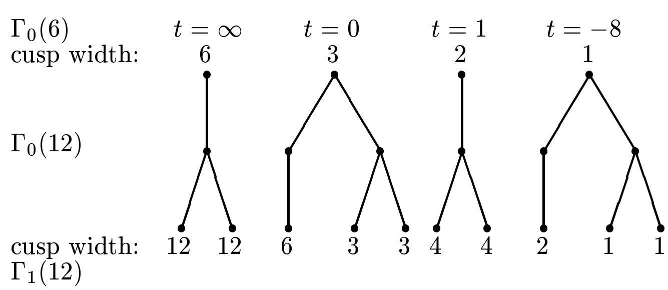
and

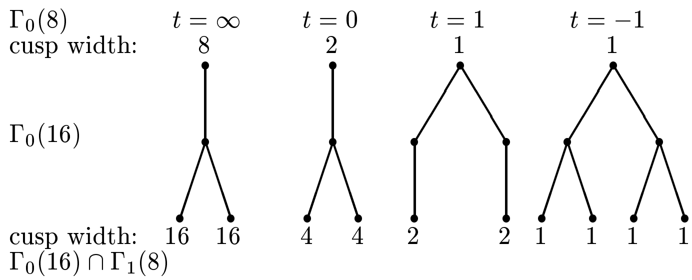
$$\Gamma_0(16)\cap\Gamma_1(8)\subset\Gamma_0(16)\subset\Gamma_0(8)$$

in which each subgroup has index two in the next one. An explicit covering morphism can be described as follows.

$$\begin{aligned}\text{for}\quad \Gamma_1(12):&\quad s\longmapsto t:=1-(s^2+3)^2/(s^2+1)^2;\\ \text{for}\quad \Gamma_0(16)\cap\Gamma_1(8):&\quad s\longmapsto t:=(s^2-1)^2/(s^2+1)^2.\end{aligned}$$

The associated covering data is expressed in the following two diagrams.





5. Some remarks. As remarked at the end of subsection 2.2, the elliptic modular surfaces of which we found equations are semi-stable and extremal. As is well known (in fact, the argument in the last paragraph of subsection 2.2 shows this), a semi-stable elliptic surface over $\mathbf{P}^1$ of geometric genus p_g is extremal if and only if the number of singular fibers equals $2p_g+4$. A result of Nori [**16**], recently also proven by Kloosterman [**10**], implies for the special case of semi-stable elliptic surfaces that such a surface is extremal precisely when its j-invariant is unramified outside $0, 1728, \infty$, and all points in $j^{-1}(0)$ have ramification index 3, and all points in $j^{-1}(1728)$ have ramification index 2. The degree of this j-map will be $12(p_g+1)$, and the types I_{n_i} of the singular fibers of the elliptic surface can be read off from the fact that the n_i are the ramification indices of the points in $j^{-1}(\infty)$.

For the case $p_g = 0$, i.e., rational elliptic surfaces, Beauville [**2**] presented all semi-stable extremal ones, see also Table 2 above. The next case $p_g = 1$, i.e., elliptic $K3$ surfaces, Miranda and Persson [**15**] showed that there exist 112 possible configurations of singular fibers of semi-stable extremal ones. In subsections 2.3.1, 2.3.2 and 2.3.3 we showed that 8 of them are obtained as quadratic base changes from the Beauville surfaces. In fact, by choosing different cusps as ramification points of the quadratic map, one finds equations for 10 more such semi-stable extremal elliptic $K3$ surfaces. Namely, also the ones with fibre configuration, using the notation from [**15**],

$$
\begin{array}{llll}
[1,1,1,1,2,18], & [1,1,1,1,10,10], & [1,1,2,2,6,12], & \\
[1,1,2,2,9,9], & [1,1,2,5,5,10], & [1,1,4,6,6,6], & \\
[2,2,2,3,3,12], & [2,2,5,5,5,5], & [2,3,3,4,6,6], & [3,3,3,3,6,6].
\end{array}
$$

If one adds the elliptic modular surface corresponding to $\Gamma_1(7)$ to this, see Table 3, plus the case $[1,1,1,1,1,19]$ constructed by Iron and Shioda, then in total 20 of the 112 possible cases have been given by explicit equations. Shioda's method [**19**] seems unfit for producing more semi-stable extremal elliptic $K3$ surfaces. It seems not unlikely, however, that Iron's method [**7**] could give several more examples. The method of Shioda, based on the work of Stothers, does however show the existence of certain semi-stable extremal elliptic surfaces of general type. More precisely, it shows that, for every integer $m > 0$, a semi-stable extremal elliptic surface exists, with $p_g = m - 1$, having $2m + 1$ fibers of type I_1 and one fiber of type I_{10m-1}.

Actually, it should be possible to obtain a few more equations by using base changes from easier (rational) surfaces. To illustrate this point, start with a rational elliptic surface with two fibers of type II, one of type I_1, and one of type I_7. The existence of this follows from the existence of the corresponding j-invariant: it suffices to show that a rational function $j(t)$ of the form

$$j(t) := \lambda \, \frac{(t-a)^3(t^2+bt+c)}{t^7(t-d)}$$

can be constructed, with a, d distinct and different from 0, and t^2+bt+c having two different zeroes both different from any of $a, d, 0$, and $j(t) = 1728$ having only roots with even multiplicity. Then an elliptic surface with this j-invariant exists, having fibers of type II over the roots of $t^2 + bt + c = 0$, and an I_7-fiber over $t = 0$ and an I_1 fiber over $t = d$ and no other singular fibers. Now, taking a cyclic base change of degree 3 ramified over the zeroes of $t^2 + bt + c$, and then twisting the resulting surface over the quadratic extension ramified only at these two zeroes, results in a semi-stable surface with $[1,1,1,7,7,7]$-configuration.

Nevertheless, for most cases such easy cyclic base changes will not exist. Therefore, we consider the problem of finding explicit equations for the remaining $112 - 20 = 92$ cases in the table of Miranda and Persson wide open. Recently, (2004), Beukers and Montanus have solved this problem for all 112 cases of the Miranda-Persson list in terms of "dessin d'enfant." See http://www.math.uu.nl/people/beukers/mirandapersson/Dessins.html for the list of defining equations that have rational coefficients.

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