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ELLIPTIC FIBRATIONS OF SOME EXTREMAL $K3$ SURFACES

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ABSTRACT. This paper is concerned with the construction of extremal elliptic $K3$ surfaces. It gives a complete treatment of those fibrations which can be derived from rational elliptic surfaces by easy manipulations of their Weierstrass equations. In particular, this approach enables us to find explicit equations for 38 semi-stable extremal elliptic $K3$ fibrations, 32 of which are indeed defined over $\mathbf{Q}$. They are realized as pull-back of non semi-stable extremal rational elliptic surfaces via base change. This is related to the work of J. Top and N. Yui which exhibited the same procedure for the semi-stable extremal rational elliptic surfaces.

1. Introduction. The aim of this paper is to find all extremal elliptic $K3$ fibrations which can be derived from rational elliptic surfaces by direct, relatively simple manipulations of their Weierstrass equations. The main technique for this purpose will be pull-back by a base change. We only exclude the general construction involving the induced J-map of the fibration (considered as a base change generally of degree 24, conf. [**10**, Section 2]). The base changes we construct will have degree at most 8. Additionally there is another effective method if we allow the extremal $K3$ surface to have nonreduced fibres. Then we can also manipulate the Weierstrass equations by adding or transferring common factors, thus changing the shape of singular fibres rather than introducing new cusps. In total this approach will enable us to realize 201 out of the 325 configurations of singular fibres which exist for extremal elliptic $K3$ surfaces due to the classification of [**15**]. Note, however, that the configuration does in general not determine the isomorphism class.

For most of this paper, we will concentrate on the extremal elliptic $K3$ fibrations with only semi-stable fibres. The determination of the

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112 possible configurations of singular fibres goes originally back to Miranda and Persson [**9**]. For 20 of them, Weierstrass equations over $\mathbf{Q}$, or in one case $\mathbf{Q}(\sqrt{5})$, have been obtained in [**4**] or [**6, 16, 19**]. These give rise to the elliptic $K3$ surface with the maximal singular fibre (the first one from the list in [**9**]), the modular ones and those coming from semi-stable extremal (hence modular) rational elliptic surfaces after a quadratic base change. By way of construction, their isomorphism classes are indeed known in advance.

The main idea of this paper consists in applying a base change of higher degree to other extremal rational elliptic surfaces (namely those with three cusps). Thereby we can substitute the nonreduced singular fibres by semi-stable ones in the pull-back surface such that it turns out to be an extremal $K3$. Indeed five of the modular rational elliptic surfaces can also be obtained in this way according to [**9**, Section 7]. Here, we investigate those base changes of the original surfaces with three cusps, which do not factor through the modular rational ones, and indeed find Weierstrass equations for 38 further extremal semi-stable elliptic $K3$ fibrations, only 6 of which are not already defined over $\mathbf{Q}$. Again their isomorphism classes are known due to the predetermined shape of the Mordell-Weil group. The surfaces over $\mathbf{Q}$ realize the following 32 configurations of singular fibres in the notation of [**10**]:

$[1,1,1,2,3,16]$	$[1,1,1,2,5,14]$	$[1,1,1,3,3,15]$	$[1,1,1,3,6,12]$
$[1,1,1,5,6,10]$	$[1,1,2,2,3,15]$	$[1,1,2,3,3,14]$	$[1,1,2,4,4,12]$
$[1,1,2,4,6,10]$	$[1,1,3,3,8,8]$	$[1,1,3,4,6,9]$	$[1,2,2,2,3,14]$
$[1,2,2,2,5,12]$	$[1,2,2,2,7,10]$	$[1,2,2,3,4,12]$	$[1,2,2,3,6,10]$
$[1,2,2,5,6,8]$	$[1,2,2,6,6,7]$	$[1,2,3,3,3,12]$	$[1,2,3,4,4,10]$
$[1,2,4,4,6,7]$	$[1,2,4,5,6,6]$	$[1,3,3,3,5,9]$	$[1,3,3,5,6,6]$
$[1,3,4,4,4,8]$	$[2,2,2,4,6,8]$	$[2,2,2,3,5,10]$	$[2,2,3,3,4,10]$
$[2,2,3,4,5,8]$	$[2,2,4,4,6,6]$	$[2,3,3,3,4,9]$	$[2,3,4,4,5,6]$

The additional fibrations which can only be defined over some quadratic or cubic extension of $\mathbf{Q}$ by this approach are

$$[1,1,2,2,4,14],\ [1,1,2,6,6,8],\ [1,2,2,4,5,10]$$
$$[1,2,2,4,7,8],\ [1,2,3,3,6,9],\ [1,2,3,4,6,8].$$

Finally, the non semi-stable extremal $K3$ fibrations derived from rational elliptic surfaces can be found in two tables at the end of this paper.

After shortly recalling some basic facts about elliptic surfaces in the next section (2), we will spend the major part of this paper with constructing the base changes and giving the resulting equations for the semi-stable extremal elliptic $K3$ surfaces (Sections 3, 4, 5, 6). Eventually we will return to the non semi-stable fibrations in the last section (7) although we will keep their treatment quite concise.

One final remark seems to be in order: There are, of course, many other ways to produce extremal (or singular) elliptic $K3$ surfaces. The perhaps best known is the concept of double sextics as introduced in [**13**]. We will not pursue this approach here, so the interested reader is also referred to [**1, 11**] for instructive applications.

2. Elliptic surfaces over $\mathbf{P}^1$. An elliptic surface over $\mathbf{P}^1$, say $Y \overset{r}{\to} \mathbf{P}^1$ with a section, is given by a minimal affine Weierstrass equation

$$y^2 = x^3 + Ax + B$$

where A and B are homogeneous polynomials in the two variables of $\mathbf{P}^1$ of degree $4M$ and $6M$, respectively, for some $M \in \mathbf{N}$. Here, the term minimal refers to the common factors of A and B: They are not allowed to have a common irreducible factor with multiplicity greater than 3 in A and greater than 5 in B, since otherwise we could cancel these factors out by an admissible change of variables. This also restricts the singularities of the Weierstrass equation to rational double points, such that Y is the minimal desingularization. In this paper we are interested in special examples where both A and B have rational coefficients (while the work of [**10**] in accordance with [**17**], will only guarantee the existence of A and B over some number field). Of course, we can also assume A and B to have (minimal) integer coefficients, but we will not go into detail with this.

As announced in the introduction, we are going to pay special attention to the singular fibres of Y. For a general choice of A and B there will be $12M$ of them (each a rational curve with a node). The types of the singular fibres, which were first classified by Kodaira in [**5**], can be read off directly from the j-function of Y (cf. [**18**, IV.9, Table

4.1]). This is the quotient of $4A^3$ by the discriminant Δ of Y which is defined as $\Delta = 4A^3 + 27B^2$:

$$j = \frac{4A^3}{\Delta}.$$

Then Y has singular fibres above the zeroes of Δ which we call the cusps of Y. The fibre above such a cusp x_0 is called *semi-stable* if and only if it is a rational curve with a node or a cycle of n lines, i.e., of type I_n in Kodaira's notation, where n is the order of vanishing of Δ at x_0. Note that above a cusp x_0 there is a semi-stable fibre if and only if A does not vanish at x_0, i.e., if and only if it is not a common zero of A and B. On the other hand, we get either a nonreduced fibre (distinguished by an *) over x_0 if A and B both vanish at x_0 to the orders at least 2 and 3, respectively, or an additive fibre II, III or IV otherwise. One common property of the singular fibres is that in every case the vanishing order of Δ at the cusp x_0 equals the Euler number of the fibre above x_0. At this point recall the well-known facts that $H^1(Y, \mathcal{O}_Y) = 0$ and $p_g = \dim H^2(Y, \mathcal{O}_Y) = M - 1$, while the canonical divisor $K_Y = (M-2)\ F$ for a general fibre F (cf., e.g., [**7**, Lecture III]). Hence, Y is $K3$, respectively rational, if and only if $M = 2$, respectively $M = 1$. Since the Euler number of Y equals the sum of the Euler numbers of its (singular) fibres, which coincides with the degree $12M$ of Δ by the above considerations, we obtain that Y is a $K3$ surface if and only if its Euler number equals 24. (On the contrary, Y is a rational surface if and only if $e(Y) = 12$.)

Before discussing the effect of a base change on the elliptic surface Y, let us at first introduce the following notation: We say that a map $\pi : \mathbf{P}^1 \to \mathbf{P}^1$ has ramification index $(n_1, \dots, n_r)$ at $x_0 \in \mathbf{P}^1$ if x_0 has r pre-images under π with respective orders n_i, $i = 1, \dots, r$. The base change of Y by π is simply defined as the pull-back surface $X \xrightarrow{\pi \circ r} \mathbf{P}^1$, i.e., we substitute π into the Weierstrass equation and j-function of Y to obtain X. It is thus immediate that a semi-stable singular fibre of type I_n above a cusp x_0 of Y where π has ramification index $(n_1, \dots, n_r)$ is substituted via π by r fibres of types $I_{n_1 n}, \dots, I_{n_r n}$. However, for a non semi-stable singular fibre the substitution process turns out to be nontrivial (especially with respect to our purposes) for two reasons: On the one hand, the Weierstrass equation might simply lose its minimality by way of the substitution. On the other hand, the minimalized Weierstrass equation can also become *inflated.*

This expression is meant to describe that the pull-back surface has more than one nonreduced fibre. Then there is a quadratic twist of the surface, sending $x \mapsto \alpha^2 x$ and $y \mapsto \alpha^3 y$, which replaces an even number of nonreduced fibres whose cusps are just the zeros of α by their reduced relatives, i.e., I_n^* by I_n and II^*, III^*, IV^* by IV, III, II, respectively. Following [**8**] this process will be called *deflation*. Since our main interest lies in elliptic ($K3$) surfaces with only semi-stable fibres, this provides an appropriate tool to construct such surfaces.

Indeed, it is exactly these two methods (minimalization and deflation after a suitable base change) which we will use to resolve the non semi-stable fibres of the base surface Y. Since the explicit behavior of the singular fibres under a base change can directly be derived from [**18**, IV.9, Table 4.1] or be found in [**9**, Section 7], we only sketch it in the next figure where the number next to an arrow denotes the order of ramification under π (of one pre-image). Note that the fibres of type I_n^* are exceptional in that they allow two possibilities of substitution by semi-stable fibres: Either by ramification of even index or by the pairwise deflation process which was described in the last paragraph.

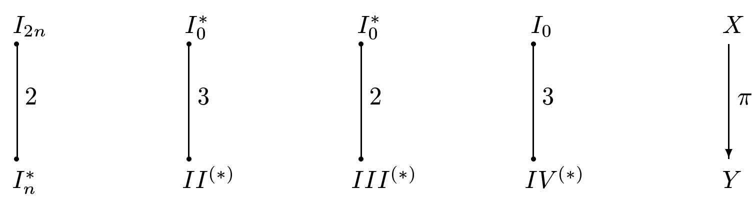

FIGURE 1. The resolution of the non semi-stable fibres.

3. The deflated base changes. Our main interest lies in finding equations over **Q** for extremal semi-stable elliptic $K3$ surfaces. By definition these are singular, i.e., the Picard number is maximal, equalling $h^{1,1}$, with finite Mordell-Weil group and only semi-stable fibres. These assumptions turn out to be quite restrictive. In fact, it is an immediate consequence that the number of cusps has to be 6, and we find the 112 possible configurations of singular fibres in the classification of [**10**, Theorem (3.1)]. The main idea of this paper is to produce some of these $K3$ fibrations by the methods described in the previous section via the pull-back of a rational elliptic surface by a base change. This approach is greatly helped by the good explicit knowledge one has of the rational elliptic surfaces (cf. [**3, 9, 14**]) such that one only has to construct suitable base changes.

The starting point for our considerations is a rational elliptic surface $Y \xrightarrow{r} \mathbf{P}^1$ with a section. It is natural to also assume Y to be extremal since its Mordell-Weil group $MW(Y)$ injects into the Mordell-Weil group of the pull-back $X \xrightarrow{\pi \circ r} \mathbf{P}^1$ via the base change $\pi : \mathbf{P}^1 \to \mathbf{P}^1$. Note, however, that by the general theory X is only guaranteed to be minimal. This has to be stressed since the process of deflation can a priori change the Mordell-Weil group. Hence it could also seem worth considering nonextremal rational elliptic surfaces, especially those with a small number of cusps as presented in [**3, 14**]. A close observation nevertheless shows that these would not produce any configurations different from those known or obtained in this paper, unless one takes π to have degree 24. Since this is equivalent to the general construction involving the J-map, making no effective use of the rational elliptic surface Y, we will skip this possibility here. See [**4**] for the only treatment of this method to date, where such a suitable degree 24 map is constructed giving rise to the configuration $[1, 1, 1, 1, 1, 19]$. With two further exceptions, all other members of the list of [**10**], which have to our knowledge been realized over $\mathbf{Q}$ until now, come from extremal rational elliptic surfaces after a base change so we turn to these now.

Extremal rational elliptic surfaces have been completely classified by Miranda-Persson in [**9**]. At first there are six semi-stable surfaces with four cusps which had previously been identified to be modular by Beauville [**2**]. As explained, these have been exhaustively treated in [**19**], giving rise to the semi-stable extremal elementary fibrations of [**13**]. Furthermore there are four surfaces with only two cusps (one of them appearing in a whole continuous family), which are of no use for our purpose since they have no fibre of type I_n or I_n^* with $n > 0$ at all. The remaining six extremal rational elliptic surfaces have three cusps and each exactly one nonreduced fibre while the other two singular fibres are semi-stable. These are the surfaces we are going to investigate for a pull-back via a base change. For the remaining part of this section we will content ourselves with those ("deflated") base changes π which give rise to a noninflated pull-back $K3$ surface

$X \stackrel{\pi_{or}}{\to} \mathbf{P}^1$ after minimalizing. The pull-back surfaces coming from inflating base changes will be dealt with in Section 6.

Given an extremal rational elliptic surface Y with three cusps there are a number of conditions on the base change π to be met (be it deflated or not). Evidently, the Euler number $e(X) = 24$ of the pull-back surface X predicts the degree of π, only depending on the type of the nonreduced fibre W^* of Y. On the one hand, if W is of additive type, i.e., $W \in \{II, III, IV\}$, we will eventually replace it by smooth fibres after minimalizing and deflating, if necessary. Hence, when the other two singular fibres are of type I_m and I_n with $m, n \in \mathbf{N}$, we have $m + n \leq 4$ and $e(X) = (\deg \pi)(m + n)$. On the other hand, if the nonreduced fibre has type I_k^* and the other singular fibres have again m and n components, we have $k + m + n = 6$ and thus $\deg \pi = 4$. Furthermore, our assumption for the pull-back X to have exactly the minimal number of six semi-stable fibres gives another sharp restriction, which will in some cases lead to a contradiction to the Hurwitz formula

$$-2 \geq -2 \deg \pi + \sum_{x \in \mathbf{P}^1} (\deg \pi - \# \pi^{-1}(x)).$$

Finally, as we decided to content ourselves to resolving the nonreduced fibre W^* by a deflated base change in this section, the ramification index at the corresponding cusp has to be divisible by 2, 3, 4, or 6 if $W = I_n, IV, III$, or II, respectively, by the last figure.

It will turn out that some of the base changes in question can only be defined over an extension of $\mathbf{Q}$ of low degree. Nevertheless, for any base change it will be immediate from the ramification at the two cusps of the semi-stable singular fibres that the pull-back surface X has at least two rational cusps. For simplicity and without loss of generality, we will choose these by Möbius transformation to be 0 and ∞ (and a further third rational cusp, if it exists, to be 1). Additionally we will be able to normalize the Weierstrass equations over $\mathbf{Q}$ for every surface such that the cusps are $0, 1$ and ∞ with the nonreduced fibre sitting over one arbitrary of these. This gives us the opportunity to construct quite generally the base changes of $\mathbf{P}^1$ which will eventually give rise to new equations of extremal elliptic $K3$ fibrations before getting too much into detail with the surfaces themselves. Our requirement that the pull-back of such a base change π should not factor through a modular rational elliptic surface is clearly equivalent to π not factoring into a composition $\pi'' \circ \pi'$ of a degree 2 map π'' and a further map π', such that the nonreduced fibre is already resolved by π'. This is because then the intermediate pull-back $X' \stackrel{\pi' \circ r}{\to} \mathbf{P}^1$ would be semi-stable and necessarily have at most 4 cusps, hence it would be modular by [**2**].

In what follows we investigate the existence and shape of deflated base changes with the above listed properties in a case-by-case analysis depending on the type of the nonreduced fibre W^* of the extremal rational elliptic surface Y with three cusps. Throughout we employ the notation of [**9**]. For the computations we wrote a straightforward Maple program which in all but two cases sufficed completely to determine the minimal base change. Meanwhile, for the remaining two, we decided to use Gröbner bases in Macaulay to compute a solution mod p for some primes p and then lift to characteristic 0.

At first assume $\mathbf{W}$ to be $\mathbf{II}$. According to [**9**, Section 5] there is up to isomorphism a unique such surface, named $\mathbf{X_{2\ 1\ 1}}$, whose other two singular fibres are both of type I_1. Hence, for the pull-back X to have Euler number $e(X) = 24$, we would need π to have degree 12 and ramification index 12 or $(6, 6)$ at the cusp of the nonreduced fibre. Then, the restriction of the other two cusps to have exactly six pre-images under π leads to a contradiction by the Hurwitz formula.

The situation is similar if $\mathbf{W} = \mathbf{III}$ which implies $Y = \mathbf{X_{3\ 2\ 1}}$ to have further singular fibres I_2, I_1. The III^* fibre requires ramification of index a multiple of 4, so π must have degree 8 with ramification index 8 or $(4, 4)$ at the cusp of this fibre. Again, the Hurwitz formula rules out the pull-back surface X to have only six (semi-stable) singular fibres. We will realize in the sixth section that it is nevertheless possible to construct adequate nondeflated base changes which eventually help resolve the nonreduced fibre and obey the Hurwitz formula.

Turning to $\mathbf{W} = \mathbf{IV}$ this gives a priori two possibilities for the elliptic surface Y, one of which, namely $\mathbf{X_{4\ 3\ 1}}$ with singular fibres of type IV^*, I_3, I_1, actually exists by [**9**]. Since a IV^* fibre requires the ramification index to be divisible by 3, we need π to have degree 6 and ramification of order $(3, 3)$ at the cusp of this fibre (since ramification index 6 would contradict the other two fibres having six pre-images again by the Hurwitz formula). The suitable maps are presented in the next paragraphs. Throughout, we assume the IV^* fibre to sit above 1. Since such an adequate map π, which was totally ramified above one of the two remaining cusps, would necessarily be composite as excluded above, we only have to deal with those maps such that 0 has two or three pre-images (and then exchange 0 and ∞).

At first let us consider those maps π of degree 6 which have ramification index $(3,3)$ at 1 such that both 0 and ∞ have three pre-images. Our restriction to maps not factoring into a degree 2 map after a degree 3 map is seen to imply at least one of the cusps to have ramification index $(3,2,1)$. Thus we assume ∞ to have this very ramification index and search for maps such that 0 has ramification index $(4,1,1)$, $(3,2,1)$ or $(2,2,2)$. However, a map with the last ramification cannot exist since the corresponding pull-back of $X_{4\ 3\ 1}$ does not appear in the list of [**10**]. Meanwhile the computations show that the second is only realizable over the cubic extension $\mathbf{Q}(x^3+12x-12)/\mathbf{Q}$. (With v a solution of $5x^3+12x^2+12x+4$ it can be given as $\tilde{\boldsymbol{\pi}}((s:t)) = (s^3(s-t)^2(s+(2+3v)t) : -(2+3v)t^3(s+(1+v)^2t)^2(s+t/(5v+2))).$) Hence, we have to content ourselves with the construction of the first base change:

Consider the map $\boldsymbol{\pi_{3,4}}$ given as

$$\begin{aligned} \boldsymbol{\pi_{3,4}}:\ \mathbf{P}^1 &\longrightarrow \mathbf{P}^1 \\ (s:t) &\longmapsto (27s^4(125t^2-90st-27s^2) : -3125t^3(t-s)^2(5t+4s)). \end{aligned}$$

We have $27s^4(125t^2-90st-27s^2)+3125t^3(t-s)^2(5t+4s) = (25t^2-10st-9s^2)^3$, so $\pi_{3,4}$ has the desired properties.

Now we come to those base changes π of degree 6 and ramification index $(3,3)$ at 1 such that 0 has only two pre-images and ∞ has four. Hence, the respective ramification indices have to be $(5,1)$, $(4,2)$ or $(3,3)$ and $(2,2,1,1)$ or $(3,1,1,1)$. Note that only the first and the last of the whole of these do not allow a compositum of a degree 3 and a degree 2 map, so at least one of these two must be at hand for a map of interest.

Let us first construct those maps with ramification index $(5,1)$ at 0. These are:

$$\begin{aligned} \boldsymbol{\pi_{5,3}}:\ \mathbf{P}^1 &\longrightarrow \mathbf{P}^1 \\ (s:t) &\longmapsto (729s^5(s-t) : -t^3(135s^3+9st^2+t^3)) \end{aligned}$$

with $729s^5(s-t)+t^3(135s^3+9st^2+t^3) = (9s^2-3st-t^2)^3$ and

$$\begin{aligned}\boldsymbol{\pi_{5,2}}: \;\; \mathbf{P}^1 &\longrightarrow \mathbf{P}^1\\ (s:t) &\longmapsto (2^6 3^3 s^5 t : -(s^2-4st-t^2)^2(125s^2+22st+t^2))\end{aligned}$$

with $2^6 3^3 s^5 t + (s^2-4st-t^2)^2(125s^2+22st+t^2) = (5s^2+10st+t^2)^3$.

The other nonfactoring maps require ramification index $(3,1,1,1)$ at ∞. However, it is immediate that there is no such map π with ramification index $(3,3)$ at 0: After exchanging 1 and ∞, the map π would have to look like $(f_0^3 g_0^3 : f_1^3 g_1^3)$ with distinct linear homogeneous factors f_i, g_i. Then, with ϱ a primitive third root of unity, $f_0^3 g_0^3 - f_1^3 g_1^3 = (f_0 g_0 - f_1 g_1)(f_0 g_0 - \varrho f_1 g_1)(f_0 g_0 - \varrho^2 f_1 g_1)$ could obviously not have a cubic factor. Hence the next map completes the list of suitable base changes for $X_{4\;3\;1}$, taking into account the permutation of 0 and ∞ via exchanging s and t:

$$\begin{aligned}\boldsymbol{\pi_{4,3}}: \;\; \mathbf{P}^1 &\longrightarrow \mathbf{P}^1\\ (s:t) &\longmapsto (729 s^4 t^2 : -(s-t)^3(8s^3+120s^2t-21st^2+t^3))\end{aligned}$$

with $729s^4t^2 + (s-t)^3(8s^3+120s^2t-21st^2+t^3) = (2s^2-8st-t^2)^3$.

We conclude this section by considering the nonreduced fibre $\mathbf{W}^*$ to equal $\mathbf{I_n^*}$ for some $n \geq 0$. By [**9**] there are three extremal rational elliptic surfaces with such a singular fibre. They have two further singular fibres, both semi-stable, and will be introduced in the next section. Independent of the surface, we have already seen that an adequate deflated base change π must have degree 4 and ramification of index $(2,2)$ or 4 at the cusp of the I_n^* fibre. In this setting, the condition on X to be extremal is equivalent to the two other cusps having 4 or 5 pre-images, respectively. Assuming the nonreduced fibre to sit over the cusp $\infty = (1:0)$, we now construct the suitable base changes π which do not factor into two maps of degree 2. By inspection, this property is equivalent to one of the cusps having two pre-images with ramification index $(3,1)$.

At first let us consider those base changes π which are totally ramified at ∞. By the above considerations, we have to assume the other two cusps to have 2 and 3 pre-images with ramification indices $(3,1)$ and (2,1,1). Up to exchanging them, for example, by

$$\phi : (s:t) \longmapsto (t-s:t),$$

the map π can be realized as

$$\begin{aligned}\boldsymbol{\pi_4}: \ \mathbf{P}^1 &\longrightarrow \mathbf{P}^1\\ (s:t) &\longmapsto (256s^3(s-t) : -27t^4),\end{aligned}$$

since $(256s^3(s-t)+27t^4) = (4s-3t)^2(16s^2+8st+3t^2)$.

We now come to the case where the ramification of π at ∞ has index $(2,2)$. Our restrictions imply ramification of index $(3,1)$ at one of the other two cusps (without loss of generality at 1). Then the last cusp also requires two pre-images and thus ramification of index $(3,1)$ or $(2,2)$. Before we conclude this section with a construction of a map π with the first ramification index, we sketch an argument why the second cannot exist: Given such a map it could be expressed as $(f_0^2g_0^2 : f_1^2g_1^2)$ with distinct homogeneous linear forms f_i, g_i in s,t. For the ramification index at 1, we would compute the difference $f_0^2g_0^2 - f_1^2g_1^2 = (f_0g_0+f_1g_1)(f_0g_0-f_1g_1)$ which obviously cannot have a cubic factor.

As announced, we conclude this section with a base change $\boldsymbol{\pi_2}$ of degree 4 which is ramified only at $0,1$ and ∞ with ramification indices $(3,1)$, $(3,1)$ and $(2,2)$, respectively:

$$\begin{aligned}\pi_2: \ \mathbf{P}^1 &\longrightarrow \mathbf{P}^1\\ (s:t) &\longmapsto (64s^3(s-t) : (8s^2-4st-t^2)^2).\end{aligned}$$

In the next two sections we will substitute the base changes $\pi_{3,4}$, $\pi_{5,3}$, $\pi_{5,2}$, $\pi_{4,3}$, π_2 and π_4 into the normalized Weierstrass equations of the extremal rational elliptic surfaces with three singular fibres in order to get equations over $\mathbf{Q}$ for extremal $K3$ surfaces with six singular fibres, all of which are semi-stable.

4. The equations coming from degree 4 maps. In this and the next section we proceed as follows to obtain equations for extremal $K3$ surface with six semi-stable fibres: Starting from the Weierstass equation given for the extremal rational elliptic surfaces in [**9**, Table 5.2] we apply the normalizing Möbius transformation which maps the cusps to 0, 1 and ∞ and then substitute s and t by the factors of the deflated base changes π_i constructed in the previous section. After minimalizing by an admissible change of variables this gives the desired

Weierstrass equations for 16 extremal elliptic $K3$ surfaces from the list of [**10**]. Throughout we choose the coefficients of the polynomials A and B involved in the Weierstrass equation to be minimal by rescaling, if necessary. Note that, for all semi-stable examples in the remainder of this paper, the pull-back surface X inherits a nontrivial section from the rational elliptic surface Y by construction. As a consequence, we are able to derive the isomorphism class of X (in terms of the intersection form on its transcendental lattice) from the classification in [**15**].

In this section we consider only the extremal rational elliptic surfaces with an $\mathbf{I_n^*}$ fibre, thus requiring a base change of degree 4. Before substituting by the base changes π_4 or π_2 we therefore choose the normalizing Möbius transformation in such a way that the I_n^* fibre sits above ∞.

Let us start with $\mathbf{X_{411}}$ which has Weierstrass equation

$$y^2 = x^3 - 3\,t^2(s^2 - 3t^2)\,x + s\,t^3(2s^2 - 9t^2)$$

locating an I_4^* fibre over ∞ and two I_1 fibres over ± 2. Substituting $(s:t) \mapsto (4s - 2t : t)$ maps the two I_1 fibres to 0 and 1, giving

$$y^2 = x^3 - 3\,t^2(16s^2 - 16st + t^2)\,x + 2\,t^3(2s - t)(32s^2 - 32st - t^2).$$

Finally, we substitute by $\boldsymbol{\pi_4}$ and get the Weierstrass equation of an extremal $K3$ surface:

$$\begin{aligned} y^2 = x^3 &- 3\,(9s^8 + 48s^7t + 48s^4t^4 + 64s^6t^2 + 128s^3t^5 + 16t^8)\,x \\ &- 2\,(3s^4 + 8s^3t + 8t^4) \\ &\quad (9s^8 + 48s^7t + 48s^4t^4 + 64s^6t^2 + 128s^3t^5 - 8t^8). \end{aligned}$$

This provides indeed a realization of the configuration **[1,1,1,2,3,16]** in the notation of [**10**], i.e., three fibres of type I_1, one of types I_2, I_3 and I_{16} each.

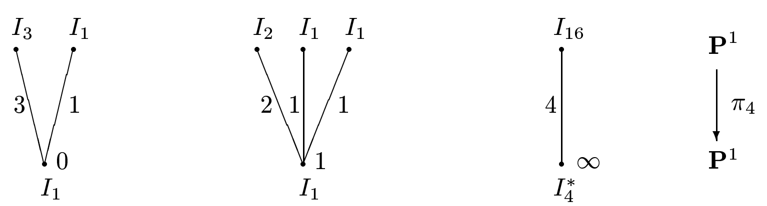

FIGURE 2. A realization of **[1,1,1,2,3,16]**.

On the other hand, we can also substitute by $\boldsymbol{\pi_2}$ in the normalized Weierstrass equation and obtain:

$$y^2 = x^3 - 3\,(16s^8 - 64s^7t - 224s^5t^3 + 392t^4s^4 + 64s^6t^2 + 112t^5s^3 + 16t^6s^2 + 8t^7s + t^8)\,x - 2\,(2s^2 - 4st - t^2)(2s^2 + t^2)(16s^8 - 64s^7t + 544s^5t^3 - 952t^4s^4 + 64s^6t^2 - 272t^5s^3 + 16t^6s^2 + 8t^7s + t^8)$$

which realizes [**1,1,3,3,8,8**].

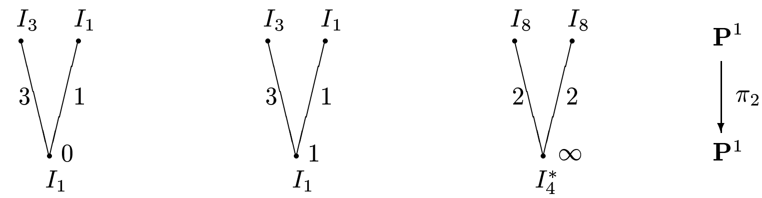

FIGURE 3. A realization of [**1,1,3,3,8,8**].

The same procedure applied to the surface $\mathbf{X_{141}}$ with Weierstrass equation

$$y^2 = x^3 - 3\,(s-2t)^2(s^2-3t^2)\,x + s(s-2t)^3(2s^2-9t^2)$$

gives three more examples, connected by the Möbius transformation ϕ :

The normalization of the cusps leads to the Weierstrass equation

$$y^2 = x^3 - 3\,t^2(16t^2 - 16st + s^2)\,x + 2\,t^3(s-2t)(s^2 + 32st - 32t^2)$$

which has the I_4 fibre sitting above 0. Substitution of $\boldsymbol{\pi_4}$ produces a realization of [**1,1,2,4,4,12**]:

$$y^2 = x^3 - 3\,(16t^8 + 48s^4t^4 - 64s^3t^5 + 9s^8 - 24s^7t + 16s^6t^2)\,x - 2\,(2t^4 + 3s^4 - 4s^3t)(-32t^8 - 96s^4t^4 + 128s^3t^5 + 9s^8 - 24s^7t + 16s^6t^2)$$

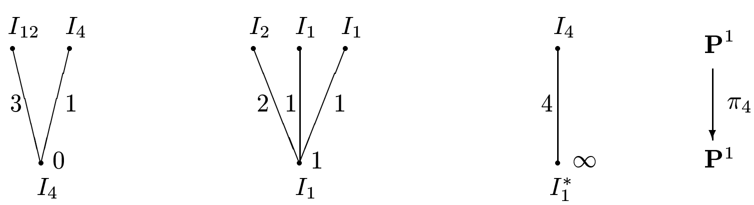

FIGURE 4. A realization of [**1,1,2,4,4,12**].

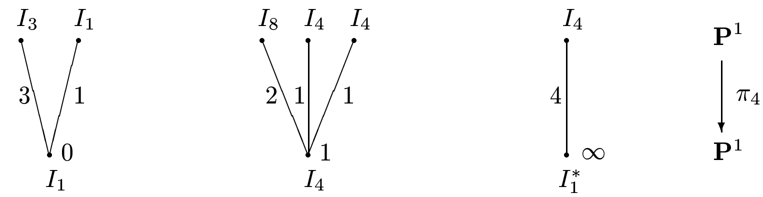

FIGURE 5. A realization of **[1,3,4,4,4,8]**.

Conjugation by ϕ implies the Weierstrass equation

$$y^2 = x^3 - 3\,t^2(t^2 + 14st + s^2)\,x - 2\,(t + s)t^3(t^2 - 34st + s^2)$$

which leads to the following realization of **[1,3,4,4,4,8]**

$$y^2 = x^3 + 3\,(-24s^7t - 16s^6t^2 - 9s^8 - t^8 + 42s^4t^4 + 56s^3t^5)\,x$$
$$- 2\,(9s^8 + 24s^7t + 16s^6t^2 + 102s^4t^4 + 136s^3t^5 + t^8)(-3s^4 - 4s^3t + t^4)$$

On the other hand, substitution of $\boldsymbol{\pi_2}$ in the normalized Weierstrass equation gives:

$$y^2 = x^3 - 3\,(s^8 - 4s^7t + 16s^5t^3 - 28t^4s^4 + 4s^6t^2 - 8t^5s^3 + 16s^2t^6 + 8st^7 + t^8)\,x$$
$$- (2s^4 - 4s^3t + 4st^3 + t^4)$$
$$(s^8 - 4s^7t - 32s^5t^3 + 56t^4s^4 + 4s^6t^2 + 16t^5s^3 - 32s^2t^6 - 16st^7 - 2t^8).$$

This is a realization of **[1,2,2,3,4,12]**, which is not changed by conjugation by ϕ.

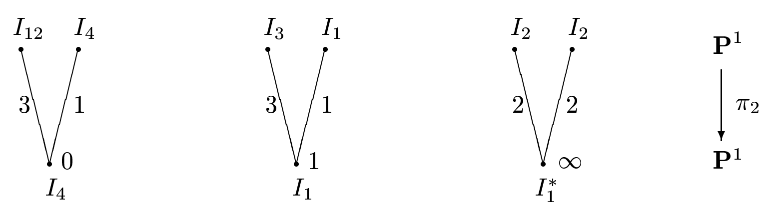

FIGURE 6. A realization of **[1,2,2,3,4,12]**.

Finally for this section of elliptic surfaces requiring a base change of degree 4, we turn to the surface $\mathbf{X_{2\ 2\ 2}}$ in the notation of [**9**]. Miranda-Persson give the Weierstrass equation

$$y^2 = x^3 - 3\,st(s-t)^2\,x + (s-t)^3(s^3 + t^3)$$

which has cusps the third roots of unity with an I_2^* fibre above 1 and I_2 fibres above the two primitive roots ω, ω^2. By mapping ω to ∞ and likewise ω^2 to 0 while fixing 1, taking ω in the upper half plane, we obtain a Weierstrass equation which is not defined over $\mathbf{Q}$. Nevertheless, the change of variables $x \mapsto \xi^2 x$, $y \mapsto \xi^3 y$ with $\xi = 3\sqrt{-3}$ leads to a Weierstrass equation over $\mathbf{Q}$ with the same cusps:

$$y^2 = x^3 - 3\,(s^2 - st + t^2)(s-t)^2 x + (s-2t)(2s-t)(t+s)(s-t)^3.$$

We exchange the cusps 1 and ∞ and subsequently substitute by $\boldsymbol{\pi_4}$ or $\boldsymbol{\pi_2}$. The first substitution produces a realization of **[2,2,2,4,6,8]**:

$$\begin{aligned} y^2 = x^3 &- 3\,(9s^8 - 24s^7t + 16s^6t^2 + 3s^4t^4 - 4s^3t^5 + t^8)\,x \\ &- (2t^4 + 3s^4 - 4s^3t)(3s^4 - 4s^3t - t^4)(6s^4 - 8s^3t + t^4) \end{aligned}$$

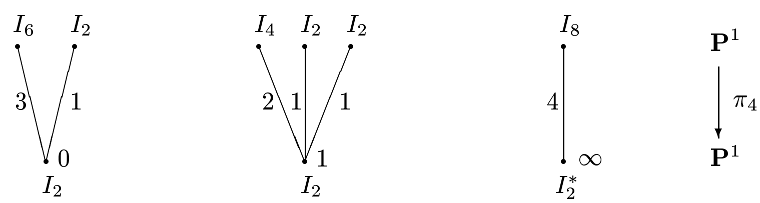

FIGURE 7. A realization of **[2,2,2,4,6,8]**.

Meanwhile the second realizes **[2,2,4,4,6,6]**:

$$\begin{aligned} y^2 = x^3 &- 3\,(16s^8 - 64ts^7 + 64t^2s^6 + 16t^3s^5 - 28t^4s^4 - 8t^5s^3 + 16t^6s^2 + 8t^7s + t^8)\,x \\ &+ 2\,(2s^2 - 4st - t^2)(8s^4 - 16s^3t + 4st^3 + t^4)(t^2 + 2s^2)(2s^4 - 4s^3t + 4st^3 + t^4). \end{aligned}$$

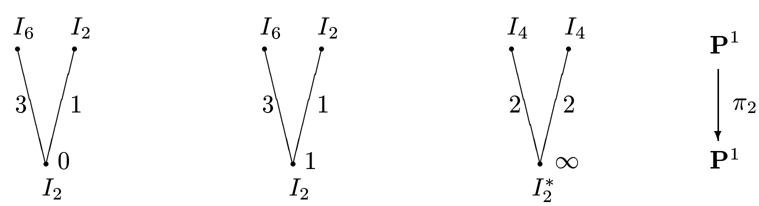

FIGURE 8. A realization of **[2,2,4,4,6,6]**.

5. The equations coming from degree 6 maps. The extremal rational elliptic surface $X_{4\ 3\ 1}$ gives rise to 9 extremal elliptic $K3$ surfaces, 8 of which can be realized over $\mathbf{Q}$ as pull-back by a base change of $\mathbf{P}^1$. They are presented in the following:

We modify the Weierstrass equation of $\mathbf{X_{4\ 3\ 1}}$ given in [**9**] by exchanging 1 and ∞ such that it becomes

$$y^2 = x^3 - 3\,(s-t)^3(s-9t)\,x - 2(s-t)^4(s^2+18st-27t^2)$$

with the I_3 fibre above 0 and the I_1 fibre above ∞ while the IV^* fibre sits above 1. Thus, we can resolve the nonreduced fibre by substituting one of the degree 6 base changes of the third section into the normalized Weierstrass equation of $X_{4\ 3\ 1}$ (which is also possible after permuting 0 and ∞).

At first let us look at the effect of $\boldsymbol{\pi_{3,4}}$. This leads to the Weierstrass equation

$$\begin{aligned} y^2 = x^3 &- 3\,(-15s^4t^2+54s^5t+81s^6+15s^2t^4-100s^3t^3+6st^5-t^6)(9s^2+2st-t^2)x \\ &- 2\,(19683s^{12}+26244s^{11}t+1458s^{10}t^2+43740s^9t^3+25785s^8t^4-16776s^7t^5 \\ &\quad -10108s^6t^6+3864s^5t^7+885s^4t^8-380s^3t^9-6s^2t^{10}+12st^{11}-t^{12}) \end{aligned}$$

which realizes [**1,2,3,3,3,12**].

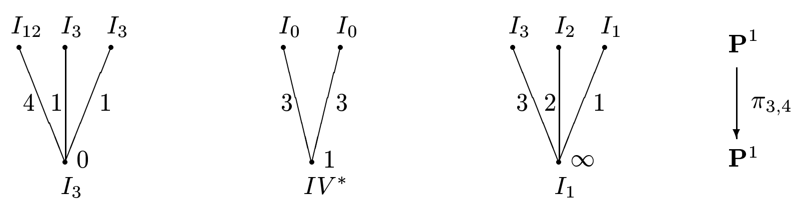

FIGURE 9. A realization of [**1,2,3,3,3,12**].

Permuting 0 and ∞ before the substitution gives a realization of [**1,1,3,4,6,9**]:

$$\begin{aligned} y^2 = x^3 &- 3\,(-t^6+6st^5+15s^2t^4-100s^3t^3-1215s^4t^2+4374s^5t+6561s^6) \\ &(9s^2+2st-t^2)\,x \\ &+ 2\,(14348907s^{12}+19131876s^{11}t+1062882s^{10}t^2-4855140s^9t^3 \\ &\quad -185895s^8t^4+452952s^7t^5-7084s^6t^6-20328s^5t^7 \\ &\quad +3405s^4t^8-380s^3t^9-6s^2t^{10}+12st^{11}-t^{12}). \end{aligned}$$

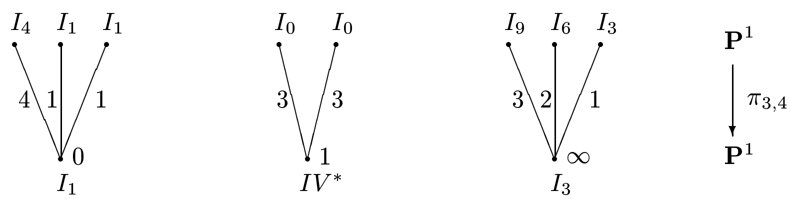

FIGURE 10. A realization of **[1,1,3,4,6,9]**.

We now turn to the substitutions by $\boldsymbol{\pi_{5,3}}$. These provide the following realizations of **[1,1,1,3,3,15]** as

$$\begin{aligned} y^2 = x^3 &- 3\,(s^2 - ts - t^2)(s^6 - 3s^5t + 45t^3s^3 - 27t^5s - 9t^6)\,x \\ &+ 2\,(-s^{12} + 6s^{11}t - 9s^{10}t^2 + 90s^9t^3 - 270t^4s^8 - 54t^5s^7 + 819t^6s^6 \\ &\quad + 54t^7s^5 - 810t^8s^4 - 270t^9s^3 + 243t^{10}s^2 + 162st^{11} + 27t^{12}); \end{aligned}$$

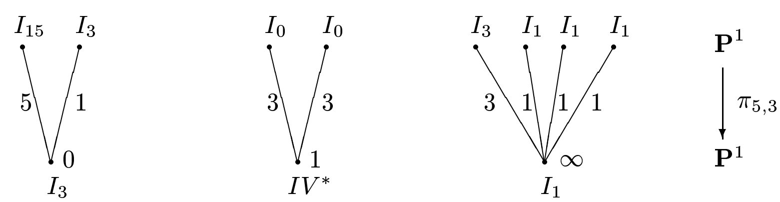

FIGURE 11. A realization of **[1,1,1,3,3,15]**.

and of **[1,3,3,3,5,9]** as

$$\begin{aligned} y^2 = x^3 &- 3,(s^2 - st - t^2)(5s^3t^3 - 3st^5 - t^6 + 9s^6 - 27s^5t)\,x \\ &+ 2\,(27s^{12} - 162s^{11}t + 243s^{10}t^2 + 90s^9t^3 - 270s^8t^4 - 54s^7t^5 \\ &\quad + 119s^6t^6 + 54s^5t^7 + 30s^4t^8 + 10s^3t^9 - 9s^2t^{10} - 6st^{11} - t^{12}). \end{aligned}$$

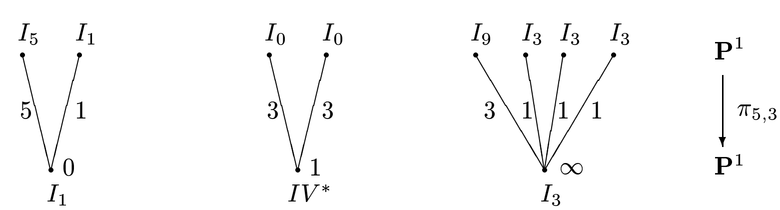

FIGURE 12. A realization of **[1,3,3,3,5,9]**.

Similarly, substitution by $\boldsymbol{\pi_{5,2}}$ allows us to realize **[1,1,2,2,3,15]** by virtue of the Weierstrass equation

$$\begin{aligned} y^2 = x^3 &- 3\,(125s^6 - 786s^5t + 1575s^4t^2 + 1300s^3t^3 + 315s^2t^4 + 30st^5 + t^6)\\ &(5s^2 + 10st + t^2)\,x\\ &+ 2\,(15625s^{12} + 112986s^{10}t^2 - 100500s^{11}t - 941300s^9t^3\\ &\quad + 1514175s^8t^4 + 3849240s^7t^5 + 2658380s^6t^6 + 912696s^5t^7\\ &\quad + t^{12} + 180375s^4t^8 + 21500s^3t^9 + 1530s^2t^{10} + 60st^{11}) \end{aligned}$$

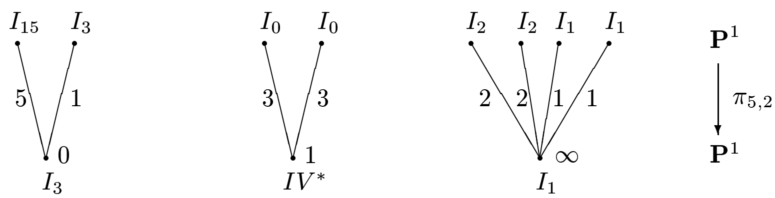

FIGURE 13. A realization of **[1,1,2,2,3,15]**.

and **[1,3,3,5,6,6]** by

$$\begin{aligned} y^2 = x^3 &- 3\,(125s^6 + 14574s^5t + 1575s^4t^2 + 1300s^3t^3 + 315s^2t^4 + 30st^5 + t^6)\\ &(5s^2 + 10st + t^2)\,x\\ &- 2\,(15625s^{12} - 4132500s^{11}t - 48851622s^{10}t^2 - 51744500s^9t^3\\ &\quad - 40418625s^8t^4 - 6311400s^7t^5 + 1690700s^6t^6 + 880440s^5t^7\\ &\quad + 180375s^4t^8 + 21500s^3t^9 + 1530s^2t^{10} + 60st^{11} + t^{12}). \end{aligned}$$

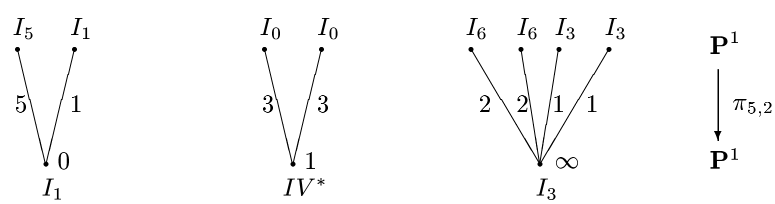

FIGURE 14. A realization of **[1,3,3,5,6,6]**.

Furthermore, we can also substitute by $\boldsymbol{\pi_{4,3}}$ and produce Weierstrass equations for **[1,1,1,3,6,12]** as

$$\begin{aligned} y^2 = x^3 &- 3\,(-276s^4t^2 + 8s^6 + 96s^5t + 416s^3t^3 - 186s^2t^4 + 24st^5 - t^6) \\ &(2s^2 + 8st - t^2)\,x \\ &+ 2\,(11160s^8t^4 + 7392s^{10}t^2 - 15232s^9t^3 - 130176s^7t^5 \\ &\quad + 220056s^6t^6 - 160416s^5t^7 + 54792s^4t^8 + 64s^{12} \\ &\quad + 1536s^{11}t - 9760s^3t^9 + 948s^2t^{10} - 48st^{11} + t^{12}) \end{aligned}$$

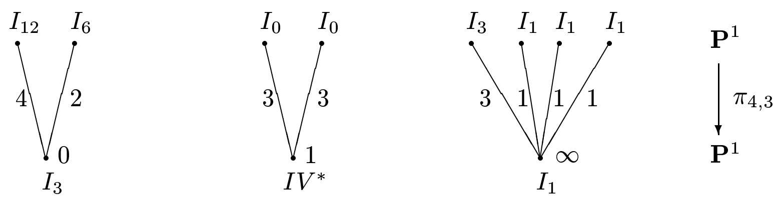

FIGURE 15. A realization of **[1,1,1,3,6,12]**.

and for **[2,3,3,3,4,9]** as

$$\begin{aligned} y^2 = x^3 &- 3\,(8s^6 + 96s^5t + 6204s^4t^2 + 416s^3t^3 - 186s^2t^4 + 24st^5 - t^6) \\ &(2s^2 + 8st - t^2)\,x \\ &- 2\,(68400s^4t^8 + 64s^{12} - 101472s^{10}t^2 + 1536s^{11}t \\ &\quad + 2751144s^6t^6 - 1321600s^9t^3 - 9460008s^8t^4 - 5791104s^7t^5 \\ &\quad - 487008s^5t^7 - 9760s^3t^9 + 948s^2t^{10} - 48st^{11} + t^{12}). \end{aligned}$$

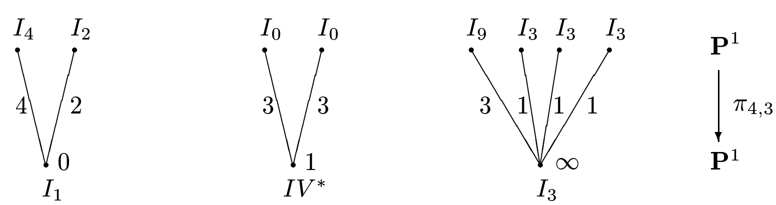

FIGURE 16. A realization of **[2,3,3,3,4,9]**.

Let us finally remark that substitution by $\tilde{\pi}$ allows us to realize the configuration [**1,2,3,3,6,9**] over the number field $\mathbf{Q}(x^3+12x-x)$, but we will not give the equations here.

6. The inflating base changes and the resulting equations. As announced, we now turn to the other possibility to resolve nonreduced fibres in the pull-back surface X of an extremal rational elliptic surface Y under a base change π. In this case we allow π to be inflating, i.e., X may contain nonreduced fibres of type I_n^* for $n \geq 0$ apart from its semi-stable fibres. The only additional assumption to be made is that their number is even, since then we can substitute them by their semi-stable reduced relatives after a quadratic twist of X via deflation.

Although one should hope to produce a number of new configurations by this method, a close inspection shows that in fact none arise from the extremal rational elliptic surfaces considered in the last two sections. Nevertheless this approach is indeed quite useful, since it enables us to work with the extremal rational elliptic surface $\mathbf{X_{3\ 2\ 1}}$ as well. The reason is that the Hurwitz formula is not violated if we choose the degree 8 base change π of $\mathbf{P}^1$ to have ramification index $(2,2,2,2)$ at the cusp of the III^* fibre (instead of being divisible by 4 before). The fibre is thus replaced by four fibres of type I_0^* which can easily be twisted away. Therefore, we have to assume the other two cusps to have six pre-images in total. Indeed there are (up to exchanging the cusps 0 and ∞) 13 such base changes which do not allow a factorization through an extremal rational elliptic surface. In the following we concentrate on the 9 of these which can be defined over $\mathbf{Q}$ and present the resulting 17 further extremal semi-stable elliptic $K3$ surfaces which arise from them by pull-back from X_{321}. The four base changes which are not defined over $\mathbf{Q}$ will be shortly sketched at the end of this section.

Remember that $\mathbf{X_{3\ 2\ 1}}$ has singular fibres of type III^*, I_2 and I_1. We normalize the Weierstrass equation for $X_{3\ 2\ 1}$ given in [**9**] such that the III^* fibre sits above 1 and the I_2 and I_1 fibre above 0 and ∞, respectively:

$$y^2 = x^3 - 3\,(s-t)^3(s-4t)\,x - 2\,(s-t)^5(s+8t).$$

Hence, in the following we investigate the degree 8 base changes of $\mathbf{P}^1$ with ramification index $(2,2,2,2)$ at 1 such that the further cusps 0 and ∞ have six pre-images in total. We will only concentrate on those adequate base changes which give rise to new examples of extremal elliptic $K3$ fibrations, i.e., the configuration of singular fibres has not been realized over $\mathbf{Q}$ yet, and assume without loss of generality that 0 does not have more pre-images than ∞ (since we can exchange these afterwards). By inspection, the three base changes which are totally ramified at 0 realize known constellations of singular fibres. So, we now turn to the base changes π such that 0 has two pre-images under π and list them by ramification indices at 0.

There are a priori four base changes with ramification index **(7,1)** at 0, since ramification $(2, 2, 2, 2)$ at both other cusps, 1 and ∞, would contradict high ramification at the third cusp by the before mentioned considerations. However, the computations show that a base change with ramification index $(4, 2, 1, 1)$ at ∞ can only be defined over the quadratic extension $\mathbf{Q}(\sqrt{-7})$. Hence we content ourselves with the remaining three base changes for the moment:

The first has ramification $(5, 1, 1, 1)$ at ∞ and can be given as

$$\begin{aligned} \boldsymbol{\pi}: \quad \mathbf{P}^1 \quad &\longrightarrow \quad \mathbf{P}^1 \\ (s:t) &\longmapsto (s^7(s-4t) : -4t^5(14s^3+14ts^2+20t^2s+25t^3)) \end{aligned}$$

since

$$\begin{aligned} s^7(s-4t) &+ 4t^5(14s^3+14ts^2+20t^2s+25t^3) \\ &= (-10t^4-4st^3-2s^2t^2+2s^3t+s^4)^2. \end{aligned}$$

Substituting π into the normalized Weierstrass equation of $X_{3\ 2\ 1}$ gives, after an appropriate change of variables, a realization of **[1,1,1,2,5,14]**:

$$\begin{aligned} y^2 = x^3 &- 3\,(16s^8+32s^7t-112t^5s^3+56t^6s^2-40t^7s+25t^8)\,x \\ &- 2\,(-5t^4+4t^3s-4s^2t^2+8s^3t+8s^4) \\ &(8s^8+16s^7t+112t^5s^3-56t^6s^2+40t^7s-25t^8). \end{aligned}$$

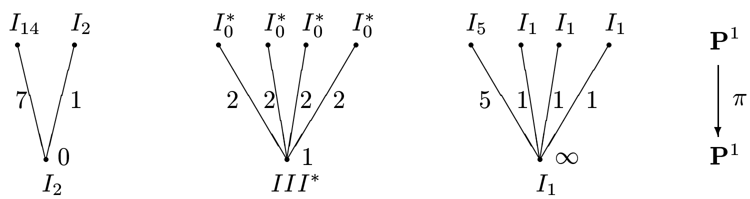

FIGURE 17. A realization of [**1,1,1,2,5,14**].

Furthermore, permuting the cusps 0 and ∞ before the substitution leads to the following realization of [**1,2,2,2,7,10**]:

$$\begin{aligned} y^2 = x^3 - 3\,(-14t^5s^3 + 14t^6s^2 - 20t^7s + 25t^8 + s^8 + 4s^7t)\,x \\ + (-10t^4 + 4t^3s - 2s^2t^2 + 2s^3t + s^4) \\ (14t^5s^3 - 14t^6s^2 + 20t^7s - 25t^8 + 2s^8 + 8s^7t) \end{aligned}$$

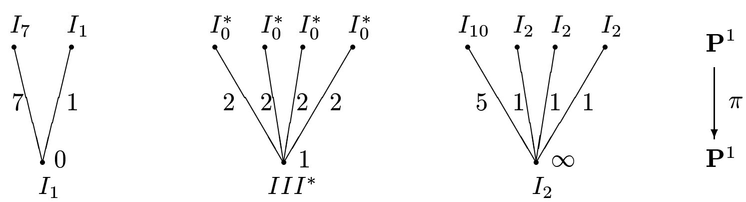

FIGURE 18. A realization of [**1,2,2,2,7,10**].

The second base change, this time with ramification index (3,3,1,1) at ∞, can be chosen as

$$\begin{aligned} \boldsymbol{\pi}: \quad & \mathbf{P}^1 \longrightarrow \mathbf{P}^1 \\ & (s:t) \longmapsto (1728s^7t : -(s^2 - 35st + 49t^2)^3(7s^2 - 13st + 7t^2)) \end{aligned}$$

since we have

$$\begin{aligned} 1728s^7t + (s^2 + 48t) - 12544t^3(7s^2 - 28st + 24t^2)^2(s-t) \\ = (384t^4 - 640t^3s + 288s^2t^2 - 24s^3t - s^4)^2. \end{aligned}$$

As pull-back of X_{321} via π we realize the constellations **[1,2,2,2,3,14]**:

$$\begin{aligned} y^2 = x^3 - 3\,(49s^8 - 316s^7t + 4018s^6t^2 - 8624s^5t^3 + 5915s^4t^4 - 1904s^3t^5 \\ + 322s^2t^6 - 28st^7 + t^8)\,x \\ + 2\,(49s^8 - 964s^7t + 4018s^6t^2 - 8624s^5t^3 + 5915s^4t^4 - 1904s^3t^5 \\ + 322s^2t^6 - 28st^7 + t^8)(t^4 - 14st^3 + 63s^2t^2 - 70s^3t - 7s^4). \end{aligned}$$

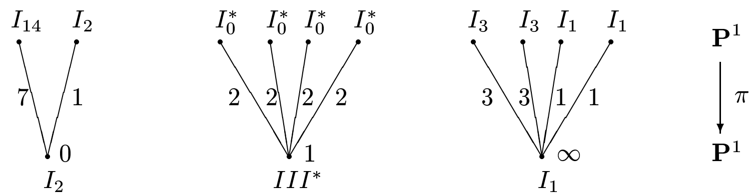

FIGURE 19. A realization of **[1,1,2,3,3,14]**.

and **[1,2,2,6,6,7]**:

$$\begin{aligned} y^2 = x^3 - 3\,(49s^8 + 6164s^7t + 4018s^6t^2 - 8624s^5t^3 \\ + 5915s^4t^4 - 1904s^3t^5 + 322s^2t^6 - 28st^7 + t^8)\,x \\ - 2\,(49s^8 - 14572s^7t + 4018s^6t^2 - 8624s^5t^3 \\ + 5915s^4t^4 - 1904s^3t^5 + 322s^2t^6 - 28st^7 + t^8) \\ (t^4 - 14st^3 + 63s^2t^2 - 70s^3t - 7s^4). \end{aligned}$$

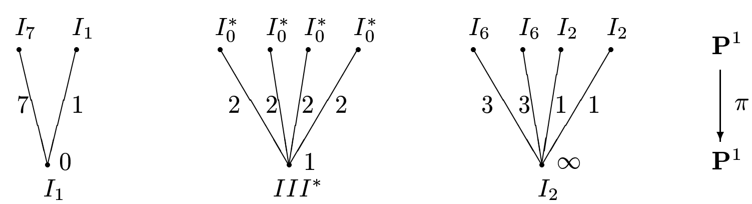

FIGURE 20. A realization of **[1,2,2,6,6,7]**.

At this point, we should recall that all our pull-back surfaces inherit the group of sections of the rational elliptic surfaces. As a consequence the above fibration of $[1,1,2,3,3,14]$ necessarily has Mordell-Weil group $\mathbf{Z}/(2)$. Hence, it differs substantially from the surface with the same configuration and $MW = (0)$, as obtained as a double sextic over $\mathbf{Q}$ in [**1**, p. 55].

For the third base change, which has ramification index $(3, 2, 2, 1)$ at ∞, we consider the map

$$\begin{aligned} \boldsymbol{\pi}: \quad \mathbf{P}^1 &\longrightarrow \mathbf{P}^1 \\ (s:t) &\longmapsto (s^7(s+48t) : 256t^3(7s^2-28st+24t^2)^2(s-t)) \end{aligned}$$

with

$$\begin{aligned} s^7(s+48t) - 12544t^3(7s^2-28st+24t^2)^2(s-t) \\ = (384t^4 - 640t^3s + 288s^2t^2 - 24s^3t - s^4)^2. \end{aligned}$$

The pull-back via π gives rise to the constellations [**1,2,2,2,3,14**]:

$$\begin{aligned} y^2 = x^3 - 3\,(s^8 + 12s^7t - 784t^3s^5 + 1764t^4s^4 - 1512t^5s^3 + 616t^6s^2 \\ - 120t^7s + 9t^8)\,x \\ + (-s^8 - 12s^7t - 1568t^3s^5 + 3528t^4s^4 - 3024t^5s^3 + 1232t^6s^2 \\ - 240t^7s + 18t^8)(-2s^4 - 12s^3t + 36s^2t^2 - 20st^3 + 3t^4). \end{aligned}$$

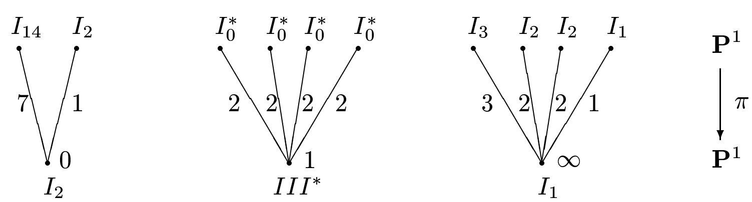

FIGURE 21. A realization of [**1,2,2,2,3,14**].

and [**1,2,4,4,6,7**]:

$$\begin{aligned} y^2 = x^3 - 3\,(s^8 - 392s^5t^3 + 1764s^4t^4 - 3024s^3t^5 + 2464s^2t^6 - 960st^7 \\ + 144t^8 + 24s^7t)\,x \\ + 2\,(-24t^4 + 80st^3 - 72s^2t^2 + 12s^3t + s^4) \\ (196s^5t^3 - 882s^4t^4 + 1512s^3t^5 - 1232s^2t^6 + 480st^7 - 72t^8 + s^8 + 24s^7t). \end{aligned}$$

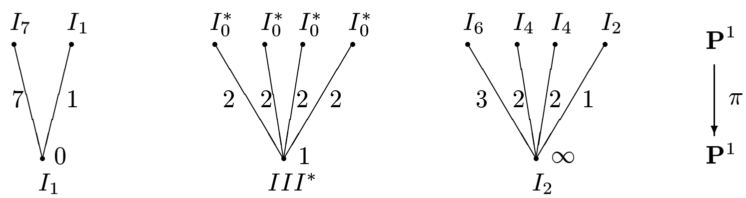

FIGURE 22. A realization of [**1,2,4,4,6,7**].

Turning to base changes with ramification index **(6,2)** at 0, we immediately conclude that all possible maps either do not exist, e.g., since the configuration of singular fibres resulting from the pull-back does not meet the criteria of [**10**] or realize configurations known from [**19**] or the previous sections. Since the situation is exactly the same for ramification index **(4,4)** at 0, the only other ramification index at 0 which we have to deal with at this point is **(5,3)**. Again we can exclude the ramification index at ∞ to be $(2,2,2,2)$ or $(3,3,1,1)$ by the above considerations, so we have three possibilities left.

For ramification index $(5,1,1,1)$ at ∞ consider the map

$$\begin{aligned} \boldsymbol{\pi}: \quad \mathbf{P}^1 &\longrightarrow \mathbf{P}^1 \\ (s:t) &\longmapsto (2^{14}s^5(s-t)^3 : t^5(2^7 3 s^3 - 2^5 11 s^2 t - 48 s t^2 - 9t^3)) \end{aligned}$$

with

$$\begin{aligned} (2^{14}s^5(s-t)^3 - t^5(2^7 3 s^3 - 2^5 11 s^2 t - 48 s t^2 - 9t^3)) \\ = (3t^4 + 8st^3 + 48s^2t^2 - 192s^3t + 128s^4)^2. \end{aligned}$$

This base change realizes [**1,1,1,5,6,10**]:

$$\begin{aligned} y^2 = x^3 - 3\,(16s^8 - 96s^7t + 192s^6t^2 - 128s^5t^3 - 48s^3t^5 + 88s^2t^6 + 24st^7 + 9t^8)\,x \\ - 2\,(3t^4 + 4st^3 + 12s^2t^2 - 24s^3t + 8s^4) \\ (8s^8 - 48s^7t + 96s^6t^2 - 64s^5t^3 + 48s^3t^5 - 88s^2t^6 - 24st^7 - 9t^8) \end{aligned}$$

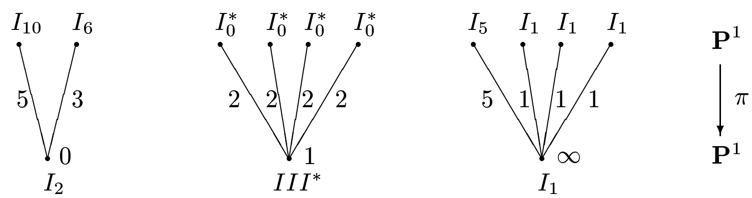

FIGURE 23. A realization of [**1,1,1,5,6,10**].

and, after exchanging 0 and ∞, also [**2,2,2,3,5,10**]:

$$y^2 = x^3 - 3\,(s^8 - 6s^3t^5 + 22s^2t^6 + 12st^7 + 9t^8 - 12s^7t + 48s^6t^2 - 64s^5t^3)\,x \\ + (6t^4 + 4st^3 + 6s^2t^2 - 6s^3t + s^4) \\ (6s^3t^5 - 22s^2t^6 - 12st^7 - 9t^8 + 2s^8 - 24s^7t + 96s^6t^2 - 128s^5t^3).$$

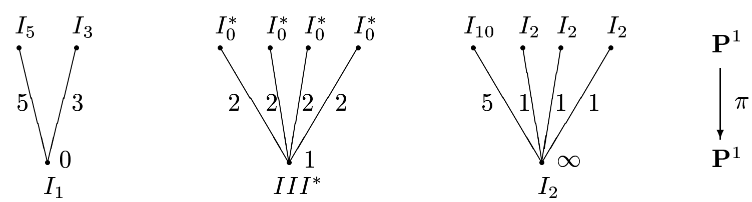

FIGURE 24. A realization of [**2,2,2,3,5,10**].

The ramification index $(4,2,1,1)$ at ∞ can be obtained by the following base change:

$$\begin{aligned} \boldsymbol{\pi}: \quad \mathbf{P}^1 &\longrightarrow \mathbf{P}^1 \\ (s:t) &\longmapsto (2^{22}s^5(s-t)^3 : t^4(24s+t)^2(320s^2 - 2^4 21st - 9t^2)) \end{aligned}$$

with

$$2^{22}s^5(s-t)^3 - t^4(24s+t)^2(320s^2 - 2^4 21st - 9t^2) \\ = (2^{11}s^4 - 3072s^3t + 768s^2t^2 + 128st^3 + 3t^4)^2.$$

This realizes [**1,1,2,4,6,10**]:

$$\begin{aligned} y^2 = x^3 - 3\,(16s^8 - 192s^7t + 768s^6t^2 - 1024s^5t^3 - 720s^4t^4 \\ + 2784s^3t^5 + 1312s^2t^6 + 192st^7 + 9t^8)\,x \\ - 2\,(8s^4 - 48s^3t + 48s^2t^2 + 32st^3 + 3t^4) \\ (8s^8 - 96s^7t + 384s^6t^2 - 512s^5t^3 + 720s^4t^4 \\ - 2784s^3t^5 - 1312s^2t^6 - 192st^7 - 9t^8). \end{aligned}$$

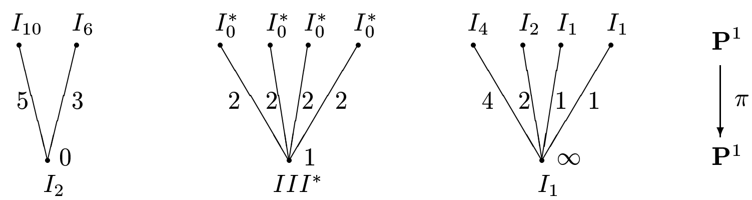

FIGURE 25. A realization of [**1,1,2,4,6,10**].

In this case the permutation of 0 and ∞ leads to the constellation [**2,2,3,4,5,8**]:

$$\begin{aligned} y^2 = x^3 - 3\,(-45s^4t^4 + 348s^3t^5 + 328s^2t^6 + 96st^7 + 9t^8 + s^8 - 24s^7t \\ + 192s^6t^2 - 512s^5t^3)\,x + (s^4 - 12s^3t + 24s^2t^2 + 32st^3 + 6t^4) \\ (45s^4t^4 - 348s^3t^5 - 328s^2t^6 - 96st^7 - 9t^8 + 2s^8 - 48s^7t + 384s^6t^2 - 1024s^5t^3) \end{aligned}$$

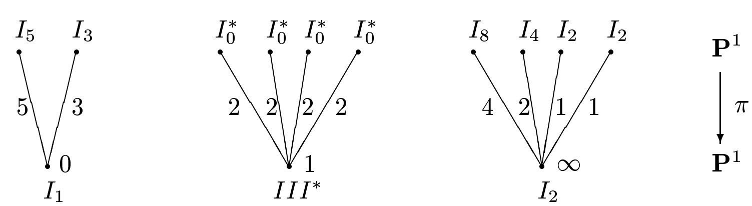

FIGURE 26. A realization of [**2,2,3,4,5,8**].

The final possible ramification index at ∞ is $(3,2,2,1)$ which is encoded in the following base change:

$$\begin{aligned} \boldsymbol{\pi} :\ \ \mathbf{P}^1 &\longrightarrow \mathbf{P}^1 \\ (s:t) &\longmapsto (2^{18}s^5(s-t)^3 : -t^3(16s+9t)(160s^2 - 2^3 27st + 81t^2)^2) \end{aligned}$$

with

$$2^{18}s^5(s-t)^3 + t^3(16s+9t)(160s^2 - 2^3 27st + 81t^2)^2 \\ = (512s^4 - 768s^3t + 192s^2t^2 + 432st^3 - 243t^4)^2.$$

This base change enables us to realize [**1,2,2,3,6,10**]:

$$y^2 = x^3 - 3\,(9s^8 - 36s^7t + 48s^6t^2 + 112s^5t^3 - 380s^4t^4 + 312s^3t^5 + 72s^2t^6 \\ - 216st^7 + 81t^8)\,x + (-6s^4 + 12s^3t - 4s^2t^2 - 12st^3 + 9t^4) \\ (-9s^8 + 36s^7t - 48s^6t^2 + 288s^5t^3 - 760s^4t^4 + 624s^3t^5 \\ + 144s^2t^6 - 432st^7 + 162t^8)$$

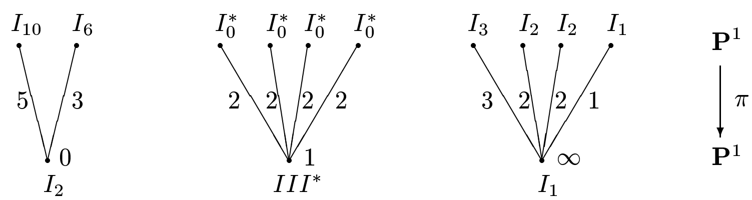

FIGURE 27. A realization of [**1,2,2,3,6,10**].

and furthermore [**2,3,4,4,5,6**]:

$$y^2 = x^3 - 3\,(-208s^5t^3 - 380s^4t^4 + 312s^3t^5 + 72s^2t^6 - 216st^7 \\ + 81t^8 + 144s^8 - 576s^7t + 768s^6t^2)\,x \\ - 2\,(-6s^4 + 12s^3t - 4s^2t^2 - 12st^3 + 9t^4) \\ (816s^5t^3 - 380s^4t^4 + 312s^3t^5 + 72s^2t^6 - 216st^7 \\ + 81t^8 - 288s^8 + 1152s^7t - 1536s^6t^2).$$

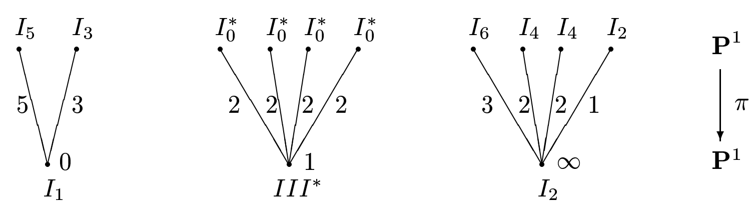

FIGURE 28. A realization of [**2,3,4,4,5,6**].

We end this part by investigating the base changes of $\mathbf{P}^1$ such that both cusps, 0 and ∞ have three pre-images. As a starting point we take ramification index **(6,1,1)** at 0. Of the three ramification indices at ∞ which prevent the base change from factorization one, namely $(4,2,2)$, cannot occur while, for another, $(4,3,1)$, a corresponding map can only be defined over the number field $\mathbf{Q}(\sqrt{-3})$. So there is only one base change, with ramification index $(5,2,1)$ at ∞, remaining to construct. It can be given as

$$\begin{aligned} \boldsymbol{\pi}: \quad \mathbf{P}^1 \quad &\longrightarrow \quad \mathbf{P}^1 \\ (s:t) &\longmapsto (4s^6(9s^2+24st+70t^2) : t^5(14s-5t)^2(4s-t)) \end{aligned}$$

with

$$\begin{aligned} 4s^6(9s^2+24st+70t^2) - t^5(14s-5t)^2(4s-t) \\ = (5t^4 - 24t^3s + 18s^2t^2 + 8s^3t + 6s^4)^2. \end{aligned}$$

The pull-back surface has the singular fibres **[1,2,2,2,5,12]**:

$$\begin{aligned} y^2 = x^3 &- 3\,(9s^8 + 24s^7t + 70s^6t^2 - 784t^5s^3 + 756t^6s^2 - 240t^7s + 25t^8)\,x \\ &+ (50t^8 2 - 9s^8 - 24s^7t - 70s^6t^2 - 1568t^5s^3 + 1512t^6s^2 - 480t^7s) \\ &(5t^4 - 24st^3 + 18s^2t^2 + 8s^3t + 6s^4) \end{aligned}$$

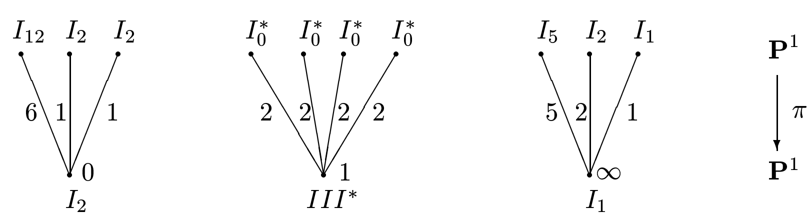

FIGURE 29. A realization of **[1,2,2,2,5,12]**.

As permuting 0 and ∞ gives rise to the constellation **[1,1,2,4,6,10]** which we already realized in the preceding paragraphs where 0 was assumed to have only two pre-images, we now come to base changes with ramification index **(5,2,1)** at 0. Again, the other cusp ∞ can be excluded to have ramification index $(4,2,2)$. Furthermore, the base change with ramification index $(5,2,1)$ at ∞ can only be defined over the number field $\mathbf{Q}(7x^3+19x^2+16x+8)$. So, we only construct the remaining base changes with ramification index $(4,3,1)$ or $(3,3,2)$ at ∞. For the first, consider the map

$$\begin{aligned}\boldsymbol{\pi}: \quad \mathbf{P}^1 &\longrightarrow \mathbf{P}^1\\ (s:t) &\longmapsto (2^8s^5(24s-49t)^2(4s+21t) : -7^7t^4(s-t)^3(15s-7t))\end{aligned}$$

with

$$\begin{aligned}&(2^8s^5(24s-49t)^2(4s+21t)+7^7t^4(s-t)^3(15s-7t))\\ &\qquad = (-2401t^4+6174t^3s-3381s^2t^2+224s^3t+384s^4)^2.\end{aligned}$$

This enables us to realize the configurations **[1,2,3,4,4,10]**:

$$\begin{aligned}y^2 = x^3 - 3\,&(36864s^8+6144s^7t-12992s^6t^2+2352t^3s^5\\ &+5145t^4s^4-2548t^5s^3+462t^6s^2-36t^7s+t^8)\,x\\ &-2\,(-t^4+18st^3-69s^2t^2+32s^3t+384s^4)\\ &(18432s^8+3072s^7t-6496s^6t^2+1176t^3s^5\\ &-5145t^4s^4+2548t^5s^3-462t^6s^2+36t^7s-t^8).\end{aligned}$$

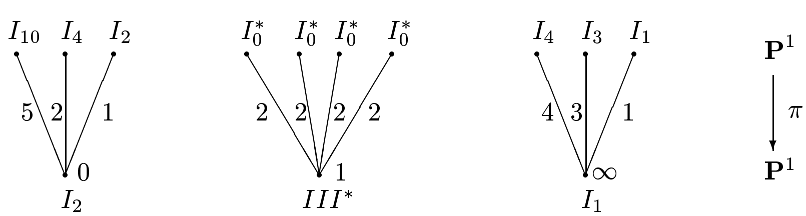

FIGURE 30. A realization of **[1,2,3,4,4,10]**.

and **[1,2,2,5,6,8]**:

$$\begin{aligned}y^2 = x^3 - 3\,&(5145t^4s^4-2548t^5s^3+462t^6s^2-36t^7s+t^8\\ &+589824s^8+98304s^7t-207872s^6t^2+37632t^3s^5)\,x\\ &+2\,(-t^4+18st^3-69s^2t^2+32s^3t+384s^4)\\ &(-5145t^4s^4+2548t^5s^3-462t^6s^2+36t^7s-t^8\\ &+1179648s^8+196608s^7t-415744s^6t^2+75264t^3s^5).\end{aligned}$$

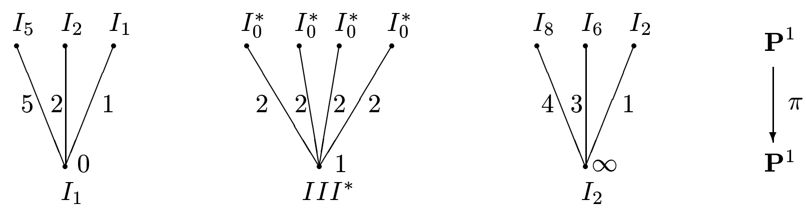

FIGURE 31. A realization of [**1,2,2,5,6,8**].

The second base change with ramification index $(3, 3, 2)$ at ∞ can be constructed in the following way

$$\begin{aligned} \boldsymbol{\pi}: \quad \mathbf{P}^1 &\longrightarrow \mathbf{P}^1 \\ (s:t) &\longmapsto (9s^5(s+6t)^2(9s+4t) : -4t^2(10s^2+24st+9t^2)^3) \end{aligned}$$

with

$$\begin{aligned} 9s^5(s+6t)^2(9s+4t) &+ 4t^2(10s^2+24st+9t^2)^3 \\ &= (9s^4+56s^3t+234s^2t^2+216st^3+54t^4)^2. \end{aligned}$$

This enables us to realize the configurations [**2,2,3,3,4,10**]:

$$\begin{aligned} y^2 = x^3 - 3\,(&1296s^8 + 8064s^7t + 77392s^6t^2 + 232992s^5t^3 + 319680s^4t^4 \\ &+ 214272s^3t^5 + 71928s^2t^6 + 11664st^7 + 729t^8)\,x \\ &- 2\,(27t^4 + 216st^3 + 468s^2t^2 + 224s^3t + 72s^4) \\ &(648s^8 + 4032s^7t - 57304s^6t^2 - 229104s^5t^3 - 319680s^4t^4 \\ &- 214272s^3t^5 - 71928s^2t^6 - 11664st^7 - 729t^8) \end{aligned}$$

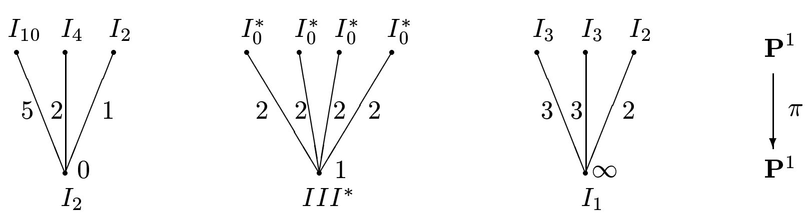

FIGURE 32. A realization of [**2,2,3,3,4,10**].

and [**1,2,4,5,6,6**]:

$$y^2 = x^3 - 3\,(4348s^6t^2 + 8496s^5t^3 + 19980t^4s^4 + 26784t^5s^3 + 17982t^6s^2 + 5832t^7s + 729t^8 + 81s^8 + 1008s^7t)\,x + (54t^4 + 216st^3 + 234s^2t^2 + 56s^3t + 9s^4)(5696s^6t^2 - 4608s^5t^3 - 19980t^4s^4 - 26784t^5s^3 - 17982t^6s^2 - 5832t^7s - 729t^8 + 162s^8 + 2016s^7t).$$

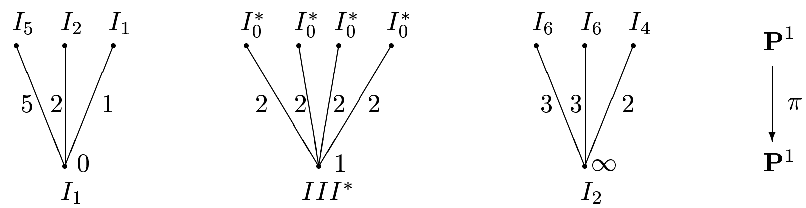

FIGURE 33. A realization of [**1,2,4,5,6,6**].

Eventually coming to the remaining base changes with ramification index **(4,3,1)**, **(4,2,2)** or **(3,3,2)** at 0, we realize that all but one of these either cannot exist or give rise to known configurations of singular fibres. The final one with ramification index $(4,3,1)$ at both cusps, 0 and ∞, however, can only be defined over the quadratic extension $\mathbf{Q}(\sqrt{2})$ of $\mathbf{Q}$.

We conclude this section by collecting the four base changes discussed above which are only defined over an extension of $\mathbf{Q}$. They are presented in the order of appearance in the course of this section:

• With v a solution of $2x^2 - 7x + 28$, a base change with ramification indices $(7,1)$ and $(4,2,1,1)$ at 0 and ∞ can be given by $\pi((s:t)) = (s^7(s+2vt) : (10633/4v - 2401)/40\,t^4(s-t)^2(s^2 + (6 - 2v)st/5 + (3v-14)t^2/10))$. It realizes the configurations [**1,1,2,2,4,14**] and [**1,2,2,4,7,8**] over $\mathbf{Q}(\sqrt{-7})$. We strongly conjecture the field of definition given above to be optimal for these isomorphism classes.

• The base change with ramification indices $(6,1,1)$ and $(4,3,1)$ can be defined over $\mathbf{Q}(\sqrt{-3})$. If v is a solution to $3x^2 - 3x + 7$, then we have $\pi((s:t)) = (s^6(s^2 + 4vst - (19v+14)/5\,t^2) : (1763v - 259)/20\,t^4(s-t)^3(s - (4v-7)/5\,t))$. On the one hand we can thereby produce the configuration [**1,1,2,6,6,8**]. Note that the field of definition of this fibration is the same as for the double sextic in [**13**, p. 313]. The other pull-back has the configuration [**1,2,2,3,4,12**], which has already been obtained over $\mathbf{Q}$ in Section 4. However, these two fibrations have different Mordell-Weil groups MW: For each fibration, consider the pull-back of a primitive section of the basic rational surface X_{321}, respectively X_{141}. It can be directly computed in terms of the actual components of the singular fibres which it meets. Comparing this shape to [**1**, Remark 0.4 (5)], we conclude that in both cases the induced section is already a generator of MW of the pull-back surface. In particular, MW of basic surface and pull-back coincide where $MW(X_{321}) = \mathbf{Z}/(2)$ and $MW(X_{141}) = \mathbf{Z}/(4)$. This proves the claim.

• A base change with ramification index (5,2,1) at both cusps, 0 and ∞, is obtained by choosing a zero v of the polynomial $7x^3 + 19x^2 + 16x + 8$ and setting $\pi((s:t)) = (167s^5(s-2t)^2(s+4(v+1)\,t) : -(15v^2+55v+52)\,t^5(4s+(3v^2+3v-4)\,t)^2(8s-(7v^2+15v+4)\,t))$. The pull-back gives rise to the configuration [**1,2,2,4,5,10**] over the extension $\mathbf{Q}(x^3 - 75x + 5150)$.

• The final base change of this section which has ramification index $(4,3,1)$ at 0 and ∞, is defined over $\mathbf{Q}(\sqrt{2})$. For this, we consider $\pi((s:t)) = (2^4 7^3 s^4 (s-t)^3 (s+(8v+3)\,t) : (8v+3)\,t^4(14s+(9v+4)\,t)^3(2s-(5v+4)\,t))$ where we let v be a root of $7x^2+8x+2$. This map leads to the extremal $K3$ surface with singular fibres [**1,2,3,4,6,8**].

It is an immediate consequence of the computations that the previous three sections together with [**19**] exhaust the extremal semi-stable elliptic $K3$ surfaces which can be realized as nongeneral pull-back of rational elliptic surfaces. Here the term “general” refers to the general pull-back construction involving the induced J-map and the rational elliptic surface with singular fibres I_1, II, III^*. Its construction as a base change of degree 24 with very restricted ramification is far beyond the scope of this paper, as it essentially makes no use of the elliptic fibration of the basic rational elliptic surface. We intend, however, to pursue it in a future project.

Let us now turn to the last section which gives a short treatment of the non semi-stable extremal elliptic $K3$ fibrations coming directly from rational elliptic surfaces.

7. The non semi-stable fibrations. The final section of this paper is devoted to a brief analysis of the non semi-stable extremal elliptic $K3$ fibrations. The treatment is significantly simplified due to the fact that every such surface has a nonreduced fibre by the classification of [**15**]. To go one step further, every $K3$ fibration with more than one nonreduced fibre is easily turned into a rational elliptic surface by way of the deflation process described in the second section. As such an extremal $K3$ surface necessarily has three or four cusps, we find either the original surfaces directly in [**12**] in the case of three cusps or the corresponding rational elliptic surfaces in [**3**] for four cusps. Note that all but one of the corresponding rational surfaces can be uniquely defined over $\mathbf{Q}$ up to $\mathbf{C}$-isomorphism. Except for the three cases given below, this also holds for the deflation processes, considered as manipulations of the Weierstrass equations.

The next table collects the extremal $K3$ fibrations with three or four cusps. The numbering refers to the classification in [**15**] and will be employed throughout this section.

TABLE 1. The extremal $K3$ fibrations with three or four cusps.

No.	Config.	No.	Config.
113	5,5,1*,1*	206	1,1,2*,8*
121	2,8,1*,1*	209	1,1,1*,9*
124	1,9,1*,1*	219	IV*,IV*,IV*
136	2*,2*,2*	220	4,4,IV*,IV*
137	4,4,2*,2*	222	2,4,IV*,IV*
153	3,6,1*,2*	226	1,7,IV*,IV*
154	1,8,1*,2*	243	4,5,1*,IV*
155	3,3,3*,3*	244	2,7,1*,IV*
167	2,6,1*,3*	245	1,8,1*,IV*
168	1,6,2*,3*	246	2*,IV*,IV*
169	2,2,4*,4*	247	3,5,2*,IV*
177	1*,1*,4*	248	1,7,2*,IV*
178	2,4,2*,4*	249	2,5,3*.IV*

TABLE 1. (Continued).

No.	Config.	No.	Config.
179	1,1,5*,5*	250	1*,3*,IV*
187	1,5,1*,5*	251	1,5,4*,IV*
195	2,3,1*,6*	252	2,3,5*,IV*
196	1,3,2*,6*	253	1,4,5*,IV*
197	1,2,3*,6*	254	1,2,7,IV*
205	1,2,1*,8*	255	1,1,8*,IV*
256	3,3,III*,III*	293	2,5,III*,IV*
257	2,4,III*,III*	294	1,6,III*,IV*
258	1,5,III*,III*	295	1*,III*,IV*
279	0*,III*,III*	296	2,2 II*,II*
280	3,5,1*,III*	297	IV,II*,II*
281	2,6,1*,III*	313	1*,1*,II*
282	1,7,1*,III*	314	2,5,1*,II*
283	3,4,2*,III*	315	1,6,1*,II*
284	1,6,2*,III*	316	3,3,2*,II*
285	1*,2*,III*	317	1,5,2*,II*
286	2,4,3*,III*	318	2,3,3*,II*
287	1,5,3*,III*	319	1,2,5*,II*
288	2,3,4*,III*	320	1,1,6*,II*
289	1,3,5*,III*	321	2,4,II*,IV*
290	1,2,6*,III*	322	1,5,II*,IV*
291	1,1,7*,III*	323	0*,II*,IV*
292	3,4,III*,IV*	324	2,3,II*,III*
325	1,4,II*,III*		

As appointed, there are three $K3$ fibrations in the above list which cannot be defined over $\mathbf{Q}$ by this approach although the corresponding rational elliptic surfaces indeed are. This is due to the appearance of cusps which are conjugate in some quadratic field. As a result, for the surfaces with No. 187, No. 245 and No. 282 fields of definition can only be given as $\mathbf{Q}(\sqrt{5})$, $\mathbf{Q}(\sqrt{-2})$ and $\mathbf{Q}(\sqrt{-7})$, respectively. Meanwhile the rational elliptic surface which corresponds to No. 294 can itself only be defined over $\mathbf{Q}(\sqrt{-3})$, thus giving rise to two nonisomorphic models.

The remaining set of extremal $K3$ fibrations which is still missing with respect to the classification in [**15**] consists of those with one nonreduced fibre. In order to derive half of them from rational elliptic surfaces, we return to our concept of base change and further make use of the "transfer of *" as explained in [**8**]. Essentially this just moves the * from one fibre to another (a priori not necessarily singular) by changing the common factor of the polynomials A and B in the Weierstrass equation. We will have to take base changes of degree ranging from 2 to 6 into account, depending on the shape of the basic rational elliptic surface. For each degree we are going to exploit one example in more detail, but only sketch the remaining quite roughly.

Degree 2. These base changes involve the rational elliptic surfaces with four singular fibres, which have one fibre of type III. For example, take the surface Y with singular fibres I_1, I_3, I_5, III. According to [**3**] this surface as well as all its cusps can be defined over $\mathbf{Q}$. Consider a quadratic base change π of $\mathbf{P}^1$ which is ramified at the cusp of the III-fibre and at one further cusp. The pull-back of Y via π is a $K3$ surface over $\mathbf{Q}$ with five semi-stable singular fibres and one of type I_0^*. Transferring the * gives rise to three extremal $K3$ surfaces with one nonreduced and four semi-stable fibres. For instance, the configuration [2,3,3,5,5,0*] can be taken to [**3,3,5,5,2***] (No. 138), [**2,3,5,5,3***] (No. 157) or [**2,3,3,5,5***] (No. 180).

Degree 3. The rational elliptic surfaces which serve for pull-back by a base change of degree 3 are X_{141}, X_{222} and X_{411} as given in Section 4. The $K3$ surfaces which they give rise to (after deflating if necessary) are seen to be extremal if and only if the base change is maximally ramified at the three cusps (such that these obtain 5 pre-images in total). We will refer to them as triple covers without specifying the particular base change.

Degree 4. The surface X_{431}, as introduced in Section 5, serves as an object for base changes of degree 4. To obtain an extremal $K3$ fibration we only have to select the ramification index $(3,1)$ at the cusp of the IV^*-fibre and minimize the number of pre-images of the other two cusps at 4. Indeed, we can adequately choose both base changes π_2 and π_4 from the third section after exchanging cusps. A further useful base change has ramification index $(3,1)$ at every cusp and can be given by $\boldsymbol{\pi_3}((s:t)) = (s^3(s-2t) : t^3(t-2s))$. This base change, for example, gives rise to the constellation [**1,3,3,9,IV***] (No. 233).

Degree 5. For these and for the next base changes the basic rational elliptic surface will be X_{321}, defined in Section 6. The base changes of degree 5 have to be chosen with ramification index (2,2,1) at 1 (the cusp of III^*), such that only one original III^* remains in the pull-back after deflation. The extremality of the resulting $K3$ fibrations is guaranteed by the other two cusps having again the minimal number of four pre-images. It turns out that there are five such base changes, all but one defined over $\mathbf{Q}$. We will only go into detail for one of them and then list the others:

- A first base change can be given as $\boldsymbol{\pi_E}((s:t)) = (s(s^2-5st+5t^2)^2 : 4t^5)$, since $s(s^2-5st+5t^2)^2-4t^5 = (s-4t)(s^2-3ts+t^2)^2$. As pull-back we are able to realize [**2,4,4,5,III***] (No. 259) and [**1,2,2,10,III***] (No. 275).
- $\boldsymbol{\pi_F}((s:t)) = (4s^3(3s-5t)^2 : t^4(15s+2t))$.
- $\boldsymbol{\pi_G}((s:t)) = (s^3(4s-5t)^2 : t^3(4t-5s)^2)$.
- $\boldsymbol{\pi_H}((s:t)) = (64t^5 : (t-s)^3(9s^2-33st+64t^2))$.
- $\boldsymbol{\pi_I}((s:t)) = (s^4(s-5t) : t^4(2i-11)(5s+(3+4i)t)$ with imaginary unit i.

Degree 6. We pull back X_{321} by a base change of degree 6 with ramification index $(2,2,2)$ at 1. By deflation and transfer of * we obtain an elliptic $K3$ fibration with singular fibres only above the pre-images of the other two cusps 0 and ∞. Restricting their number to the minimum 5, we achieve the extremality of our $K3$ surface. Due to the choice in the transfer of * which turns a distinct semi-stable fibre I_n into its nonreduced relative of type I_n^*, one base change produces indeed up to nine different fibrations. Of the six base changes with the above properties existing, exactly three cannot be factorized into the composition of two maps. We give them below:

- $\boldsymbol{\pi}_{\mathbf{A}}((s:t)) = (4(s^2-4st+t^2)^3 : 27t^4s(s-4t))$, or alternatively, $\boldsymbol{\pi}_{\mathbf{A'}}((s:t)) = (4s^3(s-2t)^3 : t^4(3s^2-6st-t^2))$.
- $\boldsymbol{\pi}_{\mathbf{B}}((s:t)) = (-4t^5(6s+t) : s^3(2s-5t)^2(s-4t))$.
- $\boldsymbol{\pi}_{\mathbf{C}}((s:t)) = (s^4(s^2+2st+5t^2)+4t^5(t-2s))$.

Let us discuss one particular example in more detail: We want to realize the configuration [**1,2,3,10,2***] (No. 148). This can be achieved as pull-back from X_{321} via the base change π_B since then we have ramification index $(3,2,1)$ at the I_1-fibre and $(5,1)$ at the I_2. The remarkable point about this construction is that we still have a choice of where to move the * after the pull-back: We can transfer it either to the I_2 at $5/2$ which sits above the original I_1 or to the I_2 over $-1/6$ which comes from the I_2-fibre of X_{321}. Indeed, we can already prove that the two resulting extremal fibrations are nonisomorphic.

For this purpose, consider the nontrivial section of the pull-back which is induced from X_{321}. It necessarily meets the fibres above the original I_1-fibre at the identity component (the one meeting the 0-section) whereas it meets the fibres above the I_2 at the "opposite" component. So the sections differ for the two fibrations as do the Mordell-Weil groups lattice-theoretically. As a consequence we compute the two distinguished intersection forms on the transcendental lattice. This fact is illustrated by the following observation: Consider the quotients of the two fibrations by the respective nontrivial (torsion) section and proceed to their minimal resolutions. Again these are extremal $K3$ fibrations (conf. [**10**, Section 5]). To be precise, we obtain the configuration [**1,2,5,6,4***] (No. 173) for the first choice of transfer of * and [**2,4,5,6,1***] (No. 116) for the second. Clearly, these are not isomorphic by virtue of the intersection form and unique up to isomorphism by the classification of [**15**]. Note, however, that this is the only example where there is such ambiguity concerning the transfer of *.

We are now in the position to compute all the remaining extremal elliptic $K3$ fibrations which can be derived from rational elliptic surfaces by our simple methods. The following table collects their configurations together with the number at which they appear in [**15**]. We further add the Mordell-Weil group MW and the reduced coefficients of the intersection form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ on the transcendental lattice which determine the isomorphism class of the surface (up to orientation if there is ambiguity in the sign of b). The right-hand part of the table consists of a very brief description of the construction and the field of definition for the fibration. For brevity we will indicate the occurrence of a transfer of * only by a * at the end of the construction while not even mentioning deflation.

TABLE 2. The extremal $K3$ fibrations with five cusps.

No.	Config.	MW	a	b	c	Construction	def.
114	1,4,6,6,1*	$\mathbf{Z}/(2)$	12	0	12	pull-back from X_{321} via π_A*	$\mathbf{Q}$
115	1,5,5,6,1*	(0)	20	0	30	double cover of $[1,3,5,III]$*	$\mathbf{Q}$
116	2,4,5,6,1*	$\mathbf{Z}/(2)$	12	0	20	pull-back from X_{321} via π_B *	$\mathbf{Q}$
117	1,2,7,7,1*	(0)	14	0	28	double cover of $[1,1,7,III]$*	$\mathbf{Q}(\sqrt{-7})$
122	2,3,4,8,1*	$\mathbf{Z}/(4)$	6	0	8	triple cover of X_{141}	$\mathbf{Q}$
123	2,2,5,8,1*	$\mathbf{Z}/(2)$	8	0	20	pull-back from X_{321} via π_C *	$\mathbf{Q}$
127	1,3,3,10,1*	(0)	6	0	60	double cover of $[1,3,5,III]$*	$\mathbf{Q}$
128	2,2,3,10,1*	$\mathbf{Z}/(2)$	2	0	60	pull-back of X_{321} via π_B *	$\mathbf{Q}$
129	1,2,4,10,1*	$\mathbf{Z}/(2)$	8	4	12	pull-back from X_{321} via π_C *	$\mathbf{Q}(\sqrt{-1})$
132	1,2,2,12,1*	$\mathbf{Z}/(4)$	2	0	6	triple cover of X_{141}	$\mathbf{Q}$
133	1,1,3,12,1*	$\mathbf{Z}/(2)$	6	0	6	triple cover of X_{141}	$\mathbf{Q}$
135	1,1,1,14,1*	(0)	6	2	10	double cover of $[1,1,7,III]$*	$\mathbf{Q}(\sqrt{-7})$

TABLE 2. (Continued).

No.	Config.	MW	a	b	c	Construction	def.
138	3,3,5,5,2*	(0)	30	0	30	double cover of $[1,3,5,III]*$	$\mathbf{Q}$
139	2,2,6,6,2*	$\mathbf{Z}/(2)\times\mathbf{Z}/(2)$	6	0	6	triple cover of X_{222}	$\mathbf{Q}$
140	2,4,4,6,2*	$\mathbf{Z}/(2)\times\mathbf{Z}/(2)$	4	0	12	triple cover of X_{222}	$\mathbf{Q}$
141	1,4,5,6,2*	$\mathbf{Z}/(2)$	4	0	30	pull-back from X_{321} via π_B *	$\mathbf{Q}$
142	1,1,7,7,2*	(0)	14	0	14	double cover of $[1,1,7,III]*$	$\mathbf{Q}(\sqrt{-7})$
144	2,3,3,8,2*	$\mathbf{Z}/(2)$	6	0	24	pull-back from X_{321} via π_A *	$\mathbf{Q}$
145	1,3,4,8,2*	$\mathbf{Z}/(2)$	4	0	24	triple cover of X_{141} *	$\mathbf{Q}$
146	1,2,5,8,2*	$\mathbf{Z}/(2)$	6	2	14	pull-back from X_{321} via π_C *	$\mathbf{Q}(\sqrt{-1})$
148	1,2,3,10,2*	$\mathbf{Z}/(2)$	6 4	0 2	10 16	pull-back from X_{321} via π_B *	$\mathbf{Q}$
149	1,1,4,10,2*	$\mathbf{Z}/(2)$	4	0	10	pull-back from X_{321} via π_C *	$\mathbf{Q}$
151	1,1,2,12,2*	$\mathbf{Z}/(2)$	4	0	6	triple cover of X_{141} *	$\mathbf{Q}$
156	3,4,4,4,3*	$\mathbf{Z}/(4)$	8	4	8	triple cover of X_{141}	$\mathbf{Q}$
157	2,3,5,5,3*	(0)	10	0	60	double cover of $[1,3,5,III]*$	$\mathbf{Q}$
161	2,2,3,8,3*	$\mathbf{Z}/(2)$	4	0	24	pull-back from X_{321} via π'_A *	$\mathbf{Q}$
162	1,2,4,8,3*	$\mathbf{Z}/(4)$	2	0	8	triple cover of X_{141}	$\mathbf{Q}$
163	1,2,2,10,3*	$\mathbf{Z}/(2)$	4	0	10	pull-back from X_{321} via π_B *	$\mathbf{Q}$
164	1,1,3,10,3*	(0)	2	0	60	double cover of $[1,3,5,III]*$	$\mathbf{Q}$
166	1,1,1,12,3*	$\mathbf{Z}/(4)$	2	1	2	triple cover of X_{141}	$\mathbf{Q}$
170	3,3,4,4,4*	$\mathbf{Z}/(2)$	12	0	12	double cover of $[2,3,4,III]*$	$\mathbf{Q}$

TABLE 2. (Continued).

No.	Config.	MW	a	b	c	Construction	def.
171	1,1,6,6,4*	$\mathbf{Z}/(2)$	6	0	6	pull-back from X_{321} via π_A *	$\mathbf{Q}$
172	2,2,4,6,4*	$\mathbf{Z}/(2)\times\mathbf{Z}/(2)$	2	0	12	triple cover of X_{222} *	$\mathbf{Q}$
173	1,2,5,6,4*	$\mathbf{Z}/(2)$	2	0	30	pull-back from X_{321} via π_B *	$\mathbf{Q}$
175	1,2,3,8,4*	$\mathbf{Z}/(2)$	2	0	24	triple cover of X_{411}	$\mathbf{Q}$
176	1,1,2,10,4*	$\mathbf{Z}/(2)$	2	0	10	pull-back from X_{321} via π_C *	$\mathbf{Q}$
180	2,3,3,5,5*	(0)	12	0	30	double cover of $[1,3,5,III]*$	$\mathbf{Q}$
181	1,2,4,6,5*	$\mathbf{Z}/(2)$	4	0	12	pull-back from X_{321} via π_B *	$\mathbf{Q}$
182	1,1,5,6,5*	(0)	4	0	30	double cover of $[1,3,5,III]*$	$\mathbf{Q}$
184	1,2,2,8,5*	$\mathbf{Z}/(2)$	4	0	8	pull-back from X_{321} via π_C *	$\mathbf{Q}$
188	2,2,4,4,6*	$\mathbf{Z}/(2)\times\mathbf{Z}/(2)$	4	0	4	triple cover of X_{222}	$\mathbf{Q}$
189	1,1,5,5,6*	(0)	10	0	10	double cover of $[1,3,5,III]*$	$\mathbf{Q}$
190	1,2,4,5,6*	$\mathbf{Z}/(2)$	2	0	20	pull-back from X_{321} via π_B *	$\mathbf{Q}$
191	2,2,2,6,6*	$\mathbf{Z}/(2)\times\mathbf{Z}/(2)$	4	2	4	triple cover of X_{222}	$\mathbf{Q}$
192	1,1,4,6,6*	$\mathbf{Z}/(2)$	2	0	12	pull-back from X_{321} via π'_A *	$\mathbf{Q}$
201	1,1,2,7,7*	(0)	6	2	10	double cover of $[1,1,7,III]*$	$\mathbf{Q}(\sqrt{-7})$
202	2,2,3,3,8*	$\mathbf{Z}/(2)$	6	0	6	pull-back from X_{321} via π_A *	$\mathbf{Q}$
203	1,2,3,4,8*	$\mathbf{Z}/(2)$	4	0	6	triple cover of X_{411} *	$\mathbf{Q}$
204	1,2,2,5,8*	$\mathbf{Z}/(2)$	2	0	4	pull-back from X_{321} via π_C *	$\mathbf{Q}$

TABLE 2. (Continued).

No.	Config.	MW	a	b	c	Construction	def.
210	1,1,3,3,10*	(0)	6	0	6	double cover of $[1,3,5,III]*$	$\mathbf{Q}$
211	1,2,2,3,10*	$\mathbf{Z}/(2)$	2	0	6	pull-back from X_{321} via π_B *	$\mathbf{Q}$
212	1,1,2,4,10*	$\mathbf{Z}/(2)$	2	0	4	pull-back from X_{321} via π_C *	$\mathbf{Q}$
215	1,1,2,2,12*	$\mathbf{Z}/(2)$	2	0	2	triple cover of X_{411}	$\mathbf{Q}$
216	1,1,1,3,12*	$\mathbf{Z}/(2)$	2	1	2	triple cover of X_{411}	$\mathbf{Q}$
218	1,1,1,1,14*	(0)	2	0	2	double cover of $[1,1,7,III]*$	$\mathbf{Q}$
223	1,3,6,6,IV*	$\mathbf{Z}/(3)$	6	0	6	pull-back from X_{431} via π_2	$\mathbf{Q}$
224	3,3,4,6,IV*	$\mathbf{Z}/(3)$	6	0	12	pull-back from X_{431} via π_4	$\mathbf{Q}$
233	1,3,3,9,IV*	$\mathbf{Z}/(3)$	6	3	6	pull-back from X_{431} via π_3	$\mathbf{Q}$
234	2,2,3,9,IV*	$\mathbf{Z}/(3)$	2	0	18	pull-back from X_{431} via π_2	$\mathbf{Q}$
241	1,1,2,12,IV*	$\mathbf{Z}/(3)$	2	0	4	pull-back from X_{431} via π_4	$\mathbf{Q}$
259	2,4,4,5,III*	$\mathbf{Z}/(2)$	4	0	20	pull-back from X_{321} via π_E	$\mathbf{Q}$
261	1,4,4,6,III*	$\mathbf{Z}/(2)$	4	0	12	pull-back from X_{321} via π_F	$\mathbf{Q}$
262	2,3,4,6,III*	$\mathbf{Z}/(2)$	6	0	12	pull-back from X_{321} via π_G	$\mathbf{Q}$
263	2,2,5,6,III*	$\mathbf{Z}/(2)$	8	2	8	pull-back from X_{321} via π_H	$\mathbf{Q}$
270	2,2,3,8,III*	$\mathbf{Z}/(2)$	2	0	24	pull-back from X_{321} via π_F	$\mathbf{Q}$
271	1,2,4,8,III*	$\mathbf{Z}/(2)$	4	0	8	pull-back from X_{321} via π_I	$\mathbf{Q}(\sqrt{-1})$

TABLE 2. (Continued).

No.	Config.	MW	a	b	c	Construction	def.
275	1,2,2,10,III*	$\mathbf{Z}/(2)$	2	0	10	pull-back from X_{321} via π_E	$\mathbf{Q}$
276	1,1,3,10,III*	$\mathbf{Z}/(2)$	4	1	4	pull-back from X_{321} via π_H	$\mathbf{Q}$
298	3,3,4,4,II*	(0)	12	0	12	double cover of $[3,4,II,III]*$	$\mathbf{Q}$
299	2,2,5,5,II*	(0)	10	0	10	double cover of $[2,5,II,III]*$	$\mathbf{Q}$
301	1,1,6,6,II*	(0)	6	0	6	double cover of $[1,6,II,III]*$	$\mathbf{Q}$

This table completes the treatment of extremal elliptic $K3$ fibrations which can be derived from rational elliptic surfaces by direct manipulation of the Weierstrass equation or as pull-back via a nongeneral base change. We would like to finish with the following remark which concerns $K3$ surfaces which possess an extremal elliptic fibration with nontrivial Mordell-Weil group. For every such surface this paper, combined with [**19**], gives at least one explicit extremal fibration which is obtained as pull-back from a rational elliptic surface. This result might be compared to the idea of elementary fibrations proposed in [**11**, Section 6]. We should, however, note that our pull-backs in general cannot be called elementary in the strict sense of [11, 13].

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