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INVERSIVE DIFFERENCE MODULES AND SOLVABILITY OF SYSTEMS OF LINEAR DIFFERENCE EQUATIONS

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ABSTRACT. In this paper we consider homological properties of inversive difference modules and apply them to the problem of solvability of a system of linear difference equations over a difference field. In particular, we prove the existence of Grothendieck's spectral sequence for the functor *Ext* in the category of inversive difference modules.

1. Introduction. Throughout the paper **N**, **Q**, and **R** denote the sets of all nonnegative integers, rational numbers and real numbers, respectively. By a ring we always mean an associative ring with a unity. Every ring homomorphism is unitary (maps unity onto unity), every subring of a ring contains the unity of the ring, and every module is unitary.

A *difference ring* is a commutative ring R together with a finite set $\sigma = \{\alpha_1, \dots, \alpha_n\}$ of mutually commuting injective endomorphisms of R into itself. The set σ is called the *basic set* of the difference ring R, and the endomorphisms $\alpha_1, \dots, \alpha_n$ are called *translations*. In other words, a difference ring R with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$, also called a *$\sigma$-ring*, is a commutative ring possessing n additional unitary operations $\alpha_i : a \mapsto \alpha_i(a)$ such that $\alpha_i(a) = 0$ if and only if $a = 0$, $\alpha_i(a+b) = \alpha_i(a) + \alpha_i(b)$, $\alpha_i(ab) = \alpha_i(a)\alpha_i(b)$, $\alpha_i(1) = 1$ and $\alpha_i(\alpha_j(a)) = \alpha_j(\alpha_i(a))$ for any $a \in R$, $1 \le i,\, j \le n$. In what follows, a difference ring R with a basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$ will also be called a σ-ring.

If $\alpha_1, \dots, \alpha_n$ are automorphisms of R, we say that R is an *inversive difference ring* with the basic set σ. In this case the set $\{\alpha_1, \dots, \alpha_n, \alpha_1^{-1}, \dots, \alpha_n^{-1}\}$ is denoted by σ^* and R is also called a

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σ^*-ring. If a difference ring with a basic set σ is a field, it is called a difference, or σ-field. An inversive difference field with a basic set σ is also called a σ^*-field.

Example 1.1. Let A be a ring of functions of n real variables defined on $\mathbf{R}^n$. (In particular, A could be one of the rings $C^p(\mathbf{R}^n)$, $p = 0, 1, \dots$, where $C^p(\mathbf{R}^n)$ denotes the ring of all functions of n real variables that are continuous on $\mathbf{R}^n$ together with all their partial derivatives up to the order p.) Let us fix some real numbers $h_1, \dots, h_n$ and consider mutually commuting automorphisms $\alpha_1, \dots, \alpha_n$ of A such that $(\alpha_i f)(x_1, \dots, x_n) = f(x_1, \dots, x_{i-1}, x_i + h_i, x_{i+1}, \dots, x_n)$, $i = 1, \dots, n$. Then A can be treated as an inversive difference ring with the basic set $\sigma = \{\alpha_1, \dots, \alpha_n\}$. Difference rings of this type arise in the theory of equations in finite differences, since the ith partial finite difference $\Delta_i f(x_1, \dots, x_n) = f(x_1, \dots, x_{i-1}, x_i + h_i, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_n)$ of a function $f(x_1, \dots, x_n) \in A$ can be written as $(\alpha_i - 1)f$.

Let R be an inversive difference ring with a basic set σ, and let Γ denote the free commutative group generated by σ. An expression of the form $\sum_{\gamma\in\Gamma} a_\gamma \gamma$ with $a_\gamma \in R$, such that all but a finite number of a_γ are equal to 0, is called an *inversive difference*, or σ^*-*operator* over R. Two σ^*-operators $\sum_{\gamma\in\Gamma} a_\gamma\gamma$ and $\sum_{\gamma\in\Gamma} b_\gamma\gamma$ are considered to be equal if and only if $a_\gamma = b_\gamma$ for all $\gamma \in \Gamma$.

The set of all σ^*-operators over R can be naturally equipped with a ring structure if one sets $\sum_{\gamma\in\Gamma} a_\gamma\gamma + \sum_{\gamma\in\Gamma} b_\gamma\gamma = \sum_{\gamma\in\Gamma}(a_\gamma + b_\gamma)\gamma$, $a\sum_{\gamma\in\Gamma} a_\gamma\gamma = \sum_{\gamma\in\Gamma}(aa_\gamma)\gamma$, $(\sum_{\gamma\in\Gamma} a_\gamma\gamma)\gamma_1 = \sum_{\gamma\in\Gamma} a_\gamma(\gamma\gamma_1)$, $\gamma_1 a = \gamma_1(a)\gamma_1$ for any σ^*-operators $\sum_{\gamma\in\Gamma} a_\gamma\gamma$, $\sum_{\gamma\in\Gamma} b_\gamma\gamma$ and for any $a \in R$, $\gamma_1 \in \Gamma$, and extends the multiplication by distributivity. The ring obtained in this way is called the *ring of inversive difference*, or σ^*-*operators over* R; it is denoted by $\mathcal{E}$.

Definition 1.2. Let R be an inversive difference ring with a basic set σ and $\mathcal{E}$ the ring of inversive difference operators over R. Then a left $\mathcal{E}$-module is said to be an inversive difference R-module or a σ^*-R-module. In other words, an R-module M is called a σ^*-R-module if elements of the set σ^* act on M in such a way that $\alpha(x+y) = \alpha x + \alpha y$,

$\alpha(\beta x) = \beta(\alpha x)$, $\alpha(ax) = \alpha(a)\alpha(x)$ and $\alpha(\alpha^{-1}x) = x$ for any $\alpha, \beta \in \sigma^*$; $x, y \in M$; $a \in R$.

If R is a σ^*-field, a σ^*-R-module M is said to be a vector σ^*-R-space (or an inversive difference vector space over R).

Let M and N be two σ^*-R-modules over an inversive difference (σ-) ring R. A homomorphism of R-modules $f : M \to N$ is said to be a *difference* (or σ-) *homomorphism* if $f(\alpha x) = \alpha f(x)$ for any $x \in M$, $\alpha \in \sigma$. A surjective, respectively, injective or bijective, difference homomorphism is called a difference (or σ-) epimorphism, respectively, a difference monomorphism or a difference isomorphism.

The theory of inversive difference modules introduced in [**3**] appeared to be very helpful in the study of difference field extensions, algebraic difference equations, and Krull dimension of difference rings, see, for example, [**2**, Chapters 6, 7]. Its generalizations and applications to systems of algebraic difference-differential equations are considered in [**4**–**6**]. In what follows we concentrate on homological properties of inversive difference modules and their applications.

2. On the functor Ext of inversive difference modules. Let R be an inversive difference ring with a basic set σ, and let M and N be σ^*-R-modules. Then each of the R-modules $\mathrm{Hom}_R(M, N)$ and $M \otimes_R N$ can be equipped with a structure of a σ^*-R-module if for any $f \in \mathrm{Hom}_R(M, N)$, $\sum_{i=1}^{k} x_i \otimes y_i \in M \otimes_R N$, $x_1, \dots, x_k \in M$; $y_1, \dots, y_k \in N$, and $\alpha \in \sigma^*$, one sets $\alpha(f) = \alpha \circ f \circ \alpha^{-1}$ and $\alpha(\sum_{i=1}^{k} x_i \otimes y_i) = \sum_{i=1}^{k} \alpha x_i \otimes \alpha y_i$. It is easy to check that $\alpha(f) \in \mathrm{Hom}_R(M, N)$ and the action of elements of σ^* on $\mathrm{Hom}_R(M, N)$ satisfies the conditions of Definition 1.2. Let us show that the action of σ^* on $M \otimes_R N$ satisfies these conditions as well. Indeed, if $u = \sum_{i=1}^{k} x_i \otimes y_i \in M \otimes_R N$, then $\alpha(au) = \alpha(\sum_{i=1}^{k} ax_i \otimes y_i) = \sum_{i=1}^{k} \alpha(a)\alpha x_i \otimes y_i = \alpha(a)\alpha(u)$ for any $a \in R$, $\alpha \in \sigma^*$, and also $\alpha(\alpha^{-1}z) = z$, $\alpha(z_1 + z_2) = \alpha z_1 + \alpha z_2$, $\alpha(\beta z) = \beta(\alpha z)$ for any $\alpha, \beta \in \sigma^*$ and $z, z_1, z_2 \in M \otimes_R N$.

In what follows, while considering the modules $\mathrm{Hom}_R(M, N)$ and $M \otimes_R N$ as σ^*-R-modules (M and N are some σ^*-R-modules) we mean the foregoing inversive difference structures of these modules.

Lemma 2.1. *Let R be an inversive difference (σ^*-) ring, and let M, N, and P be three σ^*-R-modules. Then the canonical mapping*

$$\eta : \mathrm{Hom}_R\Big(P \bigotimes_R M, N\Big) \longrightarrow \mathrm{Hom}_R(P, \mathrm{Hom}_R(M,N)),$$

defined by $[(\eta(f))x](y) = f(x \otimes y)$ for any $f \in \mathrm{Hom}_R(P \otimes_R M, N)$, $x \in P$, $y \in M$, is a σ-isomorphism of σ^*-R-modules.

Proof. The fact that η is an isomorphism of R-modules is well known (see, for example [**7**, Theorem 2.4]). If $f \in \mathrm{Hom}_R(P \otimes_R M, N)$, $x \in P$, $y \in M$, and $\alpha \in \sigma$, then $(\eta(\alpha(f))(x))(y) = (\alpha(f))(x \otimes y) = \alpha(f(\alpha^{-1}(x \otimes y))) = \alpha(f(\alpha^{-1}x \otimes \alpha^{-1}y)) = \alpha(\eta(f)(\alpha^{-1}x))(\alpha^{-1}y) = (\alpha(\eta(f)(\alpha^{-1}x)))(y) = ((\alpha\eta)(f))(x))(y)$. Thus, η is a σ-isomorphism. □

Let R be an inversive difference (σ^*-)ring and M a σ^*-R-module. Then the set $C(M) = \{x \in M \mid \alpha(x) = x \text{ for all } \alpha \in \sigma\}$ is called the *set of constants* of the module M; elements of this set are called constants. It is easy to see that $C(M)$ is a subgroup of the additive group of M and the mapping $C : M \mapsto C(M)$ is a functor from the category of σ^*-R-modules, i.e., the category of all left modules over the ring of σ^*-operators $\mathcal{E}$, to the category of Abelian groups.

Lemma 2.2. *Let R be an inversive difference (σ^*-) ring and $\mathcal{E}$ the ring of σ^*-operators over R. Then*

(i) $C(\mathrm{Hom}_R(M,N)) = \mathrm{Hom}_{\mathcal{E}}(M,N)$ *for any two σ^*-R-modules M and N.*

(ii) *The functors C and $\mathrm{Hom}_{\mathcal{E}}(R,\cdot)$ are naturally isomorphic.* (*In this case we write $C \simeq \mathrm{Hom}_{\mathcal{E}}(R,\cdot)$.*)

(iii) *The functor C is left exact and, for any positive integer p, its pth right derived functor is naturally isomorphic to the functor $\mathrm{Ext}^p_{\mathcal{E}}(R,\cdot)$.*

(iv) *If M and N are two σ^*-R-modules, then $\mathrm{Hom}_{\mathcal{E}}(\cdot \otimes_R M, N) \simeq \mathrm{Hom}_{\mathcal{E}}(\cdot, \mathrm{Hom}_R(M,N))$ and $\mathrm{Hom}_{\mathcal{E}}(M \otimes_R \cdot, N) \simeq \mathrm{Hom}_{\mathcal{E}}(M, \mathrm{Hom}_R(\cdot, N))$.*

Proof. The first statement follows from the definition of the action of elements of σ on $\mathrm{Hom}_R(M,N)$. Indeed, $\phi \in C(\mathrm{Hom}_R(M,N))$ if and only if $\alpha(\phi(\alpha^{-1}(x)) = \phi(x)$ for every $\alpha \in \sigma$, $x \in M$, that is equivalent to the inclusion $\phi \in \mathrm{Hom}_{\mathcal{E}}(M, N)$. Statement (ii) is a direct consequence of (i) and the obvious fact that the functors $C(\cdot)$ and $C(\mathrm{Hom}_R(R, \cdot))$ are naturally isomorphic.

Since the functor $\mathrm{Hom}_{\mathcal{E}}(R, \cdot)$ is left exact, statement (ii) implies that $C(\cdot)$ is left exact as well. Now, the natural isomorphism of the functors $C(\cdot)$ and $\mathrm{Hom}_{\mathcal{E}}(R, \cdot)$ implies the natural isomorphism of their pth right derived functors $\mathcal{R}^p C$ and $\mathcal{R}^p \mathrm{Hom}_{\mathcal{E}}(R, \cdot) = \mathrm{Ext}^p_{\mathcal{E}}(R, \cdot)$ for any $p > 0$.

By Lemma 2.1, $\mathrm{Hom}_R(\cdot \otimes_R M, N) \simeq \mathrm{Hom}_R(\cdot, \mathrm{Hom}_R(M, N))$, whence $C(\mathrm{Hom}_R(\cdot \otimes_R M, N)) \simeq C(\mathrm{Hom}_R(\cdot, \mathrm{Hom}_R(M, N)))$. Applying (i) we obtain that $\mathrm{Hom}_{\mathcal{E}}(\cdot \otimes_R M, N) \simeq \mathrm{Hom}_{\mathcal{E}}(\cdot, \mathrm{Hom}_R(M, N))$. The statement about the other pair of functors in (iv) can be proved in the same way.

The following result is due to Grothendieck [**1**].

Lemma 2.3. *Let A, B and C be rings, and let k_A, k_B and k_C be the categories of left modules over the rings A, B and C, respectively. Furthermore, suppose that $F : k_A \to k_B$ and $G : k_B \to k_C$ are covariant functors satisfying the following conditions.*

(i) *The functor G is left exact.*

(ii) *If M is an injective left A-module, then the B-module $F(M)$ is annihilated by any right derived functor $\mathcal{R}^q G$, $q > 0$.*

Then, for any left A-module N, there exists a spectral sequence in the category k_C that converges to $\mathcal{R}^{p+q}(GF)(N)$, $p, q \in \mathbf{N}$, and has the second term $E_2^{p,q}(N) = \mathcal{R}^p G(\mathcal{R}^q F(N))$.

Theorem 2.4. *Let R be an inversive difference ring with a basic set σ, $\mathcal{E}$ the ring of σ^*-operators over R, and M, N two σ^*-R-modules. Then, for any positive integers p and q, there exists a spectral sequence converging to $Ext_{\mathcal{E}}^{p+q}(M, N)$ whose second term is equal to $E_2^{p,q} = (\mathcal{R}^p C)(\mathrm{Ext}^q_R(M, N))$.*

Proof. Because of the statement of Lemma 2.3, it is sufficient to prove the following fact:

Let N be an injective $\mathcal{E}$-module and p a positive integer. Then

$$(\mathcal{R}^p C)(\mathrm{Hom}_R(M, N)) = 0 \quad \text{for any} \quad \mathcal{E}\text{-module } M.$$

First, let us prove the last equality for an $\mathcal{E}$-module M which is flat as an R-module. In this case the functor $\mathrm{Hom}_{\mathcal{E}}(\cdot \otimes_R M, N)$ is exact, hence the functor $\mathrm{Hom}_{\mathcal{E}}(\cdot, \mathrm{Hom}_R(M,N))$ is also exact (by Lemma 2.2 (iv) these two functors are naturally isomorphic). It follows that $\mathrm{Hom}_R(M,N)$ is an injective $\mathcal{E}$-module, hence $\mathrm{Ext}^p_{\mathcal{E}}(R, \mathrm{Hom}_R(M,N)) = 0$ for all $p > 0$. Applying Lemma 2.2 (iii) we obtain that

$$(\mathcal{R}^p C)(\mathrm{Hom}_R(M,N)) = 0 \quad \text{for all} \quad p > 0.$$

Now, let M be an arbitrary $\mathcal{E}$-module, and let $F : \cdots \to F_1 \to F_0 \to M \to 0$ be a flat, e.g., free, resolution of M as an R-module. (Each F_i is a flat R-module and the mappings are homomorphisms of R-modules.) By Lemma 2.2 (iv), $\mathrm{Hom}_{\mathcal{E}}(\mathcal{E} \otimes_R \cdot, N) \simeq \mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, \mathrm{Hom}_R(\cdot, N))$, therefore $\mathrm{Hom}_{\mathcal{E}}(\mathcal{E} \otimes_R \cdot, N) \simeq \mathrm{Hom}_R(\cdot, N)$. Since the $\mathcal{E}$-module N is injective, the functor $\mathrm{Hom}_{\mathcal{E}}(\mathcal{E} \otimes \cdot, N)$ is exact, hence $\mathrm{Hom}_R(\cdot, N)$ is also exact. Applying functor C to the injective resolution $0 \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(F,N)$ of the $\mathcal{E}$-module $\mathrm{Hom}_R(M,N)$ we obtain that $(\mathcal{R}^p C)(\mathrm{Hom}_R(M,N)) = H^p(C(\mathrm{Hom}_R(F,N)))$ is isomorphic to $H^p(\mathrm{Hom}_{\mathcal{E}}(F,N))$ for every $p > 0$. By the first part of the proof, $H^p(\mathrm{Hom}_{\mathcal{E}}(F,N)) = 0$, hence $(\mathcal{R}^p C)(\mathrm{Hom}_R(M,N)) = 0$ for any $\mathcal{E}$-module M and for any $p > 0$. This completes the proof. □

The last theorem finds its applications in the analysis of systems of linear difference equations considered in the rest of this paper.

3. Inversive difference modules and systems of linear difference equations. Let R be an inversive difference ring with a basis set $\sigma = \{\alpha_1, \dots, \alpha_n\}$, $\sigma^* = \{\alpha_1, \dots, \alpha_n, \alpha_1^{-1}, \dots, \alpha_n^{-1}\}$, Γ the free commutative group generated by σ, and $\mathcal{E}$ the ring of σ^*-operators over R. For any two σ^*-R-modules M and N, let $B(M,N)$ denote the set of all additive mappings from M to N with the following property. For every $\beta \in B(M,N)$, there exists $\gamma_\beta \in \Gamma$ such that $\beta(ax) = \gamma_\beta(a)\beta(x)$ for any $a \in R$, $x \in M$. (The mapping $\beta \mapsto \gamma_\beta$ is not supposed to be injective or surjective.) Furthermore, let $\mathcal{P}(M,N)$ denote the set of all formal sums $\sum_{\beta \in B(M,N)} a_\beta \beta$, where $a_\beta \in R$ for any element $\beta \in B(M,N)$ and only finitely many coefficients a_β are different from 0.

It is easy to see that $\mathcal{P}(M,N)$ becomes a σ^*-R-module if one defines $\alpha(\sum_{\beta \in B(M,N)} a_\beta \beta) = \sum_{\beta \in B(M,N)} \alpha(a_\beta)(\alpha\beta)$ for every $\alpha \in \sigma^*$.

(Clearly, $\alpha\beta \in B(M,N)$ if $\beta \in B(M,N)$; in this case $\gamma_{\alpha\beta} = \alpha\gamma_\beta$.) In what follows, we also treat $\mathrm{Hom}_R(M,N)$ and $\mathcal{E}\otimes_R M$ as left $\mathcal{E}$-modules, that is, σ^*-R-modules. (The corresponding structure of the first module is defined as in Section 2, and the $\mathcal{E}$-module structure on the second one is natural: $\omega(\omega_1 \otimes x) = (\omega\omega_1)\otimes x$ for every $\omega, \omega_1 \in \mathcal{E}$, $x \in M$.)

Lemma 3.1. *Let M be a σ^*-R-module and $M^* = \mathrm{Hom}_R(M,R)$. Then the $\mathcal{E}$-modules $\mathcal{P}(M,R)$ and $\mathcal{E}\otimes_R M^*$ are isomorphic.*

Proof. Consider the mapping $\phi : \mathcal{E}\otimes_R M^* \to \mathcal{P}(M,R)$ such that

$$\Big(\phi\Big(\sum_{i=1}^{k} a_i(\omega_i \bigotimes e_i^*)\Big)\Big)(e) = \sum_{i=1}^{k} a_i\omega_i(e_i^*(e)),$$
$$a_i \in R,\ \omega_i \in \mathcal{E},\ e_i^* \in M^*$$

for $i = 1,\dots,k$ and $e \in M$. It is easy to see that ϕ is a σ-homomorphism. To show that ϕ is bijective, one just needs to verify that the mapping $\psi : \mathcal{P}(M,R) \to \mathcal{E}\otimes_R M^*$ defined by

$$\psi\bigg(\sum_{i=1}^{s} a_i\beta_i\bigg) = \sum_{i=1}^{s} a_i(\gamma_{\beta_i} \bigotimes \gamma_{\beta_i}^{-1}\beta_i),$$
$$a_i \in R,\ \beta_i \in B(M,R)$$

for $i = 1,\dots,s$ is inverse of ϕ.

Let $P = (\omega_{ij})_{1\le i\le s, 1\le j\le m}$ be an $s\times m$-matrix over $\mathcal{E}$, and let $f_1,\dots,f_s$ be elements of the σ^*-ring R. We are going to consider the problem of solvability of a system of linear equations

$$Pu = f \tag{3.1}$$

with respect to unknown elements $u_1,\dots,u_m$ of the ring R (u and f denote the column of the unknowns $(u_1,\dots,u_m)^T$ and the column $(f_1,\dots,f_s)^T$, respectively). In what follows we treat the R-modules $E = R^m$ and $F = R^s$ as σ^*-R-modules such that $\alpha((a_1,\dots,a_k)^T) = (\alpha(a_1),\dots,\alpha(a_k))^T$ for any $\alpha \in \sigma^*$, $k = m$ or $k = s$, $a_1,\dots,a_k \in R$. The ring of $s\times m$-matrices $\mathcal{E}_{s\times m}$ (with entries in $\mathcal{E}$) will be also treated as a σ^*-R-module where $\alpha(\omega_{ij})_{1\le i\le s,1\le j\le m} = (\alpha(\omega_{ij}))_{1\le i\le s,1\le j\le m}$, $\alpha\in\sigma^*$, $(\omega_{ij})_{1\le i\le s,1\le j\le m}\in\mathcal{E}_{s\times m}$.

Lemma 3.2. *With the above notation, the $\mathcal{E}$-modules $\mathcal{P}(E,F)$ and $\mathcal{E}_{s\times m}$ are isomorphic.*

Proof. Let P_{ij} denote the matrix in $\mathcal{E}_{s\times m}$ whose only nonzero entry is 1 at the intersection of the ith row and jth column. Since matrices P_{ij}, $1\le i\le s$, $1\le j\le m$, generate the $\mathcal{E}$-module $\mathcal{E}_{s\times m}$, it is sufficient to define an isomorphism $\phi : \mathcal{E}_{s\times m}\to\mathcal{P}(E,F)$ on these matrices. We define $\phi(P_{ij})$ by its action on elements of E as follows. If $e=(c_1,\dots,c_m)^T\in E$, then $(\phi(P_{ij}))(e)=(0,\dots,c_j,\dots,0)^T$ (the ith coordinate is c_j, and all other coordinates are zeros). The inverse mapping $\psi : \mathcal{P}(E,F)\to\mathcal{E}_{s\times m}$ acts on generators $\beta\in B(E,F)$ of the $\mathcal{E}$-module $\mathcal{P}(E,F)$ as follows: $\psi(\beta) = (\gamma_\beta(a_{ij})\gamma_\beta)_{1\le i\le s,1\le j\le m}$ where elements $a_{ij}\in R$ are defined by the relationships $\gamma_\beta^{-1}\beta(e_k) = \sum_{j=1}^{s} a_{jk}f_j$ for the R-homomorphism $\gamma_\beta^{-1}\beta$. ($e_1,\dots,e_m$ and $f_1,\dots,f_s$ are standard bases of $E=R^m$ and $F=R^s$, respectively.) It is easy to check that $\phi\psi$ and $\psi\phi$ are identical mappings of $\mathcal{E}_{s\times m}$ and $\mathcal{P}(E,F)$, respectively. $\square$

In the rest of this section we use the notation introduced before Lemma 3.2. Furthermore, we consider $\mathcal{P}(E,F)$, $E^*=\mathrm{Hom}_R(E,R)$, $F\otimes_R\mathcal{P}(E,R)$, and $F\otimes_R(\mathcal{E}\otimes_R E^*)$ as σ^*-R-modules where the action of σ^* is defined as in Section 2 and in the beginning of this section. ($\mathcal{E}\otimes E^*$ is treated as a left $\mathcal{E}$-module with the natural structure $\omega'(\omega\otimes e^*)=\omega'\omega\otimes e^*$). In particular, if $f\otimes(\omega\otimes e^*)$ is a generator of $F\otimes_R(\mathcal{E}\otimes_R E^*)$ and $\alpha\in\sigma^*$, then $\alpha(f\otimes(\omega\otimes e^*))=\alpha(f)\otimes(\alpha\omega\otimes e^*)$.

Let us consider the diagram

$$
\begin{tikzcd}
\mathcal{P}(E,F) \arrow[rr, shift left, "\eta"] \arrow[dr, shift left, "\lambda"] & & F\bigotimes_R\mathcal{P}(E,R)) \arrow[ll, shift left, "\xi"] \arrow[dl, shift left, "\delta"] \\
& F\bigotimes_R(\mathcal{E}\bigotimes_R E^*) \arrow[ul, shift left, "\nu"] \arrow[ur, shift left, "\mu"] &
\end{tikzcd}
\tag{3.2}
$$

where all six mappings are difference homomorphisms defined at the generators as follows. $\delta(f\otimes(\omega\otimes e^*)) = f\otimes\omega(e^*(\cdot))$; $\mu(f\otimes\beta) = f\otimes(\gamma_\beta\otimes\gamma_\beta^{-1}\beta)$; $\nu(f\otimes(\omega\otimes e^*)) = \omega(e^*(\cdot))f$; $\lambda(\bar{\beta}) = \sum_{i=1}^m \bar{\beta}(e_i)\otimes(\gamma_{\bar{\beta}}\otimes e_i^*)$; $\eta(\bar{\beta}) = \sum_{i=1}^m \bar{\beta}(e_i)\otimes\gamma_{\bar{\beta}}e_i^*$; $\xi(f\otimes\beta) = \beta(\cdot)f$ for any $f\in F$, $\omega\in\mathcal{E}$, $e^*\in E^*$, $\beta\in B(E,R)$, $\bar{\beta}\in B(E,F)$ ($(e_i)_{1\le i\le m}$ denotes the standard basis of E over R, and $(e_i^*)_{1\le i\le m}$ denotes the dual basis of E^*).

Lemma 3.3. *All mappings in diagram* (3.2) *are difference isomorphisms, $\eta=\xi^{-1}$, $\mu=\delta^{-1}$, $\lambda=\nu^{-1}$, and the diagram is commutative.*

Proof. We shall prove that $\mu=\delta^{-1}$ and $\nu\mu=\xi$. (The other required relationships can be proved in a similar way.) Let $f\in F$, $e^*\in E^*$, $\sum_{\gamma\in\Gamma}a_\gamma\gamma\in\mathcal{E}$, and $\beta\in B(E,R)$. Then $(\mu\delta)(f\otimes(\sum_{\gamma\in\Gamma}a_\gamma\gamma\otimes e^*)) = \mu(f\otimes\sum_{\gamma\in\Gamma}a_\gamma\gamma(e^*(\cdot))) = \sum_{\gamma\in\Gamma}a_\gamma(f\otimes(\gamma\otimes\gamma^{-1}\gamma e^*)) = f\otimes\sum_{\gamma\in\Gamma}a_\gamma\gamma\otimes e^*)$ and $(\delta\mu)(f\otimes\beta) = \delta(f\otimes(\gamma_\beta\otimes\gamma_\beta^{-1}\beta)) = f\otimes\gamma_\beta\gamma_\beta^{-1}\beta = f\otimes\beta$, so $\mu=\delta^{-1}$. Furthermore, for any generator $f\otimes\beta$ of the $\mathcal{E}$-module $F\otimes_R\mathcal{P}(E,R)$ ($f\in F$, $\beta\in B(E,R)$), we have $(\nu\mu)(f\otimes\beta) = \nu(f\otimes(\gamma_\beta\otimes\gamma_\beta^{-1}\beta)) = \gamma_\beta(\gamma_\beta^{-1}\beta(\cdot))f = \beta(\cdot)f = \xi(f\otimes\beta)$ that proves the equality $\nu\mu=\xi$.

Let $P\in\mathcal{P}(E,F)$, and let $\overline{P}:\mathcal{P}(F,R)\to\mathcal{P}(E,R)$ be the homomorphism of $\mathcal{E}$-modules such that $\overline{P}(\beta)=\beta P$ for any $\beta\in B(E,R)$. Let $\phi_F:\mathcal{E}\otimes_R F^*\to\mathcal{P}(F,R)$ and $\psi_E:\mathcal{P}(E,R)\to\mathcal{E}\otimes_R E^*$ be difference isomorphisms defined in the proof of Lemma 3.1. Let $P^*=\psi_E\overline{P}\phi_F:\mathcal{E}\otimes_R F^*\to\mathcal{E}\otimes_R E^*$, $N=\operatorname{Ker}P^*$ and $M=\operatorname{Coker}P^*$. Applying $\operatorname{Hom}_{\mathcal{E}}(\cdot,R)$ to the exact sequence of left $\mathcal{E}$-modules

$$0\longrightarrow N\xrightarrow{i}\mathcal{E}\bigotimes_R F^*\xrightarrow{P^*}\mathcal{E}\bigotimes_R E^*\xrightarrow{j}M\longrightarrow 0 \tag{3.3}$$

(i and j are the natural injection and projection, respectively), we obtain the exact sequence of σ^*-R-modules

(3.4)
$$0\longrightarrow\operatorname{Hom}_{\mathcal{E}}(M,R)\xrightarrow{j^*}\operatorname{Hom}_{\mathcal{E}}\Big(\mathcal{E}\bigotimes_R E^*,R\Big)\xrightarrow{P^{**}}\operatorname{Hom}_{\mathcal{E}}\Big(\mathcal{E}\bigotimes_R F^*,R\Big)\xrightarrow{i^*}\operatorname{Hom}_{\mathcal{E}}(N,R)$$

Now let us consider the σ-homomorphism $\theta : \mathcal{P}(E,F) \to \mathrm{Hom}_{\mathcal{E}}(\mathcal{P}(F,R), \mathcal{P}(E,R))$ such that $(\theta(P))(P_1) = P_1P$ for every $P \in \mathcal{P}(E,F)$, $P_1 \in \mathcal{P}(F,R)$ and the exact sequence of σ^*-R-modules

$$\mathcal{P}(E,F) \xrightarrow{\lambda} F \bigotimes_R \Big(\mathcal{E} \bigotimes_R E^*\Big) \xrightarrow{\varepsilon} \mathrm{Hom}_R\Big(F^*, \mathcal{E} \bigotimes_R E^*\Big)$$
$$\xrightarrow{\rho} \mathrm{Hom}_{\mathcal{E}}\Big(\mathcal{E} \bigotimes_R F^*, \mathcal{E} \bigotimes_R E^*\Big) \xrightarrow{\pi} \mathrm{Hom}_{\mathcal{E}}(\mathcal{P}(F,R), \mathcal{P}(E,R))$$

where λ is the same as in diagram (3.2) and the other σ-homomorphisms are defined as follows: $(\varepsilon(f \otimes (\omega \otimes e^*))(f^*) = f^*(f) \otimes e^*$, $(\rho(h))(\omega \otimes f^*) = \omega h(f^*)$ and $\pi(\chi) = \phi_E \chi \psi_F$ for any $f \in F$, $\omega \in \mathcal{E}$, $e^* \in E^*$, $h \in \mathrm{Hom}_R(F^*, \mathcal{E} \otimes_R E^*)$ and $\chi \in \mathrm{Hom}_{\mathcal{E}}(\mathcal{E} \otimes_R F^*, \mathcal{E} \otimes_R E^*)$. ($\phi_E$ and ψ_F denote the σ-homomorphisms defined in the proof of Lemma 3.1.) It is easy to check that $\lambda, \varepsilon, \rho$ and π are isomorphisms of σ^*-R-modules. For example, ε^{-1} is defined by $\varepsilon^{-1}(\zeta) = \sum_{i=1}^{s} f_i \otimes \zeta(f_i^*)$ for every $\zeta \in \mathrm{Hom}_R(F^*, \mathcal{E} \otimes_R E^*)$ ($(f_i)_{1 \le i \le s}$ is the standard basis of F and $(f_i^*)_{1 \le i \le s}$ is the dual basis of the σ^*-R-module F^*).

Lemma 3.4. *With the above notation, $\theta = \pi\rho\varepsilon\lambda$.*

Proof. Clearly, it is sufficient to verify the equality at an element $\bar{\beta} \in B(E,F)$. If $\omega \in \mathcal{E}$ and $f^* \in F^*$, then $(\rho\varepsilon\lambda(\bar{\beta}))(\omega \otimes f^*) = \rho\varepsilon(\sum_{i=1}^{m} \bar{\beta}(e_i \otimes (\gamma_{\bar{\beta}} \otimes e_i^*))(\omega \otimes f^*) = \sum_{i=1}^{m} \omega f^*(\bar{\beta}(e_i))\gamma_{\bar{\beta}} \otimes e_i^*$.

Now, for any $\beta \in B(F,R)$, we have $(\pi\rho\varepsilon\lambda(\bar{\beta}))(\beta) = \phi_E(\rho\varepsilon\lambda(\bar{\beta}))\psi_F(\beta) = \phi_E(\rho\varepsilon\lambda(\bar{\beta}))(\gamma_\beta \otimes \gamma_\beta^{-1}\beta) = \phi_E \sum_{i=1}^{m} \beta(\bar{\beta}(e_i))\gamma_{\bar{\beta}} \otimes e_i^*$. To complete the proof one should note that for every $e = \sum_{k=1}^{m} c_k e_k \in E$ ($e_1, \dots, e_m$ is the standard basis of E and $c_1, \dots, c_m \in R$), $\phi_E(\sum_{i=1}^{m} \beta(\bar{\beta}(e_i))\gamma_{\bar{\beta}} \otimes e_i^*)(e) = \sum_{i=1}^{m} \beta\bar{\beta}(c_i e_i) = \beta\bar{\beta}(\sum_{i=1}^{m} c_i e_i) = \beta\bar{\beta}(e) = ((\theta(\bar{\beta})))(\beta)(e)$ whence $\theta = \pi\rho\varepsilon\lambda$. $\square$

Let us associate with every mapping $P \in \mathcal{P}(E,F)$ a set $\mathrm{Com}\, P$ consisting of all $f \in F$ such that for every pair $(G = R^t, P_1 \in \mathcal{P}(F,G))$ with the condition $P_1P = 0$, one has $P_1 f = 0$. Clearly, the image $\mathrm{Im}\, P$ of the mapping P is a subset of $\mathrm{Com}\, P$. The following example shows that the inclusion can be proper.

Example 3.5. Let $R = \mathbf{Q}(x)$ be the field of rational fractions over $\mathbf{Q}$ treated as an inversive difference field with one translation α such that $(\alpha f)(x) = f(2x)$ for every $f(x) \in R$. Let $\mathcal{E}$ denote the ring of inversive difference operators over R, $E = R^2$, $F = R^2$, and P the element of $\mathcal{P}(E, F)$ defined by the matrix $\begin{pmatrix} \alpha-1 & 0 \\ 0 & 1 \end{pmatrix}$. (By Lemma 3.2, every element of $\mathcal{P}(E, F)$ can be defined by a 2×2-matrix over $\mathcal{E}$.) Note that $\operatorname{Im} P = \left\{ \begin{pmatrix} \alpha-1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mid u_1, u_2 \in R \right\} = \left\{ \begin{pmatrix} (\alpha-1)u_1 \\ u_2 \end{pmatrix} \mid u_1, u_2 \in R \right\}$ is a proper $\mathcal{E}$-submodule of F. Indeed, it follows from the definition of α that 1 cannot be written as $(\alpha-1)u_1$ with $u_1 \in R$: if $u_1 = \Delta(a_n x^n + \cdots + a_1 x + a_0)/(b_m x^m + \cdots + b_1 x + b_0)$ (all coefficients a_i, b_j belong to R, $a_n \neq 0$, and $b_m \neq 0$), then $(\alpha - 1)u_1 = h_1(x)/h_2(x)$ where $h_1(x) = (2^n - 2^m)a_n b_m x^{m+n} + \cdots + 2(a_1 b_0 - a_0 b_1)x$ and $h_2(x) = 2^m b_m^2 x^{2m} + \cdots + 3b_1 b_0 x + b_0^2$. It is easy to see that $h_1(x) \neq h_2(x)$ (if $n \leq m$, then $\deg h_1(x) < \deg h_2(x)$; if $m < n$, then $\deg h_2(x) < \deg h_1(x)$).

On the other hand, $\operatorname{Com} P = F$. Indeed, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \operatorname{Com} P$ if and only if for every $s = 1, 2, \ldots$ and for every matrix $W = (\omega_{ij})_{1 \leq i \leq s, 1 \leq j \leq 2}$, the equality $W \begin{pmatrix} \alpha-1 & 0 \\ 0 & 1 \end{pmatrix} = 0$ implies $Wv = 0$. (In the last two equalities 0 in the right-hand sides denote the zero $s \times 2$- and $s \times 1$-matrices, respectively.) Since the ring $\mathcal{E}$ does not have zero divisors, the equality $W \begin{pmatrix} \alpha-1 & 0 \\ 0 & 1 \end{pmatrix} = 0$ implies that W is a zero matrix, so $Wv = 0$ for all $v \in F$ whence $\operatorname{Com} P = F$.

The following theorem gives a connection between $\operatorname{Im} P$ and $\operatorname{Com} P$ under the above conventions. The theorem allows one to reduce the description of the image of the operator P to the description of $\operatorname{Com} P$ using the spectral sequence from Theorem 2.4.

Theorem 3.6. *With the notation introduced before Lemma* 3.3*, for every $P \in \mathcal{P}(E, F)$, there exist isomorphisms of σ^*-R-modules $\vartheta_E : E \to \operatorname{Hom}_{\mathcal{E}}(\mathcal{E} \otimes_R E^*, R)$ and $\vartheta_F : F \to \operatorname{Hom}_{\mathcal{E}}(\mathcal{E} \otimes_R F^*, R)$ such that the following diagram is commutative.*

$$
\begin{array}{c}
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E}}\Big(\mathcal{E}\bigotimes\limits_{R} E^*, R\Big) \arrow[r, "P^{**}"] & \mathrm{Hom}_{\mathcal{E}}\Big(\mathcal{E}\bigotimes\limits_{R} F^*, R\Big) \\
E \arrow[u, "\vartheta_E"'] \arrow[r, "P"] & F \arrow[u, "\vartheta_F"']
\end{tikzcd}
\end{array}
\tag{3.5}
$$

Furthermore, we have the following properties of the exact sequence (3.4).

(i) $\mathrm{Im}\, j^* \cong \mathrm{Ker}\, P$;

(ii) $\mathrm{Ker}\, i^* \cong \mathrm{Com}\, P$;

(iii) $\mathrm{Com}\, P/\mathrm{Im}\, P \cong \mathrm{Ext}^1_{\mathcal{E}}(M, R)$.

Proof. The difference homomorphisms $\phi_E : \mathcal{E} \otimes_R E^* \to \mathcal{P}(E, R)$ and $\phi_F : \mathcal{E} \otimes_R F^* \to \mathcal{P}(F, R)$ defined in Lemma 3.1 induce homomorphisms of $\mathcal{E}$-modules $\bar{\phi}_E : \mathrm{Hom}_{\mathcal{E}}(\mathcal{E} \otimes_R E^*, R) \to \mathrm{Hom}_{\mathcal{E}}(\mathcal{P}(E, R)$ and $\bar{\phi}_F : \mathrm{Hom}_{\mathcal{E}}(\mathcal{E} \otimes_R F^*, R) \to \mathrm{Hom}_{\mathcal{E}}(\mathcal{P}(F, R)$ where $(\bar{\phi}_E(g))(\beta) = g(\gamma_\beta \otimes \gamma_\beta^{-1}\beta)$ for every $g \in \mathrm{Hom}_{\mathcal{E}}(\mathcal{E} \otimes_R E^*, R)$, $\beta \in B(E, R)$ and $\bar{\phi}_F$ acts in a similar way. Now one can define the mappings $\chi_E : E \to \mathrm{Hom}_{\mathcal{E}}(\mathcal{P}(E, R), R)$ and $\chi_F : F \to \mathrm{Hom}_{\mathcal{E}}(\mathcal{P}(F, R), R)$ by setting $(\chi_E(e))(\beta) = \beta(e)$ and $(\chi_F(f))(\beta_1) = \beta_1(f)$ for any $e \in E$, $f \in F$, $\beta \in \mathcal{P}(E, R)$, and $\beta_1 \in \mathcal{P}(F, R)$. It is easy to check that χ_E and χ_F are difference isomorphisms and the diagram

$$
\begin{array}{c}
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E}}(\mathcal{P}(E, R), R) \arrow[r, "\overline{P}^*"] & \mathrm{Hom}_{\mathcal{E}}(\mathcal{P}(F, R), R) \\
E \arrow[u, "\chi_E"'] \arrow[r, "P"] & F \arrow[u, "\chi_F"']
\end{tikzcd}
\end{array}
\tag{3.6}
$$

is commutative (the mapping $\overline{P}^*$ is obtained by applying the functor $\mathrm{Hom}_{\mathcal{E}}(\cdot, R)$ to the mapping $\overline{P} : \mathcal{P}(F, R) \to \mathcal{P}(E, R)$ considered above). The inverse difference isomorphism of χ_E is the mapping $\lambda_E = \chi_E^{-1} : \mathrm{Hom}_{\mathcal{E}}(\mathcal{P}(E, R), R) \to E$ such that $\lambda_E(g) = \sum_{i=1}^{m} g(e_i^*)e_i$ for any $g \in \mathrm{Hom}_{\mathcal{E}}(\mathcal{P}(E, R), R)$, and the inverse mapping of χ_F is defined similarly. (As before, $(e_i)_{1 \le i \le m}$ is the standard basis of the R-module E and $(e_i^*)_{1 \le i \le m} \subseteq \mathrm{Hom}_R(E, R) \subseteq \mathcal{P}(E, R)$ is the corresponding dual basis.) Indeed, for any $e \in E$, $(\lambda_E \chi_E)(e) = \sum_{i=1}^{m} (\bar{\chi}_E(e))(e_i^*) e_i = \sum_{i=1}^{m} e_i^*(e) e_i = e$, hence $\lambda_E \chi_E$ is the identity automorphism of E. Also, a routine computation shows that the mapping $\chi_E \lambda_E$ leaves fixed every element of $\mathrm{Hom}_{\mathcal{E}}(\mathcal{P}(E,R),R)$ (thus, $\lambda_E = \chi_E^{-1}$) and $\bar{P}^* \chi_E = \chi_F P$.

The commutativity of diagram (3.6) and Lemma 3.1 imply the commutativity of diagram (3.5) with $\vartheta_E = \mathrm{Hom}_{\mathcal{E}}(\phi_E, R)\chi_E$ and $\vartheta_F = \mathrm{Hom}_{\mathcal{E}}(\phi_F, R)\chi_F$ ($\phi_E : \mathcal{E} \otimes_R E^* \to \mathcal{P}(E,R)$ and $\phi_F : \mathcal{E} \otimes_R F^* \to \mathcal{P}(F,R)$ are difference isomorphisms defined in the proof of Lemma 3.1). Therefore, $\mathrm{Ker}\, P \cong \mathrm{Ker}\, P^{**} = \mathrm{Im}\, j^* = \mathrm{Hom}_{\mathcal{E}}(M,R)$ (see the exact sequence (3.3) where $M = \mathrm{Coker}\, P^*$). This proves statement (i).

To prove (ii) assume first that $\zeta \in \mathrm{Ker}\, i^*$ and $PP' = 0$ for every $P' \in \mathcal{P}(F,G)$ ($G = R^t$ for some positive integer t). Then $P^* P'^* = (P'P)^* = 0$ hence $\mathrm{Im}\, P'^* \subseteq \mathrm{Ker}\, P^* = \mathrm{Im}\, i$, and one can consider the well-defined mapping $\varrho = i^{-1} P'^* : \mathcal{E} \otimes_R G^* \to N$ (as in sequence (3.3), $N = \mathrm{Ker}\, i$). Now, if $\zeta = \vartheta_F(z) \in \mathrm{Ker}\, i^*$ ($z \in F$), then $P'^{**}(\zeta) = (i\varrho)^*(\zeta) = \varrho^* i^*(\zeta) = 0$ hence $(\vartheta_G P')(\zeta) = 0$. It follows that $P'(z) = 0$, so that $z \in \mathrm{Com}\, P$. Thus, $\mathrm{Ker}\, i^* \subseteq \vartheta_F(\mathrm{Com}\, P)$.

Conversely, let $z \in \mathrm{Com}\, P$ and $\zeta = \vartheta_F(z)$. Let us fix some $x \in N$ and consider the homomorphism of $\mathcal{E}$-modules $\delta : \mathcal{E} \to N$ such that $\delta(1) = x$. The composition of the σ^*-R-isomorphisms $\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, \mathcal{E} \otimes_R F^*) \to \mathcal{E} \otimes_R F^* \xrightarrow{\phi_F} \mathcal{P}(F,R)$, where the first mapping is the natural isomorphism, sends the element $i\delta \in \mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, \mathcal{E} \otimes_R F^*)$ to some element $P' \in \mathcal{P}(F,R)$. Denoting the natural isomorphism of $\mathcal{E}$-modules $\mathcal{E} \to \mathcal{E} \otimes_R R^*$ by τ, we obtain $i\delta = P'^* \tau$. Indeed, let $x = i\delta(1) = \sum_{k=1}^{d} a_i(\omega_k \otimes f_k^*)$ where $a_k \in R$, $\omega_k = \sum_{l=1}^{d_k} b_{kl}\gamma_{kl} \in \mathcal{E}$ ($\gamma_{kl} \in \Gamma$), and $f_k^* \in F^*$ ($1 \le k \le d$). Then $(P'^*\tau)(1) = P'^*(1 \otimes 1^*) = \psi_F \overline{P}' \phi_R(1 \otimes 1^*) = \psi_F \overline{P}'(1^*(\cdot)) = \psi_F \sum_{k=1}^{d} a_k \omega_k(f_k^*(\cdot)) = \psi_F \sum_{k=1}^{d} a_k \sum_{l=1}^{d_k} b_{kl}\gamma_{kl}(f_l^*(\cdot)) = \sum_{k=1}^{d} a_k \sum_{l=1}^{d_k} b_{kl}(\gamma_{kl} \otimes \gamma_{kl}^{-1}\gamma_{kl} f_l^* = \sum_{k=1}^{d} a_k(\omega_k \otimes f_k^*) = x = i\delta(1)$ whence $i\delta = P'^*\tau$.

Since $\mathrm{Im}\, i = \mathrm{Ker}\, P^*$, $P^*P'^*\tau = P^* i\delta = 0$ whence $P^*P'^* = 0$. Applying * we obtain that $P'P = 0$. Furthermore, since $z \in \mathrm{Com}\, P$, we have $P'(z) = 0$ and $P'^{**}(\zeta) = \zeta P'^* = 0$. Therefore, $\zeta(i(x)) = \zeta(i\delta(1)) = \zeta(P'^*\tau(1)) = 0$, that is, $(i^*(\zeta))(x) = 0$. Since x is an arbitrary element of N, $i^*(\zeta)) = 0$, that is, $\zeta \in \mathrm{Ker}\, i^*$. Thus, $\vartheta_F(\mathrm{Com}\, P)$ is contained in $\mathrm{Ker}\, i^*$ whence $\vartheta_F(\mathrm{Com}\, P) = \mathrm{Ker}\, i^*$.

In order to prove the last statement of the theorem, let us break the exact sequence (3.3) into two short exact sequences of $\mathcal{E}$-modules, $0 \to N \xrightarrow{i} \mathcal{E} \otimes_R F^* \xrightarrow{q} L \to 0$ and $0 \to L \xrightarrow{\varepsilon} \mathcal{E} \otimes_R E^* \xrightarrow{j} M \to 0$ where $L = \operatorname{Im} P^*$, q is a projection, and ε is the embedding. Applying functor $\operatorname{Hom}_{\mathcal{E}}(\cdot, R)$ to these two sequences we obtain the exact sequences

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{E}}(L, R) \xrightarrow{q^*} \operatorname{Hom}_{\mathcal{E}}\Big(\mathcal{E} \bigotimes_R F^*, R\Big) \xrightarrow{i^*} \operatorname{Hom}_{\mathcal{E}}(N, R)$$

and

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{E}}(M, R) \xrightarrow{j^*} \operatorname{Hom}_{\mathcal{E}}\Big(\mathcal{E} \bigotimes_R E^*, R\Big) \xrightarrow{\varepsilon^*} \operatorname{Hom}_{\mathcal{E}}(L, R)$$

$$0 \longrightarrow \operatorname{Ext}^1_{\mathcal{E}}(M, R) \longrightarrow \operatorname{Ext}^1_{\mathcal{E}}\Big(\mathcal{E} \bigotimes_R E^*, R\Big) = 0.$$

where the mapping q^* identifies $\operatorname{Hom}_{\mathcal{E}}(L, R)$ with $\operatorname{Ker} i^* = \vartheta_F(\operatorname{Com} P)$ and $\vartheta_F^{-1} q^* \varepsilon^* \vartheta_E = P$. Thus, $\operatorname{Ext}^1_{\mathcal{E}}(M, R) \cong \operatorname{Hom}_{\mathcal{E}}(L, R) / \varepsilon^*(\operatorname{Hom}_{\mathcal{E}}(\mathcal{E} \otimes_R E^*, R)) = \operatorname{Hom}_{\mathcal{E}}(L, R) \,/\, q^* \vartheta_E(E) (q^*)^{-1} \vartheta_F(\operatorname{Com} P) \,/\, (q^*)^{-1} \vartheta_F P(E) \cong \operatorname{Com} P \,/\, \operatorname{Im} P$.

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