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TILING THE UNIT SQUARE WITH 5 RATIONAL TRIANGLES

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ABSTRACT. There are 14 distinct ways to tile the unit square (modulo the symmetries of the square) with 5 triangles such that the 5-tiling is not a subdivision of a tiling using fewer triangles. We demonstrate how to construct infinitely many *rational* tilings in each of the 14 configurations. This stands in contrast to a long standing inability to find rational 4-tilings of the unit square in the so-called χ-configuration.

1. Introduction. Recall that a *rational triangle* is a triangle whose sides have rational length and consider the following:

Question 1. *For each $n \in \mathbf{N}$, in what ways can the unit square be tiled with n rational triangles?*

It is clear that the unit square cannot be divided into two rational triangles and, in [**7**], Guy similarly disposes of the case $n = 3$. Guy goes on to prove that there are essentially four distinct ways to tile the square with four triangles and along with Bremner [**2, 3**] proved that at least three of them admit rational tilings. The goal of this paper is to solve the $n = 5$ case. Before moving on however, it is worth noting that the remaining $n = 4$ case is the subject of problem D19 in [**8**] and can be articulated as:

Question 2. *Is there a point on the interior of the unit square that is a rational distance to each of the four corners?*

More formally, and in keeping with the language established by Guy, we say that a *proper n-tiling*, or simply an *n-tiling*, is a set of n triangles

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whose union is the unit square such that the intersection of no two triangles is a region of positive area. Observe that the group action of the dihedral group on the unit square induces a group action on a tiling. We can then define two n-tilings to be *equivalent* if they are in the same orbit under this group action and say that they are *distinct* otherwise. We say that two n-tilings, $\mathbf{T}_0$ and $\mathbf{T}_1$, are in the same *n-configuration* if $\mathbf{T}_0$ can be continuously deformed into a tiling equivalent to $\mathbf{T}_1$ through a sequence of n-tilings that preserve the underlying graph structure and keep the corner vertices fixed.

Further observe that some n-tilings are subdivisions of an m-tiling for some $m < n$. Any such n-tiling is said to be *derivative*, while all others are said to be *primitive*. Likewise, an n-configuration is said to be derivative if all the n-tilings in the n-configuration are derivative and said to be primitive otherwise. Finally, we say that an n-tiling is *rational* if all the triangles in the tiling are rational triangles.

The main theorem of this paper can now be expressed more precisely:

Theorem 1. *There are* 14 *primitive* 5*-configurations, illustrated in Figure* 2*, and each of these configurations admit infinitely many distinct rational tilings.*

2. The 14 primitive 5-configurations. We begin by outlining a proof that there are 14 primitive 5-configurations. Observe that many 5-configurations contain an edge from a corner vertex to an edge of the unit square and that this divides the unit square into a triangle and a quadrilateral. Further observe that, if such a 5-configuration is primitive, then there can be no edges interior to the triangle and hence the quadrilateral must be tiled with 4 triangles in a way that is isomorphic as a graph to a primitive 4-configuration. By enumerating all possible ways to do this and eliminating derivative configurations, we find 11 distinct primitive 5-configurations with this property.

Let us define a vertex on the interior of the unit square to be a *straight angle* vertex if it is collinear with some pair of its adjacent vertices and to be a *nonstraight angle* vertex otherwise. Let us denote the number of all, boundary, straight angle and nonstraight angle vertices in any given configuration by V, B, I_S and I_N respectively. Similarly, let E

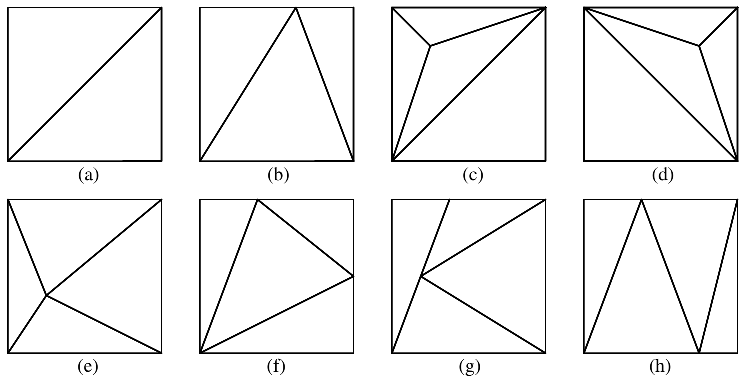

FIGURE 1. Examples of tilings in (a) the only 2-configuration, and (b) the only primitive 3-configuration; examples of (c) a derivative 4-tiling and (d) an equivalent 4-tiling; and finally examples of tilings in each of the primitive 4-configurations, named by Guy (e) the χ-configuration, (f) the δ-configuration, (g) the κ-configuration and (h) the ν-configuration.

and E_I denote the number of all edges and edges on the interior of the unit square in any given configuration, respectively. In addition to the trivial equalities $V = 4 + B + I_S + I_N$ and $E = 4 + B + E_I$, Guy [**7**] points out that, for any n-tiling, the Euler characteristic and a clever counting of the number of edges give:

$$V - E + n = 1 \quad \text{and} \quad 2E = 3n + 4 + I_S + B.$$

These relations imply that in any 5-tiling:

$$I_S + 2\ I_N = 3 - B \quad \text{and} \quad E_I = 4 + I_S + I_N,$$

and consequently, $0 \le B \le 3$. We can then prove that there are three additional primitive 5-configurations by considering each of the possible values of B and their impact on I_S, I_N, and E_I. (See [**5**] for details.) All 14 of the configurations can be seen in Figure 2.

3. Infinitely many rational tilings in each configuration.

3.1 Types of configurations. We now work toward giving explicit constructions for generating an infinite family of *rational* tilings in each of the 14 primitive 5-configurations. As a first step, we divide the configurations into a number of useful categories.

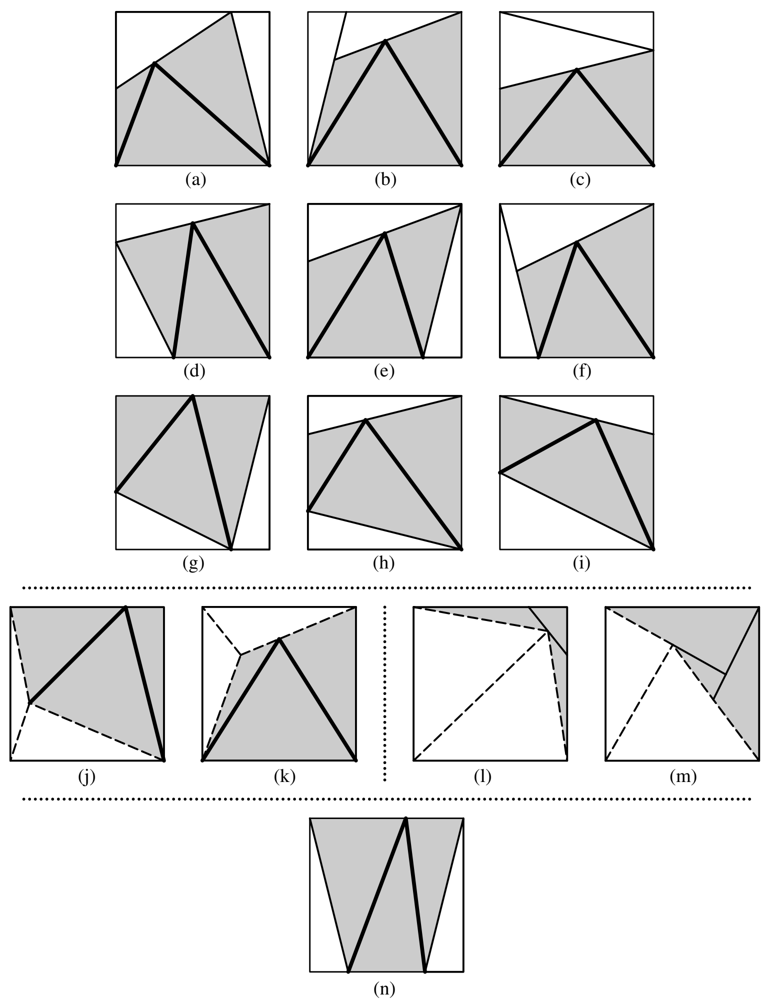

FIGURE 2. The 14 distinct, primitive, 5-configurations: (a)–(i) the simple Λ-type configurations; (j) the χ+Λ and (k) the Y+Λ configurations; (l) the dragonfly and (m) the super-χ configurations; and (n) the ω-configuration. Dashed lines are edges from a vertex to 3 corner vertices and bold edges form a Λ. The shaded regions are sub-quadrilaterals (with the top edges of those in Λ-type configurations being the apex lines). Configurations (a)–(j) and (n) contain an edge from a corner vertex to a boundary.

If any tiling in a 5-configuration contains a triangle, $\langle P_0, P_1, P, P_0\rangle$, such that removing the edges $\langle P, P_0\rangle$ and $\langle P, P_1\rangle$ leaves two triangles and a (convex) quadrilateral, $\langle P_0, P_1, P_2, P_3, P_0\rangle$, then we say that the configuration is a Λ-*type configuration*. We say that the edges $\langle P, P_0\rangle$ and $\langle P, P_1\rangle$ *form a* Λ in the tiling, the line through P_0 and P_1 is the *base line*, the line though P, P_2 and P_3 is the *apex line* and $\langle P_0, P_1, P_2, P_3, P_0\rangle$ is the *sub-quadrilateral* (relative to P). Observe that some Λ-type configurations admit more than one pair of edges that form a Λ. (In particular, see Figures 2(d), (e), (h), (i) and (n).)

We further divide the Λ-type configurations into the ω-*configuration*, the only Λ-type configuration for which the base line must be parallel to the apex line; the $Y{+}\Lambda$ and $\chi{+}\Lambda$ configurations, the only Λ-type configurations which contain a vertex adjacent to three corner vertices; and the *simple* Λ-*type configurations*, all remaining Λ-type configurations. The remaining two configurations we name the *dragonfly*, Figure 2 (l), and the *super*-χ, Figure 2 (m), configurations.

Like the $Y{+}\Lambda$ and $\chi{+}\Lambda$ configurations, the dragonfly and super-χ configurations contain a vertex, P, which is adjacent to three corner vertices, c_1, c_2 and c_3. If we let c_4 be the remaining corner vertex and let c_1 be the vertex opposite c_4, then we say that $\langle P, c_2, c_4, c_3, P\rangle$ is the *sub-quadrilateral* in these cases. Note that there is no ambiguity in the two definitions of sub-quadrilateral since they agree on the $Y{+}\Lambda$ and $\chi{+}\Lambda$ configurations. There is, however, one important difference between the sub-quadrilaterals found in the $Y{+}\Lambda$ and $\chi{+}\Lambda$ configurations versus those found in the dragonfly and super-χ configurations. In the former two configurations (and in fact, in all Λ-type configurations), the sub-quadrilateral is convex, while in the latter two configurations, the sub-quadrilateral is concave. Given this, we say that the vertex adjacent to three corner vertices is a *convex vertex* in the $Y{+}\Lambda$ and $\chi{+}\Lambda$ configurations and a *concave vertex* in the dragonfly and super-χ configurations.

Our strategy for generating infinitely many rational tilings in each of these configurations is to first produce all edges which are not interior to a sub-quadrilateral. (In the case of configurations which admit more than one pair of Λ forming edges, we simply choose a pair arbitrarily and work relative to that choice.) We then prove that given almost any sub-quadrilateral (whose edges satisfy certain rationality conditions), there are infinitely many ways to divide that sub-quadrilateral into three rational triangles.

In all cases, our method for producing rational tilings leads to equations of the form $y^2 = f(x)$, where $f(x)$ is a monic quartic polynomial with coefficients in the rationals or in some rational function field. This equation defines an elliptic curve and since we choose to work directly with this quartic model of the elliptic curve, we begin by summarizing some facts about the arithmetic of elliptic curves in this form. (These facts are discussed in greater detail in places like [**9**, Chapter II]. Still, we are motivated to include a summary since it is atypical to work with an elliptic curve in the quartic form directly.)

3.2 Arithmetic on the quartic model of an elliptic curve. Let K be any field, and recall that a *smooth curve* in $\mathbf{P}^n(\overline{K})$ is a (projective) variety of dimension one for which there exists a well defined, non-vanishing tangent at every point. Further recall that the *homogeneous ideal of a smooth curve* C, denoted $I(C)$, is the set of homogeneous polynomials defined over $\overline{K}$ which vanish on C and that we say C is *defined over* K if $I(C)$ can be generated by a set of homogeneous polynomials with coefficients in K.

Now, if C is a curve defined over K and $\{F_1, F_2, \dots, F_m\}$ is a set of generators for $I(C)$, then we call the set of equations $\{F_1 = 0,\ F_2 = 0, \dots, F_m = 0\}$ a *projective model* for the curve C and we define the K*-rational points* of C, denoted $C(K)$, to be $C \cap \mathbf{P}^n(K)$. (Note that if $K = \mathbf{Q}$, we simply say that $C(\mathbf{Q})$ is the set of *rational points* on C.) Similarly, suppose $k \in \mathbf{Z}$, $1 \le k \le n+1$, and for each $j \in \{1, 2, \dots, m\}$, we let f_j be the dehomogenization of F_j with respect to the kth coordinate, then we call the set of equations $\{f_1 = 0,\ f_2 = 0, \dots, f_m = 0\}$ an *affine model* for the curve C.

Recall that the *genus* of a curve is given by the Riemann-Roch theorem, see for example [**9**, Chapter II, Section 5], and can be thought of as a measure of the complexity of a curve. Finally, recall that an *elliptic curve*, E, *defined over* K is a smooth curve of genus one defined over K, together with some K-rational point. For example, if E is a smooth curve in $\mathbf{P}^2(\overline{K})$ with projective model:

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3,$$

where $a_1, a_2, a_3, a_4, a_6 \in K$, then E together with the K-rational point

$[X, Y, Z] = [0, 1, 0]$ is an elliptic curve defined over K. An elliptic curve in this form is said to be in *Weierstrass form*. Consider also, the curve E' in $\mathbf{P}^3(\overline{K})$ with projective model:

$$\begin{aligned} ZW &= X^2 \\ Y^2 &= W^2 + c_3WX + c_2X^2 + c_1XZ + c_0Z^2, \end{aligned} \tag{1}$$

where $c_0, c_1, c_2, c_3 \in K$. E' has genus one and contains the two K-rational points $[W, X, Y, Z] = [1, 0, 1, 0]$ and $[1, 0, -1, 0]$, denoted $\mathcal{O}'$ and $\mathcal{O}$, respectively. Therefore, if E' is smooth, then E' together with $\mathcal{O}$ is an elliptic curve defined over K. An affine model of this curve, corresponding to dehomogenizing at Z, is $y^2 = f(x)$, where $y = Y/Z$, $x = X/Z$ and $f(x) = x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$. We call this the *quartic model* of the elliptic curve and we say that $\mathcal{O}$ and $\mathcal{O}'$ are the *points at infinity*, relative to this quartic model.

Part of what makes elliptic curves defined over K special is the existence of a group operation on the set of their K-rational points. Moreover, it is well known that, for an elliptic curve in Weierstrass form, one possible group operation can be interpreted geometrically in an elegant way. It is less well known that the quartic model of an elliptic curve admits an equally elegant geometric interpretation of one of its group laws. In particular, suppose E' is defined by the quartic model (1). Now suppose $P, Q \in E'(K)$ and consider the hyperplane, H, through the points $\mathcal{O}', P$ and Q. For example, if $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, and we let $g_{P,Q}(x) = (x - x_1)^2 + A(x - x_1) + y_1$, where $A = (y_2 - y_1)/(x_2 - x_1) - (x_2 - x_1)$, then H is given by the equation $y = g_{P,Q}$. By Bezout's theorem, see [**10**, Appendix A], H must intersect the curve in four points, counting multiplicities. These four points can be found by solving the system $y^2 = f(x)$, $y = g_{P,Q}(x)$. Since three of these four points of intersection are the known K-rational points, P, Q and $\mathcal{O}'$, and since the coefficients of $f(x)$ and $g_{P,Q}(x)$ are K-rational, the fourth point of intersection, $P \star Q = [w_0, x_0, y_0, z_0]$, must be K-rational as well. If we define $P + Q$ to be $[w_0, x_0, -y_0, z_0]$, the reflection of $P \star Q$ across the x-axis when $P \star Q$ is affine, then this defines a group law on $E'(K)$. (Though proving that this defines a valid group law can be done by chasing through the elementary algebra, a more appealing way to prove this fact is to use the theory of divisors. Again, one can consult [**9**, Chapter III, Section 3] or [**4**, Section 3.1] for details.) Under this group law, $\mathcal{O}$ is the identity of the group and the negative of a point, P, is the fourth point of intersection of E' with the hyperplane which contains P and is tangent at $\mathcal{O}'$.

We conclude our summary by recalling that for any elliptic curve, E, defined over $\mathbf{Q}$, the Mordell-Weil theorem says that $E(\mathbf{Q})$ is finitely generated and Mazur's theorem says that the torsion subgroup of $E(\mathbf{Q})$ must be isomorphic to one of $\mathbf{Z}/n\mathbf{Z}$, $1 \leq n \leq 12$, $n \neq 11$ or $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2m\mathbf{Z}$, $1 \leq m \leq 4$.

3.3 The main lemmas. We are now ready to prove a few lemmas which lead to the main result. We begin with statements about the ways in which particular (non-parallel) lines interact. In the following, we let $G(x) = (x^2 - 1)/(2x)$ and interpret a line of slope $G(0)$ as a vertical line. (This is consistent with thinking of the lines as being affine pieces of a projective line.)

Lemma 2. *Let L be a line of slope μ which contains a rational point. The following conditions are equivalent*:

(1) *L contains two rational points at rational distance.*

(2) *$\mu = G(\sigma)$ for some $\sigma \in \mathbf{Q}$.*

(3) *Every pair of rational points on L are at rational distance.*

Proof. Suppose $\alpha, \beta \in \mathbf{Q}$. The equivalence follows from the fact that rational solutions to $d^2 = (x - \alpha)^2 + (y - \beta)^2$ can be parameterized by $y = G(\sigma)(x - \alpha) + \beta$, $\sigma \in \mathbf{Q}$ and by directly computing the distance between any two points on the line defined by that equation. □

Lemma 3. *Let $L_{\sigma,\beta}$ be the line defined by*

$$L_{\sigma,\beta}: \ y = G(\sigma)(x + \beta).$$

For infinitely many $\alpha, \beta, \sigma \in \mathbf{Q}$, there is a dense set of rational points on $L_{\sigma,\beta}$ at rational distance to both $(0,0)$ and $(\alpha, 0)$. Moreover, if P is any such point, L_0 is the line containing P and $(0,0)$, and L_α is the line containing P and $(\alpha, 0)$, then L_0 has slope $G(\lambda)$ and L_α has slope $G(\mu)$ for some $\lambda, \mu \in \mathbf{Q}$.

Proof. We begin by treating α, β and σ as variables and letting $K = \mathbf{Q}(\alpha, \beta, \sigma)$. If P is the point of intersection between $L_{\sigma,\beta}$ and L_0, the line of slope $G(\lambda)$ through $(0,0)$, then P is a K-rational point and by Lemma 2 is a K-rational distance to both $(0,0)$ and $(-\beta, 0)$. If we let d be the distance from P to $(\alpha, 0)$,

$$\begin{aligned}
\rho &= 2\ G(\sigma)\ (\alpha+\beta),\\
u &= \frac{\rho}{\lambda-1},\\
\kappa &= \alpha^2 + G(\sigma)^2\ \beta^2
\end{aligned}$$

and

$$v = d\left(\frac{u^2\ (\lambda^2 - 2\ G(\sigma)\ \lambda - 1)}{\rho}\right),$$

then we have that:

$$v^2 = u^4 + 2\ (\rho - 2\alpha)\ u^3 + (\rho^2 - 6\alpha\rho + 4\kappa)\ u^2 - 2\ \rho\ (\alpha\rho - 2\kappa)\ u + \rho^2\ \kappa.$$

This defines an elliptic curve, $\mathcal{E}$, in u and v, defined over K. The point $(-\beta, 0)$ on the two lines $L_{\sigma,\beta}$ and $y = 0$ corresponds to the K-rational points $Q = (-\rho/2, \rho^2/4)$, $R = (-\rho/2, -\rho^2/4)$ and the two points at infinity, $\mathcal{O}$ and $\mathcal{O}'$ on $\mathcal{E}$. With the group law defined as above, we see that R is a point of order 2 in $\mathcal{E}(K)$ and that $\mathcal{O}'$ is a point of infinite order in $\mathcal{E}(K)$. (Note that since $Q = \mathcal{O}' - R$, Q is a point of infinite order as well.)

Let us now consider the *specialization* of $\mathcal{E}$ to a curve defined over $\mathbf{Q}$ gotten by choosing particular rational values for the variables α, β and σ. We abuse notation and denote this curve by $E_{\alpha,\beta,\sigma}$. (So the same names, α, β and σ, on the one hand, represent the variables in the context of $\mathcal{E}$ defined over K and on the other hand, represent particular rational values in the context of the specialization $E_{\alpha,\beta,\sigma}$ defined over $\mathbf{Q}$.) We further the abuse by letting $\mathcal{O}'$ represent both the point in $\mathcal{E}(K)$ and its specialization to a point in $E_{\alpha,\beta,\sigma}(\mathbf{Q})$.

By Mazur's theorem, the torsion subgroup of $E_{\alpha,\beta,\sigma}(\mathbf{Q})$ can be only one of a finite number of possibilities. Therefore, there are a finite number of conditions on α, β and σ such that $\mathcal{O}' \in E_{\alpha,\beta,\sigma}(\mathbf{Q})$ is not a

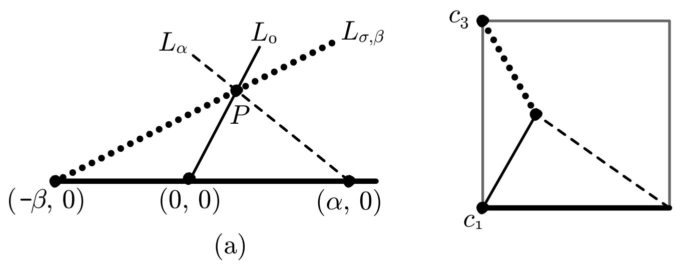

FIGURE 3. (a) A diagram of the lines and points in Lemma 3 and the special case for getting points at rational distance to the three corner vertices c_1, c_2 and c_3. In particular, the cases in which (b) P is a convex vertex and (c) P is a concave vertex.

point of infinite order. This leaves infinitely many values of $\alpha, \beta, \sigma \in \mathbf{Q}$ such that $\mathcal{O}'$ is a point of infinite order in $E_{\alpha,\beta,\sigma}(\mathbf{Q})$. (Note that one could get the same result as an immediate consequence of a theorem of Néron. See [**9**, Appendix C, Section 20] for details.) For each of these choices of $\alpha, \beta, \sigma \in \mathbf{Q}$, we get infinitely many points in $E_{\alpha,\beta,\sigma}(\mathbf{Q})$ and hence a dense set of rational points on $L_{\sigma,\beta}$ at rational distance to $(0, 0)$ and $(\alpha, 0)$.

The fact that the line, L_α through P and $(\alpha, 0)$ has slope $G(\mu)$ for some $\mu \in \mathbf{Q}$ is an immediate consequence of Lemma 2. $\square$

We now point out that this lemma can be slightly generalized as follows. Suppose $\alpha, \sigma \in \mathbf{Q}$, let L be a line of slope $G(\sigma)$ through a rational point P, and consider the translation map composed with a rotation-dilation map which takes P to $(0, 0)$ and some other rational point on L to $(\alpha, 0)$. This map is a *birational map*, giving a one-to-one and onto correspondence between rational points on L and rational points on the line $y = 0$. Therefore, the map preserves rationality of distances. Now suppose $\sigma, \tau \in \mathbf{Q}$, $\sigma \neq \tau$ and there are two lines of slope $G(\sigma)$ and $G(\tau)$ each containing a rational point. The remark above together with the lemma then imply that, in general, for each pair of rational points, P_0 and P_α, on one line, there is a dense set of rational points on the other line that are at rational distance to both P_0 and P_α.

Furthermore, we can apply this lemma to the particular case $\alpha = 1$ and $\beta = 1/G(\sigma)$, so that $L_{\sigma,\beta}$ is a line of slope $G(\sigma)$ through the point

$(0, 1)$. The lemma then gives us that, for infinitely many $\sigma \in \mathbf{Q}$, there is a dense set of rational points, P, on the line $L_{\sigma,\beta}$ at rational distance to $(0, 0)$ and $(1, 0)$. By Lemma 2, each such P is also at rational distance to $(0, 1)$ and the line containing P through $(0, 0)$ and the line containing P through $(1, 0)$ have slope $G(\sigma_1)$ and $G(\sigma_2)$, respectively, for some $\sigma_1, \sigma_2 \in \mathbf{Q}$. Furthermore, if $\sigma \in \mathbf{Q}$ is chosen so that $G(\sigma) < -1$, then P is a convex vertex and if $\sigma \in \mathbf{Q}$ is chosen so that $-1 < G(\sigma) < 0$, then P is a concave vertex. (See Figures 3(b) and (c).) An immediate consequence is then:

Corollary 4. *There are infinitely many convex and infinitely many concave rational, interior points, P, at rational distance to $c_1 = (0, 0)$, $c_2 = (1, 0)$ and $c_3 = (0, 1)$, such that for each i, there exists $\sigma_i \in \mathbf{Q}$ such that P is on a line through c_i of slope $G(\sigma_i)$.*

3.4 Infinitely many rational tilings: All cases except the $\boldsymbol{\omega}$-configuration. Immediate consequences of the proposition and lemmas above are:

Theorem 5. *There are infinitely many rational tilings in each of the simple Λ-type configurations.*

Proof. In each of the simple Λ-type configurations, we choose one pair of Λ forming edges. For any such choice, there are two non-Λ forming, interior edges. Let L_1 and L_2 be the lines given by extending these edges. One of these lines, say L_1, must go through a corner vertex, c, and some boundary vertex, b. Let L_1 have slope $G(\sigma)$. Since the edges of the unit square are lines of slope $G(0)$ or $G(1)$ through rational points, b will be a $\mathbf{Q}(\sigma)$-rational point and, by Lemma 2, at $\mathbf{Q}(\sigma)$-rational distance to c. Similarly, let L_2 be a line of slope $G(\tau)$.

We then have a (convex) sub-quadrilateral whose vertices are $\mathbf{Q}(\sigma, \tau)$-rational points and whose boundaries are determined by lines of slope $G(\sigma_i)$, $1 \le i \le 4$, where each $\sigma_i \in \{1, 0, \sigma, \tau\}$. We then have that there are infinitely many choices of $\sigma, \tau \in \mathbf{Q}$ such that the apex line and the base line of the sub-quadrilateral are not parallel and, by Lemma 3, for infinitely many such choices of σ and τ, there is a dense set of points on the apex line which yield rational 5-tilings. $\square$

Note that it is possible for the apex line to be parallel to the base line in some of the simple Λ-type tilings. This theorem does not demonstrate that such rational tilings are possible. That there are in fact rational tilings in the simple Λ-type configurations in which the apex line and base line are parallel can be deduced by following an argument like the one given below for proving the existence of rational tilings in the ω-configuration.

Theorem 6. *There are infinitely many rational tilings in the $\chi+\Lambda$ and $Y+\Lambda$ configurations.*

Proof. By Corollary 4, there are infinitely many rational, convex vertices, P, such that the edges of the sub-quadrilateral have slope $G(\sigma_i)$, $\sigma_i \in \mathbf{Q}$. By Lemma 3, we can then find infinitely many ways to divide the sub-quadrilateral into three rational triangles. $\square$

Theorem 7. *There are infinitely many rational tilings in the dragonfly configuration.*

Proof. By Corollary 4, there are infinitely many rational, concave vertices, P, at rational distance to three corners. By Lemma 2, any line of slope $G(\sigma)$, $\sigma \in \mathbf{Q}$ through P will intersect two boundaries of the unit square at rational points at rational distance to P. Therefore, for each such P, we have infinitely many rational 5-tilings in the dragonfly configuration containing P. $\square$

Theorem 8. *There are infinitely many rational tilings in the super-χ configuration.*

Proof. Let c_1, c_2, c_3 and c_4 be the corner vertices of the unit square (where c_i is adjacent to c_{i+1}). Let P be a rational, concave vertex at rational distance to c_1, c_2 and c_3 guaranteed by Corollary 4. Let L_i be the line through P and c_i of slope $G(\sigma_i)$, $\sigma_i \in \mathbf{Q}$ for $i = 1, 2, 3$, and let $M_{P,c4}$ be the slope of the line through P and c_4.

Now, let L_4 be any line through c_4 of slope $G(\sigma_4)$ where $\sigma_4 \in \mathbf{Q}$ such that $G(\sigma_4) > M_{P,c4}$, let v_{14} be the point of intersection between

L_1 with L_4, and let v_{34} be the point of intersection between L_3 and L_4. The boundary of the unit square together with the edges $\langle P, c_i \rangle$, $i = 1, 2, 3$, $\langle P, v_{14} \rangle$, $\langle v_{34}, v_{14} \rangle$ and $\langle v_{14}, c_4 \rangle$ then form a rational 5-tiling in the super-χ configuration. Hence, there are infinitely many such tilings. $\square$

3.5 Infinitely many rational tilings: The ω-configuration.

Theorem 9. *There are infinitely many rational tilings in the ω-configuration.*

Proof. Let b_1, b_2 and b_3 be the boundary points given by:

$$b_1 = (G(\tau), 0),$$
$$b_2 = (G(\lambda) + G(\tau), 1)$$

and

$$b_3 = (1 - G(\sigma), 0).$$

By Lemma 2, the distance from $(0, 1)$ to b_1, the distance from b_1 to b_2 and the distance from b_3 to $(1, 1)$ must be rational for all $\sigma, \lambda, \tau \in \mathbf{Q}$. If we let

$$v = 2\ d\ \lambda\ \sigma^2\ \tau^2,$$
$$u = \lambda\ \sigma\ \tau,$$
$$\rho = (\sigma + \tau)(\sigma\tau - 1) - 2\ \sigma\ \tau$$

and

$$\kappa = \sigma^2\ \tau^2,$$

where d is the distance from b_2 to b_3, then (u, v) must be a point on the elliptic curve, $\mathcal{E}$, defined by the equation:

$$\mathcal{E}:\ v^2 = u^4 + 2\ \rho\ u^3 + (\rho^2 + 2\kappa)\ u^2 - 2\ \rho\ \kappa\ u + \kappa^2.$$

Each K-rational point on this curve gives a K-rational distance d. $\mathcal{E}$ contains the two K-rational points at infinity, $\mathcal{O}$ and $\mathcal{O}'$, and the two affine points $(0, \pm\kappa)$. Unfortunately, all of these points are of finite order in $\mathcal{E}$. In particular, $\mathcal{O}'$ is a point of order 4 with $2\mathcal{O}' = (0, -\kappa)$.

Though there does not seem to be a K-rational point of infinite order on $\mathcal{E}$, we can still find infinitely many rational points on infinitely many specializations by setting u to be particular values in the equation defining $\mathcal{E}$ and finding infinitely many values that satisfy the resulting equation. For example, if we let $u = \sigma + \tau$, then we get a new elliptic curve, $\mathcal{E}'$, in $w = v\sigma\tau$ and τ defined over $\mathbf{Q}(\sigma)$. If we also let $\eta = \sigma^2 - 4\sigma + 8$ and $\theta = \sigma - 2$, then $\mathcal{E}'$ is defined by:

$$\mathcal{E}' : \ w^2 = \tau^4 + 2\ \theta\ \tau^3 + (\eta + 2\ \theta\ \sigma)\ \tau^2 + 2\ \sigma\ \eta\ \tau + \eta\ \sigma^2.$$

The nontrivial point at infinity is a point of infinite order in $\mathcal{E}'(\mathbf{Q}(\sigma))$. The double of this point has τ-coordinate $-(2\sigma-1)/(\sigma+2)$. Therefore, if we specialize the curve $\mathcal{E}$ at $\tau = -(2\sigma - 1)/(\sigma + 2)$ and call the new curve, defined over $\mathbf{Q}(\sigma)$, $\mathcal{E}_\tau$, then the point with u-coordinate $\sigma + \tau = (\sigma^2 + 1)/(\sigma + 2)$ gives a point of infinite order in $\mathcal{E}_\tau(\mathbf{Q}(\sigma))$. We let $E_{\tau,\sigma}$ denote the curve $\mathcal{E}_\tau$ specialized further at some rational value σ. By Lemma 3, we can find infinitely many $\sigma \in \mathbf{Q}$ such that $E_{\tau,\sigma}(\mathbf{Q})$ is infinite. Hence, there are infinitely many rational tilings in the ω-configuration. □

Note that this proof is very similar to that of Lemma 3. Suppose we replace $L_{\sigma,\beta}$ with the line $L_\gamma :\ y = \gamma$ in Lemma 3. The two lines $y = 0$ and L_γ do not intersect in affine space, but rather at a (rational) point, S, at infinity. The proof of Lemma 3 could then proceed fairly unchanged using this S in place of $(-\beta, 0)$. This is precisely what the proof above does. As we see in the proof, the problem is that this S gives rise to points of finite order on the curve $\mathcal{E}$ and hence, does not yield an infinitude of examples. This is the reason for choosing $u = \sigma + \tau$.

This choice of u is by no means unique, nor does it necessarily yield the simplest equation. It does however, give a nicely varying family of rational tilings in the ω-configuration – the position of the vertices seem to be well distributed over the range of all possibilities.

Finally, we point out that this same technique can be used to produce infinitely many rational simple Λ-type 5-tilings in which the apex line is parallel to the base line. As above, the point S will be a point of infinite order, but making a substitution analogous to the $u = \sigma + \tau$ substitution made above yields similar results, namely, infinitely many rational tilings. (We note, however, that there are some cases where the apex line and base line are parallel for which it is not possible to divide the sub-quadrilateral into three rational triangles.)

4. Observations, questions and examples.

4.1 Improper tilings and duality. In each of the configurations, we produce a rational tiling by finding (rational) points on some elliptic curve (or pair of curves). In many cases, these points correspond to vertices which actually lie outside of the unit square. In [**7**], Guy calls such tilings *improper tilings*. The condition required to avoid improper tilings amounts to a set of restrictions on the parameters, σ, τ, etc., that control the slopes of the lines in the tilings. Because the subgroup of rational points generated by a point of infinite order on an elliptic curve is dense on the curve, we can always find infinitely many *proper* rational tilings.

Moreover, suppose a point is at a rational distance to three corners of the unit square, but is outside the unit square. Guy points out that one can "invert" the picture and construct a new point at rational distance to three corners of the unit square that is inside the unit square. This inversion could be applied to any 5-tiling that contains a vertex adjacent to three corner vertices. The inversion transformation would bring such a vertex inside the unit square and it may also transform the (entire) improper tiling into a proper tiling.

More generally, there are more interesting relationships between 4-tilings, other than those induced by the group action of the dihedral group. This suggests that there may be more interesting relationships among the 5-tilings as well. In particular, Bremner and Guy in [**3**] point out that the κ and δ 4-configurations are "dual" in the sense that there is a (geometric, invertible, "rationality preserving") transformation from each tiling in one configuration to a tiling in the other. It seems likely that this notion of duality could be extended to pairs of 5-configurations. For example, as mentioned in Section 2, many of the 5-configurations consist of an edge from a corner vertex to a boundary which divides the unit square into a triangle and a quadrilateral. In these configurations, the quadrilateral is divided in a way isomorphic to one of the 4-tilings and it seems that the same duality that holds for 4-tilings of the unit square also holds for the 4-tilings of these quadrilaterals. However, not having worked out the details, we leave as a question:

Question 3. *In what ways and for which configurations are particular pairs of* 5*-configurations "dual" to each other?*

4.2 Derivative tilings, Heron tilings and n-tilings for $\mathbf{n > 5}$. Recall that there can be no rational 2-tilings or rational 3-tilings of the unit square. Hence, the only rational 4-tilings are primitive. This is not the case for 5-tilings. It is the case that rational tilings in each of the 4-configurations with the exception of the χ-configuration can be constructed using lines of slope $G(\sigma)$ for $\sigma \in \mathbf{Q}$. Given this, we can prove that every derivative 5-configuration, with the possible exception of those derivative of the χ-configuration, admits a rational tiling. A line of slope $G(\sigma)$, $\sigma \in \mathbf{Q}$ through one of the vertices in a rational 4-tiling will intersect one of the other lines in a rational point at rational distance. By taking all such lines through each of the possible vertices, we get rational tilings in all possible derivative 5-configurations.

Given this, one might want to answer only:

Question 4. *For each $n \in \mathbf{N}$, which* primitive *n-configurations admit a rational tiling of the unit square?*

However, as n grows, even if for all $m < n$, all primitive m-configurations admit rational tilings, it is not immediately apparent that all (derivative) n-configurations admit rational n-tilings. As n grows, more complicated tilings of triangles are admissible. One would need to also find all primitive m-tilings of rational triangles. This means that restricting our attention to only primitive tilings may not be sufficient to answer Question 1. Still, given that, as n grows, one also gains some freedom to move vertices around, it seems remotely possible that for each $n \geq 4$, one might find a rational n-tiling of the unit square in each (primitive and derivative) configuration. However, there is very little evidence beyond the 5-tiling work done here to suggest that this

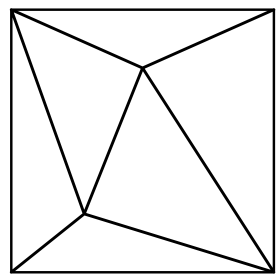

FIGURE 4. A primitive 6-configuration in which it may be challenging to find a rational tiling.

is true. For example, it may be as challenging to find a rational 6-tiling in the primitive 6-configuration shown in Figure 4 as it is to find a rational 4-tiling in the χ-configuration.

Note. David Rusin has recently communicated that the 6-configuration shown in Figure 4 in fact does admit infinitely many rational tilings.

Observe that we have proved in this paper something slightly stronger than the fact that there are infinitely many *rational* tilings in each of the possible 5-configurations (both primitive and derivative). Recall that a *Heron triangle* is a rational triangle whose area is also rational and define a *Heron n-tiling* to be an n-tiling in which each triangle is a Heron triangle. Since the vertices of all the rational primitive 5-tilings we find have rational coordinates, it is easy to see that at least four of the five triangles in each 5-tiling "obviously" has rational area. The last triangle then, necessarily has rational area. Furthermore, it is not difficult to see that if a Heron triangle has edges on lines of slope $G(\sigma_i)$, $\sigma_i \in \mathbf{Q}$, then any line of slope $G(\tau), \tau \in \mathbf{Q}$ dividing the triangle into two triangles, divides the triangle into two Heron triangles. Ultimately then, Theorem 1 can in fact be strengthened to state:

Theorem 10. *Each of the* 5*-configurations, with the possible exception of those derivative of the* χ*-configuration, admit infinitely many distinct Heron tilings.*

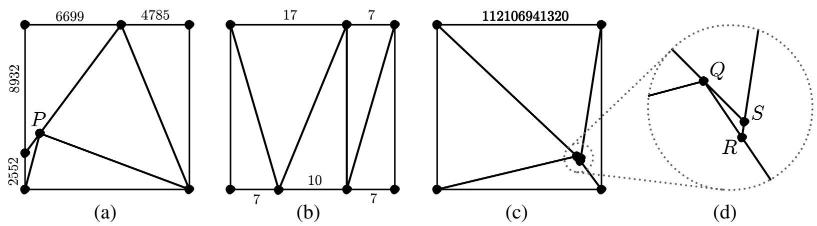

FIGURE 5. Examples of rational 5-tilings in (a) a simple Λ-type configuration, where $P = (1110, 4032)$, (b) the ω-configuration, (c) the super-χ configuration with (d) a detail near the internal vertices where $Q = (95920377600, 21582084960)$, $R = (96067544595, 21385862300)$ and $S = (96076231680, 21434997672)$.

It seems even less likely that it is possible to find Heron n-tilings of the unit square for *all* possible n-configurations and so we simply ask:

Question 5. *For each $n \in \mathbf{N}$, which n-configurations admit a Heron tiling of the unit square?*

4.3. A few examples and the χ-configuration 4-tiling. Figure 5 illustrates three examples of rational 5-tilings. Note that these examples were found with the help of *Mathematica* [**11**] (and in general, Mathematica, and occasionally `mwrank` [**6**] and *PARI/GP* [**1**], were used to perform many of the calculations presented here). Observe that the example in the super-χ configuration is "very close" to a χ-configuration 4-tiling.

This observation led us to ask if rational 5-tilings can be "arbitrarily close" to a 4-tiling in the χ-configuration and it is fairly clear that this is indeed the case. More specifically, we have:

Observation 1. *The super-χ, $\chi+\Lambda$ and $Y+\Lambda$ configurations admit families of rational 5-tilings such that*

(1) *the vertex adjacent to three corner vertices is fixed in each family* (*and hence two of the five triangles are fixed in each family*) *and*

(2) *for all $\epsilon > 0$, each family has tilings containing one triangle whose perimeter is less than ϵ.*

It is our hope that we can find a way to somehow exploit this fact to say something more definitive about the possibility of rational tilings in the χ-configuration.

In conclusion, we point out that finding a rational 4-tiling in the χ-configuration in which the interior vertex has rational coordinates is equivalent to finding a point on a particular hypersurface. More specifically, suppose we let $\rho = 2(\sigma^2 - 2\sigma - 1)$ and let

$$f_\sigma(x) = x^4 + \rho\, x^3 + (\rho\, \sigma^2 + 4\sigma + 2)\, x^2 - \rho\, \sigma^2\, x + \sigma^4.$$

Each nontrivial, affine point on the hypersurface defined by

$$\begin{aligned} y^2 &= f_\sigma(x) \\ z^2 &= f_\sigma(x) + (x^2 - \sigma^4)(x^2 - 1), \end{aligned}$$

corresponds to a rational (not necessarily proper) 4-tiling in the χ-configuration.

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