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SUFFICIENT CONDITIONS FOR OSCILLATION OF LINEAR SECOND ORDER MATRIX DIFFERENTIAL SYSTEMS

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ABSTRACT. Sufficient conditions in terms of *trace* are obtained for the oscillation of all nontrivial prepared solutions of second order self-adjoint differential matrix systems

$$\big(P(t)Y'\big)' + Q(t)Y = 0, \quad t \geq \sigma \geq 0,$$

where P and Q are $n \times n$ real continuous symmetric matrix functions on $[\sigma, \infty)$ with $P(t)$ positive definite. Our results generalize earlier results on oscillation of scalar second order equation

$$\big(p(t)y'\big)' + q(t)y = 0, \quad t \geq \sigma \geq 0,$$

where $p, q \in C([\sigma, \infty), (-\infty, \infty))$ with $p(t) > 0$, and are applicable to Euler's second order matrix equations.

1. Introduction. Many oscillation criteria for self-adjoint second order linear differential equation

$$(1) \qquad \big(p(t)y'\big)' + q(t)y \;=\; 0$$

are known, where $p \in C([\sigma, \infty), (0, \infty))$, $q \in C([\sigma, \infty), (-\infty, \infty))$ and $\sigma \geq 0$. If $p(t) \equiv 1$, then (1) takes the form

$$(2) \qquad y'' + q(t)y \;=\; 0.$$

A solution of (1) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory. Equation (1) is oscillatory if all its solutions are oscillatory. We use the following condition often:

(C_1) Let $D = \{(t, s) : t \geq s \geq \sigma\}$ and $D_0 = \{(t, s) : t > s \geq \sigma\}$. Let $h \in C(D, [0, \infty))$ satisfy the following conditions:

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(i) $h(t,t) = 0$ for $t \geq \sigma$, $h(t,s) > 0$ for $t > s \geq \sigma$.

(ii) h has a continuous and nonpositive partial derivative on D_0 with respect to the second variable. Suppose that $g \in C(D_0, [0,\infty))$ is defined by

$$-\frac{\partial h(t,s)}{\partial s} = g(t,s)\sqrt{h(t,s)}, \quad (t,s) \in D_0.$$

In 1989, Philos [**7**] obtained oscillation criteria for (2) which extended earlier criteria due to Kamenev [**3**] and Yan [**8**]. One of his results is stated in the following:

Theorem 1.1 (Philos [**7**]). *Suppose* (C_1) *holds. If*

$$\limsup_{t\to\infty} \frac{1}{h(t,\sigma)} \int_\sigma^t \Big[h(t,s)q(s) - \frac{1}{4}\, g^2(t,s)\Big]\, ds = \infty,$$

then (2) *is oscillatory.*

However, Theorem 1.1 cannot be applied to the Euler differential equation

$$y'' + \frac{\gamma}{t^2}\, y = 0, \tag{3}$$

where $\gamma > 0$ is a constant. It is well known that equation (3) is oscillatory if $\gamma > 1/4$ and nonoscillatory if $\gamma \leq 1/4$. In [**4**], Li gave oscillation criteria for (1) which generalize Theorem 1.1 and is applicable to equation (3). In the following we state three of his results:

Theorem 1.2. *Let* (C_1) *hold. If there exists a function* $f \in C^1([\sigma,\infty),(-\infty,\infty))$ *such that*

$$\limsup_{t\to\infty} \frac{1}{h(t,\sigma)} \int_\sigma^t \Big[h(t,s)\psi(s) - \frac{1}{4}\tilde{f}(s)p(s)g^2(t,s)\Big]\, ds = \infty, \tag{4}$$

where

$$\tilde{f}(t) = \exp\Big(-2\int_\sigma^t f(s)\, ds\Big)$$

and

$$\psi(t) \;=\; \tilde{f}(t)\big(q(t) + p(t)f^2(t) - (p(t)f(t))'\big),$$

then equation (1) *is oscillatory.*

Theorem 1.3. *Let* (C_1) *hold, and let*

$$0 \;<\; \inf_{s\geq\sigma}\Big[\liminf_{t\to\infty}\frac{h(t,s)}{h(t,\sigma)}\Big]. \tag{5}$$

Suppose there exist two functions $f \in C^1([\sigma,\infty),(-\infty,\infty))$ *and* $a \in C([\sigma,\infty),(-\infty,\infty))$ *such that (4) holds and the conditions*

$$\limsup_{t\to\infty}\frac{1}{h(t,\sigma)}\int_\sigma^t \tilde{f}(s)p(s)g^2(t,s)\,ds \;<\; \infty$$

and

$$\int_\sigma^\infty \frac{a_+^2(t)}{\tilde{f}(t)p(t)}\,dt \;=\; \infty \tag{6}$$

hold and, for every $t_0 \geq \sigma$,

$$\limsup_{t\to\infty}\frac{1}{h(t,t_0)}\int_{t_0}^t \Big[h(t,s)\psi(s) - \frac{1}{4}\,\tilde{f}(s)p(s)g^2(t,s)\Big]\,ds \;\geq\; a(t_0),$$

where $\tilde{f}$ *and* ψ *are the same as in Theorem 1.2 and* $a_+(t) = \max(a(t),0)$. *Then equation (1) is oscillatory.*

Theorem 1.4. *Suppose that* (C_1) *holds. Let* (5) *and* (6) *be satisfied. If*

$$\liminf_{t\to\infty}\frac{1}{h(t,\sigma)}\int_\sigma^t h(t,s)\psi(s)\,ds \;<\; \infty$$

and

$$\liminf_{t\to\infty}\frac{1}{h(t,t_0)}\int_{t_0}^t \Big[h(t,s)\psi(s) - \frac{1}{4}\,\tilde{f}(s)p(s)g^2(t,s)\Big]\,ds \;\geq\; a(t_0),$$

for every $t_0 \geq \sigma$, *then equation* (1) *is oscillatory.*

Erbe, Kong and Ruan [**1**] generalized Theorem 1.1 to linear second order differential system

$$(\mathrm{E}_1) \qquad Y'' + Q(t)Y = 0, \quad t \in [\sigma, \infty),$$

where Y and Q are $n \times n$ real continuous matrix functions with $Q(t)$ symmetric. They obtained the following result.

Theorem 1.5. *Suppose* (C_1) *holds. If*

$$(7) \qquad \limsup_{t\to\infty} \frac{1}{h(t,\sigma)}\, \lambda_1\Big[\int_\sigma^t \Big(h(t,s)Q(s) - \frac{1}{4}\, g^2(t,s)I\Big)\, ds\Big] = \infty,$$

where $\lambda_1[B] \ge \lambda_2[B] \ge \cdots \ge \lambda_n[B]$ *denotes the usual ordering of the eigenvalues of the symmetric matrix* B *and* I *is the* $n \times n$ *identity matrix, then system* (E_1) *is oscillatory* (*the definition is given below*). *They have also considered* (E), (*see below*).

However, if $Q(t) = \operatorname{diag}(\gamma/t^2, \alpha/t^2)$ in (E_1), where $\gamma \ge \alpha > 0$ are constants, then (7) fails to hold (see [**5**]). Thus, Theorem 1.5 cannot be applied to the Euler differential system

$$(\mathrm{E}_2) \qquad Y'' + \operatorname{diag}\big(\gamma/t^2,\, \alpha/t^2\big)Y = 0,$$

where Y is a 2×2 matrix, $\gamma \ge \alpha > 0$ are constants. It is shown that (see [**5**]) the Euler differential system (E_2) is oscillatory if $\gamma > 1/4$. In a recent paper [**5**], Meng, Wang and Zheng have generalized Theorem 1.5 so as to be applicable to the Euler differential system (E_2). They established the following result

Theorem 1.6 (Theorem 1, [**5**]). *Let* (C_1) *hold. If there exists a function* $f \in C([\sigma,\infty),(-\infty,\infty))$ *such that*

$$(8) \qquad \limsup_{t\to\infty} \frac{1}{h(t,\sigma)}\, \lambda_1\Big[\int_\sigma^t \Big(h(t,s)R(s) - \frac{1}{4}\, \tilde{f}(s) g^2(t,s)I\Big)\, ds\Big] = \infty,$$

where

$$\tilde{f}(t) = \exp\Big(-2\int_\sigma^t f(s)\, ds\Big)$$

and $R(t) = \tilde{f}(t)[Q(t)+f^2(t)I-f'(t)I]$, *then equation* (E_1) *is oscillatory.*

In this paper we have generalized Theorems 1.2–1.4 to self-adjoint linear second order differential system

$$(\mathrm{E}) \qquad \big(P(t)Y'\big)' + Q(t)Y = 0, \quad t \in [\sigma, \infty),$$

where $Y(t), P(t)$ and $Q(t)$ are $n \times n$ real, continuous matrix functions on $[\sigma, \infty)$ such that $Q(t)$ is symmetric and $P(t)$ is positive definite. A solution $Y(t)$ of (E) on $[\sigma, \infty)$ is said to be nontrivial if $\det Y(t) \neq 0$ for at least one $t \in [\sigma, \infty)$. It is said to be prepared or self-conjugate if

$$Y^*(t)\big(P(t)Y'(t)\big) = \big(P(t)Y'(t)\big)^* Y(t)$$

holds for $t \in [\sigma, \infty)$, where for any matrix B, the transpose of B is denoted by B^*. By a solution of (E) we understand a nontrivial prepared solution of (E). A solution $Y(t)$ of (E) is said to be oscillatory if, for every $t_0 \geq \sigma$, it is possible to find a $t_1 > t_0$ such that $\det Y(t_1) = 0$; otherwise, $Y(t)$ is called nonoscillatory. Equation (E) is said to be oscillatory if every nontrivial prepared solution of the equation is oscillatory. Oscillation of equation (E) must be studied separately from equation (E_1) since, like the scalar case, there is no oscillation-preserving transformation of the independent variable that allows the passage between the two forms. The oscillation of equation (E) is defined through its nontrivial prepared solutions because it is possible that (E) admits a nontrivial nonprepared nonoscillatory solution (see [**6**]). For a solution $Y(t)$ of (E),

$$Y^*(t)\big(P(t)Y'(t)\big) - \big(P(t)Y'(t)\big)^* Y(t) = C, \quad t \geq \sigma,$$

where C is an $n \times n$ constant matrix. Hence, for $t_0 \geq \sigma$,

$$Y^*(t_0)\big(P(t_0)Y'(t_0)\big) - \big(P(t_0)Y'(t_0)\big)^* Y(t_0) = C.$$

It is possible to choose $Y(t_0) = Y_0$ and $Y'(t_0) = Y_0'$ such that $C = 0$. Thus the initial value problem

$$\big(P(t)Y'\big)' + Q(t)Y = 0, \qquad Y(t_0) = Y_0, \qquad Y'(t_0) = Y_0'$$

always admits a nontrivial prepared solution. If S is the real linear space of all real symmetric $n \times n$ matrices, then $\mathrm{tr} : S \to (-\infty, \infty)$

is a linear functional with $(\mathrm{tr}\, A)^2 \le n\mathrm{tr}\,(A^2)$ for every $A \in S$. For $A, B, C \in S$, we write $A \ge B$ to mean that $A - B \ge 0$, that is, $A - B$ is a positive semi-definite matrix, and $A \ge B$ implies that $\mathrm{tr}\, A \ge \mathrm{tr}\, B$ and $CAC \ge CBC$. Further,

$$\mathrm{tr} \int_\sigma^t Q(s)\, ds = \int_\sigma^t \mathrm{tr}\, Q(s)\, ds$$

for every $n \times n$ real symmetric matrix function Q whose entries are integrable.

2. Sufficient conditions for oscillation. In this section we obtain sufficient conditions for oscillation of equation (E). We list the assumptions in the following that are needed for our work in the sequel.

(C_2) There exists a function $f \in C([\sigma, \infty), (-\infty, \infty))$ such that $(\tilde{f}(t)P(t))^{-1} \ge n(\tilde{f}(t)\mathrm{tr}\, P(t))^{-1} I$, where I is the $n \times n$ identity matrix and

$$\tilde{f}(t) = \exp\Big(-2\int_\sigma^t f(s)\, ds\Big).$$

(C_3) There exists a function $f \in C([\sigma, \infty), (-\infty, \infty))$ such that $f(t)P(t)$ is continuously differentiable and

$$\limsup_{t\to\infty} \frac{1}{h(t,t_0)}\Big[\int_{t_0}^t \Big(h(t,s)\mathrm{tr}\, R(s) - \frac{1}{4}\tilde{f}(s)\mathrm{tr}\, P(s) g^2(t,s)\Big)\, ds\Big] = \infty$$

for every $t_0 \ge \sigma$, where

$$R(t) = \tilde{f}(t)\big[Q(t) + f^2(t)P(t) - \big(f(t)P(t)\big)'\big]$$

(C_4) $$0 < \inf_{s \ge t_0}\Big[\liminf_{t\to\infty} \frac{h(t,s)}{h(t,t_0)}\Big] \quad \text{for every } t_0 \ge \sigma,$$

(C_5) $$\limsup_{t\to\infty} \frac{1}{h(t,t_0)}\int_{t_0}^t g^2(t,s)\tilde{f}(s)\mathrm{tr}\, P(s)\, ds < \infty$$

for every $t_0 \ge \sigma$,

(C_6) There exists a function $a \in C([\sigma,\infty),(-\infty,\infty))$ such that

$$\int_{t_0}^{\infty} a_+^2(t)\big(\tilde{f}(t)\mathrm{tr}\,P(t)\big)^{-1}\,dt \;=\; \infty \quad \text{for every } t_0 \ge \sigma,$$

where $a_+(t) = \max(a(t),0)$.

(C_7) There exists a function $a \in C([\sigma,\infty),(-\infty,\infty))$ such that, for every $t_0 \ge \sigma$,

$$\limsup_{t\to\infty} \frac{1}{h(t,t_0)} \int_{t_0}^{t} \Big[h(t,s)\mathrm{tr}\,R(s) - \frac{1}{4}\,g^2(t,s)\tilde{f}(s)\mathrm{tr}\,P(s)\Big]\,ds \;\ge\; a(t_0)$$

(C_8) $$\liminf_{t\to\infty} \frac{1}{h(t,t_0)} \int_{t_0}^{t} h(t,s)\mathrm{tr}\,R(s)\,ds \;<\; \infty \quad \text{for every } t_0 \ge \sigma.$$

(C_9) There exists a function $a \in C([\sigma,\infty),(-\infty,\infty))$ such that, for every $t_0 \ge \sigma$,

$$\liminf_{t\to\infty} \frac{1}{h(t,t_0)} \int_{t_0}^{t} \Big[h(t,s)\mathrm{tr}\,R(s) - \frac{1}{4}\,g^2(t,s)\tilde{f}(s)\mathrm{tr}\,P(s)\Big]\,ds \;\ge\; a(t_0).$$

Remark. We may note that (C_3) implies (C_7).

Theorem 2.1. *If* (C_1), (C_2) *and* (C_3) *hold, then equation* (E) *is oscillatory.*

Proof. Let $Y(t)$ be a nonoscillatory solution of (E). Hence there exists a $t_1 \ge \sigma$ such that $\det Y(t) \ne 0$ for $t \ge t_1$. For $t \ge t_1$, we set

$$V(t) \;=\; \tilde{f}(t)P(t)\big[Y'(t)Y^{-1}(t) + f(t)I\big]. \tag{9}$$

Hence, $V(t)$ is symmetric because $Y(t)$ is prepared and

$$\begin{aligned} V' = {} & \tilde{f}'(t)P(t)\big(Y'(t)Y^{-1}(t) + f(t)I\big) \\ & + \tilde{f}(t)\big[\big(P(t)Y'(t)\big)'Y^{-1}(t) + \big(f(t)P(t)\big)'\big] \\ & - \tilde{f}(t)P(t)Y'(t)Y^{-1}(t)Y'(t)Y^{-1}(t). \end{aligned}$$

Since $\tilde{f}'(t) = -2f(t)\tilde{f}(t)$ and $(P(t)Y'(t))' = -Q(t)Y(t)$, then

$$\begin{aligned} V'(t) = &-2f(t)V(t) - \tilde{f}(t)\big(Q(t) - (f(t)P(t))'\big) \\ &- \tilde{f}(t)P(t)\big(Y'(t)Y^{-1}(t)Y'(t)Y^{-1}(t)\big). \end{aligned}$$

However,

$$\begin{aligned} &Y'(t)Y^{-1}(t)Y'(t)Y^{-1}(t) \\ &\quad = \big(Y'(t)Y^{-1}(t) + f(t)I\big)^2 - 2f(t)Y'(t)Y^{-1}(t) - f^2(t)I \\ &\quad = \big(Y'(t)Y^{-1}(t) + f(t)I\big)^2 - 2f(t)\big(Y'(t)Y^{-1}(t) + f(t)I\big) \\ &\qquad + f^2(t)I \\ &\quad = \big(Y'(t)Y^{-1}(t) + f(t)I\big)\big(\tilde{f}(t)P(t)\big)^{-1}V(t) \\ &\qquad - 2f(t)\big(\tilde{f}(t)P(t)\big)^{-1}V(t) + f^2(t)I \end{aligned}$$

implies that

$$\begin{aligned} &\tilde{f}(t)P(t)\big(Y'(t)Y^{-1}(t)Y'(t)Y^{-1}(t)\big) \\ &\quad = V(t)\big(\tilde{f}(t)P(t)\big)^{-1}V(t) - 2f(t)V(t) + f^2(t)\tilde{f}(t)P(t). \end{aligned}$$

Hence,

$$V'(t) = -V(t)\big(\tilde{f}(t)P(t)\big)^{-1}V(t) - \tilde{f}(t)\big(Q(t) + f^2(t)P(t) - \big(f(t)P(t)\big)'\big),$$

that is,

$$V'(t) = -V(t)\big(\tilde{f}(t)P(t)\big)^{-1}V(t) - R(t). \tag{10}$$

Multiplying (10), where t is replaced for s, through by $h(t,s)$ and then integrating from t_1 to t, we obtain

$$\begin{aligned} &\int_{t_1}^{t} h(t,s)R(s)\,ds \\ &\quad = h(t,t_1)V(t_1) + \int_{t_1}^{t}\Big[\frac{\partial h(t,s)}{\partial s}V(s) - h(t,s)V(s)\big(\tilde{f}(s)P(s)\big)^{-1}V(s)\Big]\,ds. \end{aligned}$$

The use of (C_1) and (C_2) yields

$$\begin{aligned}
&\frac{1}{h(t,t_1)}\int_{t_1}^{t} h(t,s)R(s)\,ds \\
&\qquad = V(t_1) - \frac{1}{h(t,t_1)}\int_{t_1}^{t}\big[h(t,s)V(s)\big(\tilde f(s)P(s)\big)^{-1}V(s) \\
&\qquad\qquad\qquad + g(t,s)\big(h(t,s)\big)^{1/2}V(s)\big]\,ds \\
&\qquad \le V(t_1) - \frac{1}{h(t,t_1)}\int_{t_1}^{t}\big[nh(t,s)\big(\tilde f(s)\mathrm{tr}\,P(s)\big)^{-1}V^2(s) \\
&\qquad\qquad\qquad + g(t,s)\big(h(t,s)\big)^{1/2}V(s)\big]\,ds \\
&\qquad = V(t_1) - \frac{1}{h(t,t_1)}\int_{t_1}^{t}\Big[\big(nh(t,s)\big(\tilde f(s)\mathrm{tr}\,P(s)\big)^{-1}\big)^{\frac12}V(s) \\
&\qquad\qquad\qquad + \frac12\, g(t,s)\,\frac{\big(\tilde f(s)\mathrm{tr}\,P(s)\big)^{1/2}}{n^{1/2}}I\Big]^2\,ds \\
&\qquad\quad + \frac{1}{4nh(t,t_1)}\int_{t_1}^{t} g^2(t,s)\tilde f(s)\mathrm{tr}\,P(s)I\,ds.
\end{aligned}$$

Since $R(t)$ is symmetric, then

$$\begin{aligned}
&\frac{1}{h(t,t_1)}\int_{t_1}^{t}\mathrm{tr}\Big[h(t,s)R(s) - \frac{1}{4n}\,g^2(t,s)\tilde f(s)\mathrm{tr}\,P(s)I\Big]\,ds \\
&\quad \le \mathrm{tr}\,V(t_1) - \frac{1}{h(t,t_1)}\int_{t_1}^{t}\mathrm{tr}\Big[\big(nh(t,s)\big(\tilde f(s)\mathrm{tr}\,P(s)\big)^{-1}\big)^{1/2}V(s) \\
&\qquad\qquad + \frac12 g(t,s)\frac{\big(\tilde f(s)\mathrm{tr}\,P(s)\big)^{\frac12}}{n^{\frac12}}I\Big]^2 ds,
\end{aligned}$$

that is,

$$\begin{aligned}
(11)\quad &\frac{1}{h(t,t_1)}\int_{t_1}^{t}\Big[h(t,s)\mathrm{tr}\,R(s) - \frac14\,g^2(t,s)\tilde f(s)\mathrm{tr}\,P(s)\Big]\,ds \\
&\quad \le \mathrm{tr}\,V(t_1) - \frac{1}{h(t,t_1)}\int_{t_1}^{t}\mathrm{tr}\Big[\big(nh(t,s)\big(\tilde f(s)\mathrm{tr}\,P(s)\big)^{-1}\big)^{1/2}V(s) \\
&\qquad\qquad + \frac12\,g(t,s)\,\frac{\big(\tilde f(s)\mathrm{tr}\,P(s)\big)^{1/2}}{n^{1/2}}I\Big]^2\,ds.
\end{aligned}$$

From (11), it follows that

$$\limsup_{t\to\infty} \frac{1}{h(t,t_1)} \int_{t_1}^{t} \Big[h(t,s)\operatorname{tr} R(s) - \frac{1}{4}\, g^2(t,s)\tilde{f}(s)\operatorname{tr} P(s)\Big]\, ds \le |\operatorname{tr} V(t_1)| \;<\; \infty.$$

This contradicts (C_3) and hence the theorem is proved. □

Remark. (i) If $P(t) \equiv I$, then (C_2) is satisfied and $R(t)$ in (C_3) has the following form:

$$R(t) \;=\; \tilde{f}(t)\big[Q(t) + f^2(t)I - f'(t)I\big].$$

(ii) If $n = 1$, then (C_2) is satisfied trivially and (C_3) reduces to (4) with $t_0 = \sigma$, $Q(t) = q(t)$ and $P(t) = p(t)$. Hence, Theorem 2.1 is a generalization of Theorem 1.2 to systems (E).

In the following we consider an example to which Theorem 1 in [**5**] cannot be applied but where the above theorem holds.

Example. Consider equation (E) for $t \ge 0$ with

$$P(t) = I \quad \text{and} \quad Q(t) = \begin{bmatrix} e^{2t}(1+t^2\cos t) & 0 \\ 0 & e^{2t}(1-t^2\cos t) \end{bmatrix}.$$

Define $h(t,s) = (t-s)^2$, $t \ge s \ge 0$. Hence, $g(t,s) = 2$. Let $f(t) \equiv 1$, $t \ge 0$. Then $\tilde{f}(t) = e^{-2t}$. The conditions (C_1) and (C_2) are satisfied trivially. Here

$$R(t) = e^{-2t}\big(Q(t) + I\big) = \begin{bmatrix} 1+t^2\cos t + e^{-2t} & 0 \\ 0 & 1-t^2\cos t + e^{-2t} \end{bmatrix},$$

and hence

$$h(t,s)\operatorname{tr} R(s) - \frac{1}{4}\,\tilde{f}(s)\operatorname{tr} P(s) g^2(t,s) = 2(t-s)^2(1+e^{-2s}) - 2e^{-2s}.$$

Thus

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{h(t,t_0)} \int_{t_0}^{t} &\Big[h(t,s)\operatorname{tr} R(s) - \frac{1}{4}\,\tilde{f}(s)\operatorname{tr} P(s) g^2(t,s)\Big]\, ds \\
&= \limsup_{t\to\infty} \frac{1}{(t-t_0)^2} \int_{t_0}^{t} \big[2(t-s)^2 + 2(t-s)^2 e^{-2s} - 2e^{-2s}\big]\, ds \\
&\ge \limsup_{t\to\infty} \frac{1}{(t-t_0)^2} \Big[\frac{2}{3}\,(t-t_0)^3 + e^{-2t} - e^{-2t_0}\Big] = \infty,
\end{aligned}$$

that is, condition (C_3) holds. From Theorem 2.1 it follows that all nontrivial prepared solutions of equation (E) with $P(t)$ and $Q(t)$ as defined above are oscillatory. Clearly,

$$\int_0^t \Big[h(t,s)R(s) - \frac{1}{4}\, g^2(t,s)\tilde{f}(s)I\Big]\, ds \;=\; \begin{bmatrix} \lambda(t) & 0 \\ 0 & \mu(t) \end{bmatrix} \tag{12}$$

where

$$\begin{aligned} \lambda(t) &= \int_0^t \big[(t-s)^2(1+s^2\cos s) + e^{-2s}\big((t-s)^2 - 1\big)\big]\, ds \\ &= \frac{t^3}{3} + \frac{t^2}{2} - \frac{25}{2}\, t + \frac{1}{4}\, e^{-2t} - \frac{1}{4} - 2t^2 \sin t - 12t\cos t + 24\sin t \end{aligned}$$

and

$$\begin{aligned} \mu(t) &= \int_0^t \big[(t-s)^2(1-s^2\cos s) + e^{-2s}\big((t-s)^2 - 1\big)\big]\, ds \\ &= \frac{t^3}{3} + \frac{t^2}{2} + \frac{23}{2}\, t + \frac{1}{4}\, e^{-2t} - \frac{1}{4} + 2t^2 \sin t + 12t\cos t - 24\sin t. \end{aligned}$$

Hence $\lambda(t)$ and $\mu(t)$ are eigenvalues of the matrix given by (12).

Setting $\lambda_1(t) = \max\{\lambda(t), \mu(t)\}$ and $\lambda_2(t) = \min\{\lambda(t), \mu(t)\}$, we notice that $\lambda_1(t)$ and $\lambda_2(t)$ are, respectively, the largest and smallest eigenvalues of the matrix given by (12) and

$$\lambda_1(t) = \begin{cases} \mu(t), & t \in \big[2m\pi, (2m+1)\pi\big] \\ \lambda(t), & t \in \big[(2m+1)\pi, (2m+2)\pi\big], \end{cases}$$

for large positive integer m. However, $\lambda_1(t)$ is discontinuous at $2m\pi$. Indeed,

$$\mu(2m\pi) = \frac{(2m\pi)^3}{3} + \frac{(2m\pi)^2}{2} + \frac{23}{2}\,(2m\pi) + \frac{1}{4}\, e^{-4m\pi} + 24m\pi$$

and

$$\lambda(2m\pi) = \frac{(2m\pi)^3}{3} + \frac{(2m\pi)^2}{2} - \frac{25}{2}\,(2m\pi) + \frac{1}{4}\, e^{-4m\pi} - 24m\pi$$

imply that $\mu(2m\pi) > \lambda(2m\pi)$.

Remark. If the symmetric matrix $B(t) \geq 0$, $t \in [t_0, \infty)$, $t_0 \geq \sigma$, and its eigenvalues are put in decreasing order where

$$B(t) = \int_{t_0}^{t} \Big[h(t,s)R(s) - \frac{1}{4n}\, g^2(t,s)\tilde{f}(s)\mathrm{tr}\, P(s)I\Big]\, ds,$$

then the condition (C_3) is equivalent to

$$\limsup_{t\to\infty} \frac{1}{h(t,t_0)}\lambda_1\Big[\int_{t_0}^{t} \Big(h(t,s)R(s) - \frac{1}{4n}\, g^2(t,s)\tilde{f}(s)\mathrm{tr}\, P(s)I\Big)\, ds\Big] = \infty,$$

in view of the property $\lambda_1[B(t)] \leq \mathrm{tr}\,[B(t)] \leq n\lambda_1[B(t)]$. This is the same as equation (5) in [**5**] if $P(t) = I$.

Remark. Theorem 2.1 holds if the condition (C_3) is replaced by

(C_3'). There exists a function $f \in C([\sigma,\infty),(-\infty,\infty))$ such that $f(t)P(t)$ is continuously differentiable and

$$\limsup_{t\to\infty} \frac{1}{h(t,\sigma)} \int_{\sigma}^{t} \Big[h(t,s)\mathrm{tr}\, R(s) - \frac{1}{4}\, \tilde{f}(s)\mathrm{tr}\, P(s) g^2(t,s)\Big]\, ds = \infty$$

where $R(t)$ is the same as in (C_3). Indeed, for $t > t_1 > \sigma$,

$$\begin{aligned}
\int_{\sigma}^{t} \Big[h(t,s)\mathrm{tr}\, R(s) - \frac{1}{4}\, \tilde{f}(s)\mathrm{tr}\, P(s)g^2(t,s)\Big]\, ds \\
&= \int_{\sigma}^{t_1} \Big[h(t,s)\mathrm{tr}\, R(s) - \frac{1}{4}\, \tilde{f}(s)\mathrm{tr}\, P(s)g^2(t,s)\Big]\, ds \\
&\quad + \int_{t_1}^{t} \Big[h(t,s)\mathrm{tr}\, R(s) - \frac{1}{4}\, \tilde{f}(s)\mathrm{tr}\, P(s)g^2(t,s)\Big]\, ds \\
&< \int_{\sigma}^{t_1} h(t,s)|\mathrm{tr}\, R(s)|\, ds + h(t,t_1)|\mathrm{tr}\, V(t_1)| \\
&< h(t,\sigma) \int_{\sigma}^{t_1} |\mathrm{tr}\, R(s)|\, ds + h(t,\sigma)|\mathrm{tr}\, V(t_1)|,
\end{aligned}$$

where we used (C_1), (11) and $P(t) > 0$. We may note that $t_1 > \sigma$ is such that $Y^{-1}(t)$ exists for $t \geq t_1$. Thus

$$\limsup_{t\to\infty} \frac{1}{h(t,\sigma)} \int_{\sigma}^{t} \Big[h(t,s)\mathrm{tr}\, R(s) - \frac{1}{4}\, \tilde{f}(s)\, \mathrm{tr}\, P(s)g^2(t,s)\Big]\, ds < \infty,$$

which contradicts (C_3').

Example. Consider the Euler differential system (E_2), $t \geq 1$, with $\gamma > 1/4$. We take $\alpha = 1/4$. Let $f(t) = -(1/2t)$, $t \geq 1$. Hence $\tilde{f}(t) = t$. Let $h(t,s) = (t-s)^\beta$, $t \geq s \geq 1$ and $\beta > 1$. Then $g(t,s) = \beta(t-s)^{(\beta/2)-1}$. Clearly, ($C_1$) and ($C_2$) are satisfied. Since $n = 2$, then

$$\begin{aligned}
\int_1^t \Big[h(t,s)\operatorname{tr} R(s) &- \frac{1}{2}\,\tilde{f}(s)g^2(t,s)\Big]\,ds \\
&= \int_1^t \Big[(t-s)^\beta\Big(\frac{4\gamma-1}{4s}\Big) - \frac{1}{2}\,\beta^2 s(t-s)^{\beta-2}\Big]\,ds \\
&= \Big(\frac{4\gamma-1}{4}\Big)\int_1^t \frac{(t-s)^\beta}{s}\,ds - \frac{1}{2}\,\beta^2\int_1^t s(t-s)^{\beta-2}\,ds \\
&\geq \Big(\frac{4\gamma-1}{4}\Big)\int_1^t \frac{1}{s}\,(t^\beta - \beta\, st^{\beta-1})\,ds - \frac{1}{2}\,\beta^2\int_1^t s(t-s)^{\beta-2}\,ds \\
&= \Big(\frac{4\gamma-1}{4}\Big)t^\beta\Big(\log t - \beta + \frac{\beta}{t}\Big) \\
&\quad - \frac{1}{2}\,\beta^2(t-1)^{\beta-1}\Big(\frac{t}{\beta(\beta-1)} + \frac{1}{\beta}\Big),
\end{aligned}$$

where the inequality $(t-s)^\beta \geq t^\beta - \beta s t^{\beta-1}$ for $t \geq s \geq 1$ is used (see [**2**, Theorem 41]). Hence,

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{h(t,1)}\int_1^t &\Big[h(t,s)\operatorname{tr} R(s) - \frac{1}{2}\,\tilde{f}(s)g^2(t,s)\Big]\,ds \\
&\geq \limsup_{t\to\infty}\Big[\Big(\frac{4\gamma-1}{4}\Big)\Big(\frac{t^\beta\log t}{(t-1)^\beta} - \frac{\beta t^{\beta-1}}{(t-1)^{\beta-1}}\Big) \\
&\qquad - \frac{1}{2}\,\beta^2\frac{1}{(t-1)}\Big(\frac{t}{\beta(\beta-1)} + \frac{1}{\beta}\Big)\Big] \\
&= \infty.
\end{aligned}$$

From Theorem 2.1 and the above remark, it follows that the system (E_2) oscillates.

Theorem 2.2. *If* (C_1), (C_2) *hold and*

$$(C_{10}) \qquad \limsup_{t\to\infty}\frac{1}{h(t,\sigma)}\int_\sigma^t h(t,s)\operatorname{tr} R(s)\,ds = \infty$$

and

$$\text{(C}_{11}\text{)}\qquad \limsup_{t\to\infty} \frac{1}{h(t,\sigma)} \int_\sigma^t \tilde{f}(s)\operatorname{tr} P(s) g^2(t,s)\, ds < \infty,$$

then the system (E) *is oscillatory.*

The theorem holds because (C_{10}) and (C_{11}) imply (C_3').

Theorem 2.3. *Suppose that* (C_1), (C_2), (C_4)–(C_7) *hold. Then the system* (E) *is oscillatory.*

Proof. If possible, let $Y(t)$ be a nonoscillatory solution of (E). Hence, $\det Y(t) \neq 0$ for $t \geq t_1 \geq \sigma$. Setting $V(t)$ as in (9) for $t \geq t_1$, we obtain (10) and $V(t) = V^*(t)$. Proceeding as in the proof of Theorem 2.1, we get from (11) that

(13)
$$\begin{aligned} a(t^*) &\leq \limsup_{t\to\infty} \frac{1}{h(t,t^*)} \int_{t^*}^t \Big[h(t,s)\operatorname{tr} R(s) - \frac{1}{4}\, g^2(t,s)\tilde{f}(s)\operatorname{tr} P(s)\Big]\, ds \\ &\leq \operatorname{tr} V(t^*) \\ &\quad - \liminf_{t\to\infty} \frac{1}{h(t,t^*)} \int_{t^*}^t \operatorname{tr} \Big[(nh(t,s)(\tilde{f}(s)\operatorname{tr} P(s))^{-1})^{1/2} V(s) \\ &\qquad\qquad + \frac{1}{2} g(t,s) \frac{(\tilde{f}(s)\operatorname{tr} P(s))^{1/2}}{n^{1/2}} I\Big]^2\, ds, \end{aligned}$$

for $t > t^* \geq t_1$, by (C_7). Thus $\operatorname{tr} V(t^*) \geq a(t^*)$. Consequently,

$$\text{(14)}\qquad \big(\operatorname{tr} V(t^*)\big)^2 \geq a_+^2(t^*) \quad \text{for every } t^* \geq t_1.$$

Further, (13) yields

(15)
$$\begin{aligned} \liminf_{t\to\infty} \frac{1}{h(t,t_1)} \int_{t_1}^t \operatorname{tr} \Big[\big(nh(t,s)\big(\tilde{f}(s)\operatorname{tr} P(s)\big)^{-1}\big)^{1/2} V(s) \\ + \frac{1}{2}\, g(t,s)\, \frac{\big(\tilde{f}(s)\operatorname{tr} P(s)\big)^{1/2}}{n^{1/2}} I\Big]^2\, ds \\ \leq \operatorname{tr} V(t_1) - a(t_1) < \infty. \end{aligned}$$

Setting

$$B(t) = \frac{n}{h(t,t_1)} \int_{t_1}^{t} h(t,s)\big(\tilde{f}(s)\mathrm{tr}\,P(s)\big)^{-1} V^2(s)\,ds, \quad t \ge t_1$$

and

$$C(t) = \frac{1}{h(t,t_1)} \int_{t_1}^{t} g(t,s)\sqrt{h(t,s)}\,V(s)\,ds, \quad t \ge t_1,$$

we have

$$B(t) + C(t) < \frac{1}{h(t,t_1)} \int_{t_1}^{t} \Big[\big(nh(t,s)\big(\tilde{f}(s)\mathrm{tr}\,P(s)\big)^{-1}\big)^{1/2} V(s) + \frac{1}{2}\,g(t,s)\,\frac{\big(\tilde{f}(s)\mathrm{tr}\,P(s)\big)^{1/2}}{n^{1/2}}\,I\Big]^2\,ds.$$

Hence,

$$\liminf_{t\to\infty} \big[\mathrm{tr}\,B(t) + \mathrm{tr}\,C(t)\big] < \infty \tag{16}$$

by (15). Since $V(t)$ is symmetric, then $V^2(t) \ge 0$, $t \ge t_1$. We claim that

$$\int_{t_1}^{\infty} \big(\tilde{f}(s)\mathrm{tr}\,P(s)\big)^{-1}\,\mathrm{tr}\,V^2(s)\,ds < \infty.$$

If not, then

$$\int_{t_1}^{\infty} \big(\tilde{f}(s)\mathrm{tr}\,P(s)\big)^{-1}\,\mathrm{tr}\,V^2(s)\,ds = \infty. \tag{17}$$

From (C_4) it follows that

$$\inf_{s\ge t_1}\Big[\liminf_{t\to\infty} \frac{h(t,s)}{h(t,t_1)}\Big] > \eta > 0.$$

Hence, for $s \ge t_1$,

$$\liminf_{t\to\infty} \frac{h(t,s)}{h(t,t_1)} > \eta. \tag{18}$$

Let μ be an arbitrary positive number. Then there exists a $t_2 > t_1$ such that, for $t > t_2$,

$$\int_{t_1}^{t} \big(\tilde{f}(s)\mathrm{tr}\,P(s)\big)^{-1}\,\mathrm{tr}\,V^2(s)\,ds \;>\; \Big(\frac{\mu}{\eta}\Big) \tag{19}$$

due to (17). Hence, integration by parts yields, for $t > t_2$,

$$\begin{aligned}
\mathrm{tr}\,B(t) &= \frac{n}{h(t,t_1)}\int_{t_1}^{t} h(t,s)\big(\tilde{f}(s)\mathrm{tr}\,P(s)\big)^{-1}\,\mathrm{tr}\,V^2(s)\,ds\\
&= \frac{n}{h(t,t_1)}\int_{t_1}^{t} h(t,s)\,\frac{d}{ds}\Big[\int_{t_1}^{s}\big(\tilde{f}(\theta)\mathrm{tr}\,P(\theta)\big)^{-1}\mathrm{tr}\,V^2(\theta)\,d\theta\Big]\,ds\\
&= -\,\frac{n}{h(t,t_1)}\int_{t_1}^{t} \frac{\partial}{\partial s}\,h(t,s)\Big[\int_{t_1}^{s}\big(\tilde{f}(\theta)\mathrm{tr}\,P(\theta)\big)^{-1}\,\mathrm{tr}\,V^2(\theta)\,d\theta\Big]\,ds\\
&> -\,\frac{n}{h(t,t_1)}\int_{t_2}^{t} \frac{\partial h(t,s)}{\partial s}\Big[\int_{t_1}^{s}\big(\tilde{f}(\theta)\mathrm{tr}\,P(\theta)\big)^{-1}\,\mathrm{tr}\,V^2(\theta)\,d\theta\Big]\,ds\\
&> -\,\frac{n\mu}{\eta h(t,t_1)}\int_{t_2}^{t} \frac{\partial h(t,s)}{\partial s}\,ds
\end{aligned}$$

by (19). Thus, for $t \ge t_2$,

$$\mathrm{tr}\,B(t) \;>\; \frac{n\mu}{\eta}\,\frac{h(t,t_2)}{h(t,t_1)}.$$

It is possible to choose $t_3 > t_2$ such that

$$\frac{h(t,t_2)}{h(t,t_1)} \;>\; \eta \quad \text{for } t \ge t_3$$

by (18). Hence, for $t \ge t_3$, $\mathrm{tr}\,B(t) > n\mu$. Since μ is arbitrary, then

$$\lim_{t\to\infty} \mathrm{tr}\,B(t) \;=\; \infty. \tag{20}$$

From (16) it follows that there exists a sequence $\langle\sigma_k\rangle$ such that $\sigma_k \to \infty$ as $k \to \infty$ and

$$\mathrm{tr}\,B(\sigma_k) + \mathrm{tr}\,C(\sigma_k) \;<\; M$$

for large k, where M is a real number. Hence,

$$\lim_{k\to\infty} \mathrm{tr}\,C(\sigma_k) \;=\; -\,\infty \tag{21}$$

by (20). Further, for large k,

$$1+\frac{\operatorname{tr} C(\sigma_k)}{\operatorname{tr} B(\sigma_k)} \;<\; \frac{M}{\operatorname{tr} B(\sigma_k)} \;<\; \frac{1}{2},$$

that is,

$$\frac{\operatorname{tr} C(\sigma_k)}{\operatorname{tr} B(\sigma_k)} \;<\; -\frac{1}{2}. \tag{22}$$

Clearly, (21) and (22) imply that

$$\lim_{k\to\infty} \frac{[\operatorname{tr} C(\sigma_k)]^2}{\operatorname{tr} B(\sigma_k)} \;=\; \infty. \tag{23}$$

On the other hand, use of the Cauchy-Schwarz inequality yields

$$\begin{aligned}
[\operatorname{tr} C(\sigma_k)]^2 &= \Big[\frac{1}{h(\sigma_k,t_1)} \int_{t_1}^{\sigma_k} g(\sigma_k,s)\sqrt{h(\sigma_k,s)}\operatorname{tr} V(s)\,ds\Big]^2 \\
&\le \Big[\frac{1}{h(\sigma_k,t_1)} \int_{t_1}^{\sigma_k} g^2(\sigma_k,s)\big(\tilde f(s)\operatorname{tr} P(s)\big)\,ds\Big] \\
&\quad \cdot \Big[\frac{1}{h(\sigma_k,t_1)} \int_{t_1}^{\sigma_k} h(\sigma_k,s)\big(\tilde f(s)\operatorname{tr} P(s)\big)^{-1}\big(\operatorname{tr} V(s)\big)^2\,ds\Big] \\
&\le \big(\operatorname{tr} B(\sigma_k)\big)\Big[\frac{1}{h(\sigma_k,t_1)} \int_{t_1}^{\sigma_k} g^2(\sigma_k,s)\big(\tilde f(s)\operatorname{tr} P(s)\big)\,ds\Big],
\end{aligned}$$

where we have used $(\operatorname{tr} V(t))^2 \le n\operatorname{tr} V^2(t)$. Since $B(t) > 0$, then

$$\lim_{k\to\infty} \frac{1}{h(\sigma_k,t_1)} \int_{t_1}^{\sigma_k} g^2(\sigma_k,s)\big(\tilde f(s)\operatorname{tr} P(s)\big)\,ds \;=\; \infty$$

by (23). This contradicts (C_5). Hence our claim holds. Thus, by (14),

$$\begin{aligned}
\int_{t_1}^{\infty} a_+^2(s)\big(\tilde f(s)\operatorname{tr} P(s)\big)^{-1}\,ds &\le \int_{t_1}^{\infty} \big(\operatorname{tr} V(s)\big)^2\big(\tilde f(s)\operatorname{tr} P(s)\big)^{-1}\,ds \\
&\le n\int_{t_1}^{\infty} \big(\tilde f(s)\operatorname{tr} P(s)\big)^{-1}\operatorname{tr} V^2(s)\,ds < \infty.
\end{aligned}$$

This contradicts (C_6). Hence the theorem is proved. □

Remark. Since $h(t,s)$ is monotonically decreasing in s, then the assumptions (C_4), (C_5) and (C_6) are equivalent, respectively, to

$$(C_4') \qquad 0 < \inf_{s\geq\sigma}\Big[\liminf_{t\to\infty} \frac{h(t,s)}{h(t,\sigma)}\Big]$$

$$(C_5') \qquad \limsup_{t\to\infty} \frac{1}{h(t,\sigma)}\int_\sigma^t g^2(t,s)\big(\tilde{f}(s)\operatorname{tr} P(s)\big)\,ds < \infty$$

and

(C_6') There exists a function $a \in C([\sigma,\infty),(-\infty,\infty))$ such that

$$\int_\sigma^\infty a_+^2(t)\big(\tilde{f}(t)\operatorname{tr} P(t)\big)^{-1}\,dt = \infty,$$

where $a_+(t) = \max(a(t),0)$.

Remark. Theorem 2.3 generalizes Theorem 1.3 to the system (E).

Theorem 2.4. *If* (C_1), (C_2), (C_4), (C_6), (C_8) *and* (C_9) *hold, then the system* (E) *oscillates.*

Proof. Suppose that $Y(t)$ is a nonoscillatory solution of (E). Then $\det Y(t) \neq 0$ for $t \geq t_1 \geq \sigma$. Setting $V(t)$ as in (9) for $t \geq t_1$, one may obtain (10). Clearly, $V(t)$ is symmetric. From (11) we get, using (C_9), that

$$\begin{aligned} a(t^*) &\leq \liminf_{t\to\infty} \frac{1}{h(t,t^*)}\int_{t^*}^t \Big[h(t,s)\operatorname{tr} R(s) - \frac{1}{4}\, g^2(t,s)\tilde{f}(s)\operatorname{tr} P(s)\Big]\,ds \\ &\leq \operatorname{tr} V(t^*) - \limsup_{t\to\infty} \frac{1}{h(t,t^*)}\int_{t^*}^t \operatorname{tr}\Big[\big(nh(t,s)\big(\tilde{f}(s)\operatorname{tr} P(s)\big)^{-1}\big)^{1/2} V(s) \\ &\qquad + \frac{1}{2}\, g(t,s)\,\frac{\big(\tilde{f}(s)\operatorname{tr} P(s)\big)^{\frac{1}{2}}}{n^{\frac{1}{2}}}\, I\Big]^2\,ds \end{aligned}$$

for $t > t^* \geq t_1$. Hence, (14) holds and

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{h(t,t_1)}\int_{t_1}^t \operatorname{tr}\Big[&(nh(t,s)(\tilde{f}(s)\operatorname{tr} P(s))^{-1})^{1/2} V(s) \\ &+ \frac{1}{2} g(t,s)\frac{(\tilde{f}(s)\operatorname{tr} P(s))^{1/2}}{n^{1/2}} I\Big]^2\,ds \\ &\leq \operatorname{tr} V(t_1) - a(t_1) < \infty, \end{aligned}$$

that is,

$$\limsup_{t\to\infty}\,[\operatorname{tr} B(t) + \operatorname{tr} C(t)] \;<\; \infty,$$

where $B(t)$ and $C(t)$ are the same as in the proof of Theorem 2.3. From (C_8) and (C_9) it follows that

$$\begin{aligned} a(t_1) &\le \liminf_{t\to\infty} \frac{1}{h(t,t_1)} \int_{t_1}^{t} \Big[h(t,s)\operatorname{tr} R(s) - \frac{1}{4}\, g^2(t,s)\tilde{f}(s)\operatorname{tr} P(s)\Big]\, ds \\ &\le \liminf_{t\to\infty} \frac{1}{h(t,t_1)} \int_{t_1}^{t} h(t,s)\operatorname{tr} R(s)\, ds \\ &\quad - \frac{1}{4}\, \liminf_{t\to\infty} \frac{1}{h(t,t_1)} \int_{t_1}^{t} g^2(t,s)\tilde{f}(s)\operatorname{tr} P(s)\, ds, \end{aligned}$$

that is,

$$\liminf_{t\to\infty} \frac{1}{h(t,t_1)} \int_{t_1}^{t} g^2(t,s)\tilde{f}(s)\operatorname{tr} P(s)\, ds \;<\; \infty. \tag{24}$$

Let

$$\int_{t_1}^{\infty} \big(\tilde{f}(s)\operatorname{tr} P(s)\big)^{-1} \operatorname{tr} V^2(s)\, ds \;=\; \infty.$$

Since (C_4) holds, then proceeding as in the proof of Theorem 2.3, we obtain

$$\lim_{t\to\infty} \operatorname{tr} B(t) \;=\; \infty,$$

and hence there exists a sequence $\langle \sigma_k \rangle$ such that $\sigma_k \to \infty$ as $k \to \infty$ and

$$\lim_{k\to\infty} \frac{1}{h(\sigma_k,t_1)} \int_{t_1}^{\sigma_k} g^2(\sigma_k,s)\tilde{f}(s)\operatorname{tr} P(s)\, ds \;=\; \infty,$$

which contradicts (24). Thus,

$$\int_{t_1}^{\infty} (\tilde{f}(s)\operatorname{tr} P(s))^{-1}\operatorname{tr} V^2(s)\, ds < \infty. \tag{25}$$

However, (14) and (25) together contradict (C_6). Thus the theorem is proved.

Remark. Theorem 2.4 generalizes Theorem 1.4 to the system (E).

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