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INVERSION OF THE DUNKL INTERTWINING OPERATOR AND ITS DUAL USING DUNKL WAVELETS

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ABSTRACT. We consider in this work the Dunkl intertwining operator V_k and its dual ${}^tV_k$ on $\mathbf{R}^d$. Using these operators we give relations between the Dunkl continuous wavelet transform on $\mathbf{R}^d$ and the classical continuous wavelet transform on $\mathbf{R}^d$, and we deduce the formulas which give the inverse operators of V_k and ${}^tV_k$.

1. Introduction. We consider the differential-difference operators T_j, $j = 1, 2, \dots, d$, on $\mathbf{R}^d$ introduced by Dunkl in [**3**] and called Dunkl operators in the literature. These operators are very important in pure mathematics and in physics. They provide a useful tool in the study of special functions with root systems [**2, 4, 6**] and they are closely related to certain representations of degenerate affine Hecke algebras [**1, 16**]; moreover, the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum mechanics, namely, the Calogero-Sutherland-Moser models which deal with systems of identical particles in a one-dimensional space, (see [**8, 11, 13**]).

Dunkl has proved in [**5**] that a unique isomorphism V_k exists from the space of homogeneous polynomials $\mathcal{P}_n$ on $\mathbf{R}^d$ of degree n onto itself satisfying the relations

$$T_j V_k = V_k \frac{\partial}{\partial x_j}, \quad j = 1, 2, \dots, d \tag{1}$$

and

$$V_k(1) = 1. \tag{2}$$

This operator is called the Dunkl intertwining operator.

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Rösler has considered this operator and has proved in [**19**] that for each $x \in \mathbf{R}^d$ a probability measure μ_x exists on $\mathbf{R}^d$ with support in the closed ball $B(0, \|x\|)$ of center 0 and radius $\|x\|$ such that, for all polynomials p on $\mathbf{R}^d$, we have

$$V_k(p)(x) = \int_{\mathbf{R}^d} p(y)\, d\mu_x(y).$$

Next Trimèche has extended in [**24**] the operator V_k to an isomorphism from $\mathcal{E}(\mathbf{R}^d)$, the space of C^∞-functions on $\mathbf{R}^d$, onto itself satisfying the relations (1) and (2) and has shown that, for each $x \in \mathbf{R}^d$, a unique distribution η_x exists in $\mathcal{E}'(\mathbf{R}^d)$, the space of distributions on $\mathbf{R}^d$ of compact support, with support in $B(0, \|x\|)$ such that

$$(V_k)^{-1}(f)(x) = \langle \eta_x, f \rangle, \quad f \in \mathcal{E}(\mathbf{R}^d).$$

We have studied also in [**24**] the transposed operator ${}^tV_k$ of the operator V_k and we have proved that it has the integral representation, for all $y \in \mathbf{R}^d$,

$${}^tV_k(f)(y) = \int_{\mathbf{R}^d} f(x)\, d\nu_y(x), \quad f \in \mathcal{D}(\mathbf{R}^d),$$

where ν_y is a positive measure on $\mathbf{R}^d$ with support in the set $\{x \in \mathbf{R}^d : \|x\| \geq \|y\|\}$ and f in $\mathcal{D}(\mathbf{R}^d)$, the space of C^∞-functions on $\mathbf{R}^d$ with compact support. This operator is called the dual Dunkl intertwining operator.

We have proved in [**24**] that the operator ${}^tV_k$ is an isomorphism from $\mathcal{D}(\mathbf{R}^d)$ onto itself, satisfying the relations, for all $y \in \mathbf{R}^d$,

$${}^tV_k(T_j f)(y) = \frac{\partial}{\partial y_j} {}^tV_k(f)(y), \quad j = 1, 2, \dots, d,$$

and we have shown that for each $y \in \mathbf{R}^d$ a unique distribution Z_y exists in $\mathcal{S}'(\mathbf{R}^d)$, the space of tempered distributions on $\mathbf{R}^d$, such that

$$({}^tV_k)^{-1}(f)(y) = \langle Z_y, f \rangle, \quad f \in \mathcal{D}(\mathbf{R}^d).$$

The purpose of this paper is to establish for the Dunkl intertwining operator V_k and its dual ${}^tV_k$ the following results.

(i) We define and characterize spaces of functions other than $\mathcal{E}(\mathbf{R}^d)$ and $\mathcal{D}(\mathbf{R}^d)$ on which these transforms are bijective.

(ii) We give the following inversion formulas for V_k and ${}^tV_k$ on some spaces of functions

$$f = V_k\mathcal{K}^tV_k(f), \qquad f = \mathcal{K}^tV_kV_k(f),$$
$$f = {}^tV_k\mathcal{K}_DV_k(f), \quad f = \mathcal{K}_DV_k{}^tV_k(f),$$

where $\mathcal{K}$ and $\mathcal{K}_D$ are pseudo-differential operators.

(iii) We define and study the Dunkl continuous wavelet transform on $\mathbf{R}^d$ and we prove for this transform a Plancherel and an inversion formula.

(iv) We obtain formulas which give the inverse operators V_k^{-1} and $({}^tV_k)^{-1}$ using Dunkl wavelets.

Analogous results to the precedent have been studied in the case of differential operators and partial differential operators, (see [**22, 23**]).

The content of this paper is as follows:

In the first section we give results on the Dunkl operators and on their eigenfunction called the Dunkl kernel.

We define in the second section the Dunkl intertwining operator V_k and its dual ${}^tV_k$ and we give their main properties.

We study in the third and fourth sections the harmonic analysis associated with Dunkl operators (Dunkl transform, Dunkl translation operators and Dunkl convolution product).

In the fifth section we give spaces other than $\mathcal{E}(\mathbf{R}^d)$ and $\mathcal{D}(\mathbf{R}^d)$ on which the Dunkl intertwining operator V_k and its dual ${}^tV_k$ are bijective, and we establish inversion formulas for these operators.

We define and study in the sixth section Dunkl wavelets and the Dunkl continuous wavelet transform, and we give for this transform Plancherel and inversion formulas.

Using Dunkl wavelets we obtain in the last section formulas which give the inverse operators of the operators V_k and ${}^tV_k$.

1. The eigenfunction of the Dunkl operators. In this section we collect some notations and results on Dunkl operators and the Dunkl kernel, (see [**4, 5, 7, 9, 10**]).

1.1 *Reflection groups, root systems and multiplicity functions.* We consider $\mathbf{R}^d$ with the Euclidean scalar product $\langle \cdot, \cdot \rangle$ and $\|x\| = \sqrt{\langle x, x \rangle}$. On $\mathbf{C}^d$, $\|\cdot, \cdot\|$ also denotes the Hermitian norm, while $\langle z, w \rangle = \sum_{j=1}^d z_j \overline{w_j}$.

For $\alpha \in \mathbf{R}^d \setminus \{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbf{R}^d$ orthogonal to α, i.e.,

$$\sigma_\alpha(x) = x - (2\langle \alpha, x \rangle / \|\alpha\|^2)\alpha. \tag{1.1}$$

A finite set $R \subset \mathbf{R}^d \setminus \{0\}$ is called a root system if $R \cap \mathbf{R} \cdot \alpha = \{\pm\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. For a given root system R the reflections σ_α, $\alpha \in R$, generate a finite group $W \subset O(d)$, the reflection group associated with R. All reflections in W correspond to suitable pairs of roots. For a given $\beta \in \mathbf{R}^d \setminus \cup_{\alpha \in R} H_\alpha$, we fix the positive subsystem $R_+ = \{\alpha \in R : \langle \alpha, \beta \rangle > 0\}$, then for each $\alpha \in R$ either $\alpha \in R+$ or $-\alpha \in R_+$.

A function $k : R \to \mathbf{C}$ on a root system R is called a multiplicity function if it is invariant under the action of the associated reflection group W. If one regards k as a function on the corresponding reflections, this means that k is constant on the conjugacy classes of reflections in W. For abbreviation, we introduce the index

$$\gamma = \gamma(R) = \sum_{\alpha \in R_+} k(\alpha). \tag{1.2}$$

Moreover, let ω_k denote the weight function

$$\omega_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}, \tag{1.3}$$

which is W-invariant and homogeneous of degree 2γ.

For $d = 1$ and $W = \mathbf{Z}_2$, the multiplicity function k is a single parameter denoted $\gamma > 0$ and, for all $x \in \mathbf{R}$,

$$\omega_k(x) = |x|^{2\gamma}. \tag{1.4}$$

We introduce the Mehta-type constant

$$C_k = \left(\int_{\mathbf{R}^d} e^{-\|x\|^2} \omega_k(x)\, dx \right)^{-1}, \tag{1.5}$$

which is known for all Coxeter groups W, (see [**3, 14, 16**]).

1.2 *Dunkl operators and Dunkl kernel.* The Dunkl operators T_j, $j = 1, \dots, d$, on $\mathbf{R}^d$ associated with the finite reflection group W and multiplicity function k, are given for a function f of class C^1 on $\mathbf{R}^d$ by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}. \tag{1.6}$$

In the case $k = 0$, the T_j, $j = 1, 2, \dots, d$, reduce to the corresponding partial derivatives. In this paper we will assume throughout that $k \geq 0$ and $\gamma > 0$.

For f of class C^1 on $\mathbf{R}^d$ with compact support and g of class C^1 on $\mathbf{R}^d$, we have

$$\int_{\mathbf{R}^d} T_i f(x) g(x) \omega_k(x)\, dx = -\int_{\mathbf{R}^d} f(x) T_j g(x) \omega_k(x)\, dx, \qquad j = 1, 2, \dots, d. \tag{1.7}$$

For $y \in \mathbf{R}^d$, the system

$$\begin{cases} T_j u(x, y) = y_j u(x, y) & j = 1, 2, \dots, d \\ u(0, y) = 1 \end{cases} \tag{1.8}$$

admits a unique analytic solution on $\mathbf{R}^d$, denoted by $K(x, y)$ and called Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbf{C}^d \times \mathbf{C}^d$.

Example. If $d = 1$ and $W = \mathbf{Z}_2$, the Dunkl kernel is given by

$$K(z, t) = j_{\gamma - 1/2}(izt) + \frac{zt}{2\gamma + 1} j_{\gamma + 1/2}(izt), \quad z, t \in \mathbf{C}, \tag{1.9}$$

where for $\alpha \geq -1/2$, j_α is the normalized Bessel function defined by

$$\begin{aligned} j_\alpha(u) &= 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(u)}{u^\alpha} \\ &= \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n (u/2)^{2n}}{n! \Gamma(n + \alpha + 1)}, \quad u \in \mathbf{C}, \end{aligned} \tag{1.10}$$

with J_α is the Bessel function of first kind and index α.

The Dunkl kernel possesses the following properties.

i) For all $z, t \in \mathbf{C}^d$, we have $K(z,t) = K(t,z)$; $K(z,0) = 1$ and $K(\lambda z, t) = K(z, \lambda t)$ for all $\lambda \in \mathbf{C}$.

ii) For all $\nu \in \mathbf{Z}_+^d$, $x \in \mathbf{R}^d$ and $z \in \mathbf{C}^d$, we have

$$|D_z^\nu K(x,z)| \leq \|x\|^{|\nu|} \exp[\max_{w \in W} \langle wx, \operatorname{Re} z\rangle]. \tag{1.11}$$

In particular,

$$|D_z^\nu K(x,z)| \leq \|x\|^{|\nu|} \exp[\|x\| \|\operatorname{Re} z\|], \tag{1.12}$$

$$|K(x,z)| \leq \exp[\|x\| \|\operatorname{Re} z\|], \tag{1.13}$$

and, for all $x, y \in \mathbf{R}^d$:

$$|K(ix,y)| \leq 1, \tag{1.14}$$

with $D_z^\nu = (\partial^{|\nu|}/\partial z_1^{\nu_1} \cdots \partial z_d^{\nu_d})$ and $|\nu| = \nu_1 + \nu_2 + \cdots + \cdots + \nu_d$.

iii) For all $x, y \in \mathbf{R}^d$ and $w \in W$, we have

$$K(-ix, y) = \overline{K(ix,y)} \quad \text{and} \quad K(wx, wy) = K(x,y). \tag{1.15}$$

iv) The function $K(x,z)$ admits for all $x \in \mathbf{R}^d$ and $z \in \mathbf{C}^d$ the following Laplace type integral

$$K(x,z) = \int_{\mathbf{R}^d} e^{\langle y,z\rangle} \, d\mu_z(y), \tag{1.16}$$

where μ_x is a probability measure on $\mathbf{R}^d$ with support in the closed ball $B(0, \|x\|)$ of center 0 and radius $\|x\|$, (see [**19**]).

Remark. When $d = 1$ and $W = \mathbf{Z}_2$, for all $x \in \mathbf{R}^d \backslash \{0\}$, the relation (1.16) is of the form

$$K(x,z) = \frac{\Gamma(\gamma + 1/2)}{\sqrt{\pi}\Gamma(\gamma)} |x|^{-2\gamma} \int_{-|x|}^{|x|} (|x| - y)^{\gamma - 1} (|x| + y)^\gamma e^{yz} \, dy. \tag{1.17}$$

Then, in this case, for all $x \in \mathbf{R}^d \backslash \{0\}$, the measure μ_x is given by $d\mu_x(y) = \kappa(x,y)\,dy$ with

$$\kappa(x,y) = \frac{\Gamma(\gamma+1/2)}{\sqrt{\pi}\Gamma(\gamma)} |x|^{-2\gamma}(|x|-y)^{\gamma-1}(|x|+y)^{\gamma} 1_{]-|x|,|x|[}(y), \tag{1.18}$$

where $1_{]-|x|,|x|[}$ is the characteristic function of the interval $]-|x|,|x|\,[$.

We remark that by change of variables the relation (1.17) takes the following form, for all $x \in \mathbf{R}^d$, for all $z \in \mathbf{C}^d$,

$$K(x,z) = \frac{\Gamma(\gamma+1/2)}{\sqrt{\pi}\Gamma(\gamma)} \int_{-1}^{1} e^{txz}(1-t^2)^{\gamma-1}(1+t)\,dt. \tag{1.19}$$

2. The Dunkl intertwining operator and its dual.

Notations. We denote by $C(\mathbf{R}^d)$, respectively $C_c(\mathbf{R}^d)$, the space of continuous functions on $\mathbf{R}^d$, respectively with compact support; $C^p(\mathbf{R}^d)$, respectively $C_c^p(\mathbf{R}^d)$, the space of functions of class C^p on $\mathbf{R}^d$, respectively with compact support; $\mathcal{E}(\mathbf{R}^d)$, the space of C^∞-functions on $\mathbf{R}^d$; $\mathcal{D}(\mathbf{R}^d)$, the space of C^∞-functions on $\mathbf{R}^d$ with compact support; $\mathcal{S}(\mathbf{R}^d)$, the space of C^∞-functions on $\mathbf{R}^d$ which are rapidly decreasing as their derivatives. We provided these spaces with the classical topology.

We consider also the following spaces. $\mathcal{E}'(\mathbf{R}^d)$, the space of distributions on $\mathbf{R}^d$ with compact support. It is the topological dual of $\mathcal{E}(\mathbf{R}^d)$; $\mathcal{S}'(\mathbf{R}^d)$, the space of tempered distributions on $\mathbf{R}^d$. It is the topological dual of $\mathcal{S}(\mathbf{R}^d)$.

The Dunkl intertwining operator V_k is defined on $C(\mathbf{R}^d)$ by

$$V_k(f)(x) = \int_{\mathbf{R}^d} f(y)\,d\mu_x(y), \tag{2.1}$$

where μ_x is the measure given by the relation (1.16), (see [**24**]).

We have, for all $x \in \mathbf{R}^d$, for all $z \in \mathbf{C}^d$,

$$K(x,z) = V_k(e^{\langle \cdot, y\rangle})(x). \tag{2.2}$$

The operator ${}^tV_k$ satisfying for f in $C_c(\mathbf{R}^d)$ and g in $C(\mathbf{R}^d)$, the relation

$$\int_{\mathbf{R}^d} {}^tV_k(f)(y)g(y)\,dy = \int_{\mathbf{R}^d} V_k(g)(x)f(x)\omega_k(x)\,dx, \tag{2.3}$$

is given by

$$ {}^tV_k(f)(y) = \int_{\mathbf{R}^d} f(x)\,d\nu_y(x), \tag{2.4}$$

where ν_y is a positive measure on $\mathbf{R}^d$ with support in the set $\{x \in \mathbf{R}^d : \|x\| \geq \|y\|\}$. This operator is called the dual Dunkl intertwining operator, (see [**24**]).

The following theorems give some properties of the operators V_k and ${}^tV_k$, (see [**24**]).

Theorem 2.1. i) *The operator V_k is a topological isomorphism from $\mathcal{E}(\mathbf{R}^d)$ onto itself satisfying the relations, for all $x \in \mathbf{R}^d$,*

$$T_jV_k(f)(x) = V_k\left(\frac{\partial}{\partial \chi_j} f\right)(x), \quad j = 1, 2, \dots, d, \quad f \in \mathcal{E}(\mathbf{R}^d). \tag{2.5}$$

ii) *For each $x \in \mathbf{R}^d$, a unique distribution η_x in $\mathcal{E}'(\mathbf{R}^d)$ exists with support in the closed ball $B(0, \|x\|)$ of center 0 and radius $\|x\|$ such that, for all f in $\mathcal{E}(\mathbf{R}^d)$, we have*

$$V_k^{-1}(f)(x) = \langle \eta_x, f \rangle. \tag{2.6}$$

Theorem 2.2. i) *The operator ${}^tV_k$ is a topological isomorphism from $\mathcal{D}(\mathbf{R}^d)$, respectively $\mathcal{S}(\mathbf{R}^d)$, onto itself, satisfying the relations, for all $y \in \mathbf{R}^d$,*

$$ {}^tV_k(T_jf)(y) = \frac{\partial}{\partial y_j} {}^tV_k(f)(y),\ j = 1, 2, \dots, d,\ f \in \mathcal{D}(\mathbf{R}^d). \tag{2.7}$$

ii) *For each* $y \in \mathbf{R}^d$, *a unique distribution* Z_y *exists in* $\mathcal{S}'(\mathbf{R}^d)$ *with support in the set* $\{x \in \mathbf{R}^d : \|x\| \geq \|y\|\}$ *such that for all* f *in* $\mathcal{D}(\mathbf{R}^d)$, *we have*

$$({}^tV_k)^{-1}(f)(y) = \langle Z_y, f\rangle. \tag{2.8}$$

Examples. 1) When $d = 1$ and $W = \mathbf{Z}_2$, the Dunkl intertwining operator V_k is defined by (2.1) with κ given by the relation (1.18). We have shown in [**15**] that it can also be written for all f in $\mathcal{E}(\mathbf{R})$, in the form, for all $x \in \mathbf{R}$,

$$V_k(f)(x) = \mathbf{R}_{\gamma-1/2}(f_e)(x) + \frac{d}{dx}\mathbf{R}_{\gamma-1/2}I(f_0)(x), \tag{2.9}$$

where f_ε, respectively f_0, the even, respectively the odd, part of $f, \mathbf{R}_{\gamma-1/2}$ the Riemann-Liouville integral transform defined for all even C^∞-function g on $\mathbf{R}$ by, for all $r > 0$,

$$\mathbf{R}_{\gamma-1/2}(g)(r) = \frac{2\Gamma(\gamma+1/2)}{\sqrt{\pi}\Gamma(\gamma)} r^{-2\gamma+1}\int_0^r g(t)(r^2-t^2)^{\gamma-1}\,dt \tag{2.10}$$

(see [**23**, pages 26–27] and [**21**, page 74]), and I the operator given by, for all $x \in \mathbf{R}$,

$$I(f_0)(x) = \int_0^{|x|} f_0(t)\,dt. \tag{2.11}$$

Using properties of the transform $\mathbf{R}_{\gamma-1/2}$, we prove that the inverse operator V_k^{-1} can be written for all f in $\mathcal{E}(\mathbf{R})$, in the following form, for all $x \in \mathbf{R}$,

$$V_k^{-1}(f)(x) = \mathbf{R}^{-1}_{\gamma-1/2}(f_e)(x) + \frac{1}{x}\mathbf{R}^{-1}_{\gamma-1/2}(yf_0(y))(x). \tag{2.12}$$

It is an integro-differential operator.

The dual Dunkl intertwining operator ${}^tV_k$ is defined by (2.4) with $d\nu_y(x) = \kappa(x,y)\omega_k(x)\,dx$,where κ and ω_k given respectively by the relations (1.18) and (1.4). It can also be written for all f in $\mathcal{D}(\mathbf{R})$, in the form, for all $y \in \mathbf{R}$,

$$ {}^tV_k(f)(y) = W_{\gamma-1/2}(f_e)(y) + \frac{d}{dy} W_{\gamma-1/2} J(f_0)(y), \tag{2.13} $$

where $W_{\gamma-1/2}$ is the Weyl integral transform defined for all even C^∞-function g on $\mathbf{R}$ with compact support, by, for all $t \geq 0$,

$$ W_{\gamma-1/2}(g)(t) = \frac{2\Gamma(\gamma+1/2)}{\sqrt{\pi}\Gamma(\gamma)} \int_t^\infty g(r)(r^2-t^2)^{\gamma-1}\, dr \tag{2.14} $$

(see [**23**, pages 41–50] and [**21**, pages 81, 85]), and the operator J is given by, for all $x \in \mathbf{R}$,

$$ J(f_0)(x) = \int_{-\infty}^x f_0(t), dt. \tag{2.15} $$

From properties of the transform $W_{\gamma-1/2}$ we deduce that the inverse operator $({}^tV_k)^{-1}$ possesses for all f in $\mathcal{D}(\mathbf{R})$, the following form, for all $y \in \mathbf{R}$,

$$ ({}^tV_k)^{-1}(f)(y) = W^{-1}_{\gamma-1/2}(f_e)(y) + yW^{-1}_{\gamma-1/2}\left(\frac{1}{x}f(x)\right)(y). \tag{2.16} $$

It is also an integro-differential operator.

2) The Dunkl intertwining operator V_k of index $\gamma = \sum_{i=1}^d \alpha_i$, $\alpha_i > 0$, associated with the reflection group $\mathbf{Z}_3 \times \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$ on $\mathbf{R}^d$ is given for all f in $\mathcal{E}(\mathbf{R}^d)$ by, for all $x \in \mathbf{R}^d$,

(2.17)
$$ V_k(f)(x) = \prod_{i=1}^d \left(\frac{\Gamma(\alpha_i+(1/2))}{\sqrt{\pi}\Gamma(\alpha_i)}\right) \times \int_{[-1,1]^d} f(t_1x_1, t_2x_2, \ldots, t_dx_d) \prod_{i=1}^d (1-t_i^2)^{\alpha_i-1}(1+t_i)dt_1 \cdots dt_d, $$

(see [**26**]). It can also be written in the form, for all $x \in \mathbf{R}^d$,

$$ V_k(f)(x) = (V_k)^1 \otimes (V_k)^2 \otimes \cdots \otimes (V_k)^d(f)(x), \tag{2.18} $$

where for all g in $\mathcal{E}(\mathbf{R})$ we have, for all $x_i \in \mathbf{R}$,

$$(V_k)^i(g)(x_i) = \frac{\Gamma(\alpha_i + (1/2))}{\sqrt{\pi}\Gamma(\alpha_i)} \int_{-1}^{1} g(t_i x_i)(1 - t_i^2)^{\alpha_i - 1}(1 + t_i)\, dt_i.$$

Using the results of the previous example, we determine the inverse operator V_k^{-1}, the dual operator ${}^tV_k$ and its inverse $({}^tV_k)^{-1}$. We deduce from (2.12) and (2.16) that the operators V_k^{-1} and $({}^tV_{kk})^{-1}$ are integro-differential operators. We remark that if one or many α_i is equal to zero, we replace the corresponding $(V_k)^i$ in the definition (2.18) of the operator V_k by the operator identity.

3. Dunkl transform. In this section we define the Dunkl transform and we give the main results satisfied by this transform, (see [**5, 9, 10, 25**]).

Notations. We denote by $\mathcal{S}^0(\mathbf{R}^d)$ the subspace of $\mathcal{S}(\mathbf{R}^d)$ consisting of functions f such that, for all $\nu \in \mathbf{N}^d$,

$$D^\nu f(0) = 0,$$

where for $\nu = (\nu_1, \nu_2, \dots, \nu_d) \in \mathbf{N}^d$,

$$D^\nu = \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} \cdots \partial x_d^{\nu_d}},$$

and $|\nu| = \nu_1 + \cdots + \nu_d$.

$\mathcal{S}_0(\mathbf{R}^d)$ is the subspace of $\mathcal{S}(\mathbf{R}^d)$ consisting of functions f such that, for all $\nu \in \mathbf{N}^d$,

$$\int_{\mathbf{R}^d} f(x) x^\nu\, dx = 0,$$

where for $\nu = (\nu_1, \nu_2, \dots, \nu_d) \in \mathbf{N}^d$ and $x = (x_1, x_2, \dots, x_d) \in \mathbf{R}^d$, we have $x^\nu = x_1^{\nu_1} \cdot x_2^{\nu_2} \cdot \cdots \cdot x_d^{\nu_d}$.

$\mathcal{S}_0^0(\mathbf{R}^d)$ is the subspace of $\mathcal{S}(\mathbf{R}^d)$ consisting of functions f such that, for all $\nu \in \mathbf{N}^d$,

$$\int_{\mathbf{R}^d} f(x) m_\nu(x) \omega_k(x)\, dx = 0,$$

where, for all $x \in \mathbf{R}^d$,

$$m_\nu(x) = V_k\left(\frac{u^\nu}{\nu!}\right)(x),$$

where for $\nu = (\nu_1, \nu_2, \dots, \nu_d) \in \mathbf{N}^d$, we have $\nu! = \nu_1!\nu_2!\cdots\nu_d!$.

$L_k^p(\mathbf{R}^d)$, $p \in [1, +\infty]$, the space of measurable functions on $\mathbf{R}^d$ such that

$$\|f\|_{k,p} = \left(\int_{\mathbf{R}^d} |f(x)|^p \omega_k(x)\, dx\right)^{1/p} < +\infty, \quad \text{if } 1 \leq p < +\infty,$$
$$\|f\|_{k,\infty} = \operatorname*{ess\,sup}_{x\in\mathbf{R}^d} |f(x)| < +\infty.$$

$\mathbf{H}(\mathbf{C}^d)$ is the space of entire functions on $\mathbf{C}^d$ which are of exponential type and rapidly decreasing.

We provide these spaces with the classical topology.

The Dunkl transform of a function f in $\mathcal{D}(\mathbf{R}^d)$ is given by, for all $y \in \mathbf{R}^d$,

$$\mathcal{F}_D(f)(y) = \int_{\mathbf{R}^d} f(x) K(x, -iy) \omega_k(x)\, dx. \tag{3.1}$$

This transform has the following properties

i) For f in $L_k^1(\mathbf{R}^d)$, we have

$$\|\mathcal{F}_D(f)\|_{k,\infty} \leq \|f\|_{k,1}. \tag{3.2}$$

ii) Let f be in $\mathcal{D}(\mathbf{R}^d)$. If $f^-(x) = f(-x)$ and $f_w(x) = f(wx)$ for $x \in \mathbf{R}^d$, $w \in W$, then for all $y \in \mathbf{R}^d$, we have

$$\mathcal{F}_D(f^-)(y) = \overline{\mathcal{F}_D(f)(y)} \quad \text{and} \quad \mathcal{F}_D(f_w)(y) = \mathcal{F}_D(f)(wy). \tag{3.3}$$

iii) For all f in $\mathcal{S}(\mathbf{R}^d)$, we have

$$\mathcal{F}_D(f) = \mathcal{F} o\, {}^t V_k(f), \tag{3.4}$$

where $\mathcal{F}$ is the classical Fourier transform on $\mathbf{R}^d$ given by, for all $y \in \mathbf{R}^d$,

$$\mathcal{F}(f)(y) = \int_{\mathbf{R}^d} f(x) e^{-i\langle x,y\rangle}\, dx, \quad f \in \mathcal{D}(\mathbf{R}^d). \tag{3.5}$$

Theorem 3.1. *The transform $\mathcal{F}_D$ is a topological isomorphism from $\mathcal{D}(\mathbf{R}^d)$ onto $\mathbf{H}(\mathbf{C}^d)$, from $\mathcal{S}(\mathbf{R}^d)$ onto itself, from $\mathcal{S}_0^0(\mathbf{R}^d)$ onto $\mathcal{S}^0(\mathbf{R}^d)$. The inverse transform is given by, for all $x \in \mathbf{R}^d$,*

$$\mathcal{F}_D^{-1}(h)(x) = \frac{C_k^2}{2^{2\gamma+d}} \int_{\mathbf{R}^d} h(y) K(x, iy) \omega_k(y)\, dy. \tag{3.6}$$

Theorem 3.2. *Let f be in $L_k^1(\mathbf{R}^d)$ such that the function $\mathcal{F}_D(f)$ belongs to $L_k^1(\mathbf{R}^d)$. Then we have the following inversion formula for the transform $\mathcal{F}_D$:*

$$f(x) = \frac{C_k^2}{2^{2\gamma+d}} \int_{\mathbf{R}^d} \mathcal{F}_D(f)(y) K(x, iy) \omega_k(y)\, dy, \ \text{a.e.} \tag{3.7}$$

Theorem 3.3. i) *Plancherel formula for $\mathcal{F}_D$. For all f in $\mathcal{D}(\mathbf{R}^d)$ we have*

$$\int_{\mathbf{R}^d} |f(x)|^2 \omega_k(x)\, dx = \frac{C_k^2}{2^{2\gamma+d}} \int_{\mathbf{R}^d} |\mathcal{F}_D(f)(y)|^2 \omega_k(y)\, dy. \tag{3.8}$$

ii) *Plancherel theorem for $\mathcal{F}_D$. The renormalized Dunkl transform $f \to 2^{-\gamma-d/2} C_k \mathcal{F}_D(f)$ can be uniquely extended to an isometric isomorphism on $L_k^2(\mathbf{R}^d)$.*

From Theorem 3.1, the relation (3.4) and properties of the classical Fourier transform $\mathcal{F}$, we deduce the following result.

Theorem 3.4. *The transform ${}^tV_k$ is a topological isomorphism from $\mathcal{S}_0^0(\mathbf{R}^d)$ onto $\mathcal{S}_0(\mathbf{R}^d)$.*

Using Theorem 3.3 we obtain the following proposition.

Proposition 3.1. *Let f and g be two measurable functions on $\mathbf{R}^d$ satisfying: $p, q \in \mathbf{N}$ exist such that the functions $(1+\|x\|^2)^{-p} f$ and $(1+\|x\|^2)^{-q} g$ belong to $L^1_k(\mathbf{R}^d)$. We suppose that for all ψ in $\mathcal{S}(\mathbf{R}^d)$ we have*

$$\int_{\mathbf{R}^d} f(x)\mathcal{F}_D(\psi)(x)\omega_k(x)\,dx = \int_{\mathbf{R}^d} g(x)\psi(x)\omega_k(x)\,dx. \tag{3.9}$$

Then the function f belongs to $L^2_k(\mathbf{R}^d)$ if and only if the function g belongs to $L^2_k(\mathbf{R}^d)$, and we have

$$\mathcal{F}_D(f) = g. \tag{3.10}$$

4. Dunkl translation operators and Dunkl convolution product. In this section we define and study the Dunkl translation operators and the Dunkl convolution product, and we give some of their properties, (see [**25**]).

4.1 *Dunkl translation operators.* The Dunkl translation operators τ_x, $x \in \mathbf{R}^d$, are defined on $\mathcal{E}(\mathbf{R}^d)$ by, for all $y \in \mathbf{R}^d$,

$$\tau_x f(y) = (V_k)_x (V_k)_y [(V_k)^{-1}(f)(x+y)], \tag{4.1}$$

where V_k is the Dunkl intertwining operator given by the relation (2.1).

In the following propositions we give some properties of the operators τ_x, $x \in \mathbf{R}^d$.

Proposition 4.1. i) *For all $x \in \mathbf{R}^d$, the operator τ_x is continuous from $\mathcal{E}(\mathbf{R}^d)$ into itself.*

ii) *The function $x \to \tau_x$ is of class C^∞ on $\mathbf{R}^d$.*

iii) *For all $x, y \in \mathbf{R}^d$ and $z \in \mathbf{C}^d$ we have the product formula*

$$\tau_x K(y,z) = K(x,z)K(y,z). \tag{4.2}$$

iv) *For all f in $\mathcal{E}(\mathbf{R}^d)$, we have*

$$\tau_x f(0) = f(x); \quad \tau_x f(y) = \tau_y f(x). \tag{4.3}$$

$$\begin{cases} T_j(\tau_x f) = \tau_x(T_j f) & j = 1, 2, \dots, d, \\ (T_j)_x(\tau_x f) = \tau_x(T_j f) & j = 1, 2, \dots, d, \end{cases} \tag{4.4}$$

where T_j, $j = 1, 2, \dots, d$, are the Dunkl operators.

Proposition 4.2. *Let f be given in $\mathcal{E}(\mathbf{R}^d)$. We put*

$$u(x, y) = \tau_x f(y). \tag{4.5}$$

Then the function u is the unique solution of class C^∞ with respect to each variable of the system

$$\begin{cases} (T_j)_x u(x, y) = (T_j)_y u(x, y) & j = 1, 2, \dots, d, \\ u(x, 0) = f(x). \end{cases} \tag{4.6}$$

Proposition 4.3. *For f in $\mathcal{D}(\mathbf{R}^d)$ and $x \in \mathbf{R}^d$, the function $y \to \tau_x f(y)$ belongs to $\mathcal{D}(\mathbf{R}^d)$, and we have, for all $y \in \mathbf{R}^d$,*

$$\mathcal{F}_D(\tau_x f)(y) = K(ix, y)\mathcal{F}_D(f)(y). \tag{4.7}$$

Remark. From the relation (4.7), which is also true for functions in $\mathcal{S}(\mathbf{R}^d)$ and Theorem 3.1, we deduce that, for f in $\mathcal{S}(\mathbf{R}^d)$ and $x, t \in \mathbf{R}^d$, we have

$$\tau_x f(t) = \frac{C_k^2}{2^{2\gamma+d}} \int_{\mathbf{R}^d} K(ix, y)K(iy, t)\mathcal{F}_D(f)(y)\omega_k(y)\, dy. \tag{4.8}$$

Rösler has given this relation in [**18**, page 535].

In the following, we give definitions of the Dunkl translation operators on the spaces $L^p_k(\mathbf{R}^d)$, $p = 1, 2$:

i) The relation (4.8) permits also to define the Dunkl translation operators τ_x, $x \in \mathbf{R}^d$, on the space of functions f in $L^1_k(\mathbf{R}^d)$ such that $\mathcal{F}_D(f)$ belongs to $L^1_k(\mathbf{R}^d)$.

ii) Using Theorem 3.3 we define the Dunkl translation operators τ_x, $x \in \mathbf{R}^d$, on $L^2_k(\mathbf{R}^d)$ by the relation

$$\mathcal{F}_D(\tau_x f) = K(ix, .)\mathcal{F}_D(f). \tag{4.9}$$

The function $\tau_x f$ belongs to $L^2_k(\mathbf{R}^d)$, and we have

$$\|\tau_x f\|_{k,2} \leq \|f\|_{k,2}.$$

Remark. When $d = 1$ and $W = \mathbf{Z}_2$, Rösler [**17**] has shown that the Dunkl translation operators τ_x, $x \in \mathbf{R}$, possess the integral representation

$$\tau_x f(y) = \int_{\mathbf{R}} f(t)\, d\mu_{x,y}(t),$$

where $\mu_{x,y}$ is a finite signed measure on $\mathbf{R}$, of total mass less than or equal to 4, and supported in $[-|x|-|y|, -\||x|-|y|\|] \cup [\||x|-|y|\|, |x|+|y|]$.

4.2 *Dunkl convolution product.* The Dunkl convolution product of two functions f and g in $\mathcal{D}(\mathbf{R}^d)$ is the function $f *_D g$ defined by, for all $x \in \mathbf{R}^d$,

$$f *_D g(x) = \int_{\mathbf{R}^d} \tau_x f(-y) g(y) \omega_k(y)\, dy, \tag{4.10}$$

(see [**25**]).

This convolution product is commutative and associative and satisfies the properties given in the following propositions.

Proposition 4.4. i) *Let f be in $\mathcal{D}(\mathbf{R}^d)$ with support in the ball $B(0, a)$, $a > 0$, of center 0 and radius a and g in $\mathcal{D}(\mathbf{R}^d)$ with support in $B(0, b)$, $b > 0$. Then the function $f *_D g$ belongs to $\mathcal{D}(\mathbf{R}^d)$ with support in $B(0, a + b)$.*

ii) *Let f and g be in $\mathcal{S}(\mathbf{R}^d)$. Then $f *_D g$ belongs to $\mathcal{S}(\mathbf{R}^d)$ and we have, for all $y \in \mathbf{R}^d$,*

$$\mathcal{F}_D(f *_D g)(y) = \mathcal{F}_D(f)(y).\mathcal{F}_D(g)(y). \tag{4.11}$$

iii) *For all f and g in $\mathcal{S}(\mathbf{R}^d)$, respectively $\mathcal{D}(\mathbf{R}^d)$, we have*

$${}^tV_k(f *_D g) = {}^tV_k(f) * {}^tV_k(g) \tag{4.12}$$

where $*$ is the classical convolution product of $\mathbf{R}^d$ given by, for all $x \in \mathbf{R}^d$,

$$f * g(x) = \int_{\mathbf{R}^d} f(x-y)g(y)\,dy. \tag{4.13}$$

Proposition 4.5. i) *For f in $\mathcal{S}_0(\mathbf{R}^d)$ and g in $\mathcal{S}(\mathbf{R}^d)$ the function $f * g$ belongs to $\mathcal{S}_0(\mathbf{R}^d)$.*

ii) *For f in $\mathcal{S}_0^0(\mathbf{R}^d)$ and g in $\mathcal{S}(\mathbf{R}^d)$ the function $f *_D g$ belongs to $\mathcal{S}_0^0(\mathbf{R}^d)$.*

Proof. We deduce these results from the relation (4.11), Theorem 3.1 and properties of the classical Fourier transform $\mathcal{F}$.

We define in the following the Dunkl convolution production on the spaces $L_k^p(\mathbf{R}^d)$, $p = 1, 2$.

i) Let f be in $L_k^1(\mathbf{R}^d)$ and g in $L_k^2(\mathbf{R}^d)$. The Dunkl convolution product of f and g is the function $f *_D g$ of $L_k^2(\mathbf{R}^d)$ satisfying

$$\mathcal{F}_D(f *_D g) = \mathcal{F}_D(f)\mathcal{F}_D(g). \tag{4.14}$$

ii) Let f and g be in $L_k^2(\mathbf{R}^d)$. For each $x \in \mathbf{R}^d$, we define $f *_D g(x)$ by

$$f *_D g(x) = \frac{C_k^2}{2^{2\gamma+d}} \int_{\mathbf{R}^d} K(ix, y)\mathcal{F}_D(f)(y)\mathcal{F}_D(g)(y)\omega_k(y)\,dy. \tag{4.15}$$

This function belongs to $L_k^\infty(\mathbf{R}^d)$.

Proposition 4.6. *Let f and g be in $L_k^2(\mathbf{R}^d)$. Then*

i) *The function $f *_D g$ belongs to $L_k^2(\mathbf{R}^d)$ if and only if the function $\mathcal{F}_D(f)\mathcal{F}_D(g)$ belongs to $L_k^2(\mathbf{R}^d)$ and we have*

$$\mathcal{F}_D(f *_D g) = \mathcal{F}_D(f)\mathcal{F}_D(g). \tag{4.16}$$

ii) *We have*

$$\int_{\mathbf{R}^d} |f *_D g(x)|^2 \omega_k(x)\, dx = \frac{C_k^2}{2^{2\gamma+d}} \int_{\mathbf{R}^d} |\mathcal{F}_D(f)(y)|^2 |\mathcal{F}_D(g)(y)|^2 \omega_k(y)\, dy. \tag{4.17}$$

The two sides are finite or infinite.

Proof. i) We deduce these results from the fact that for all ψ in $\mathcal{S}(\mathbf{R}^d)$ we have

$$\int_{\mathbf{R}^d} f *_D g(x) \mathcal{F}_D^{-1}(\psi)(x)\omega_k(x)\, dx = \int_{\mathbf{R}^d} \mathcal{F}_D(f)(y)\mathcal{F}_D(g)(y)\psi(y)\omega_k(y)\, dy,$$

and Proposition 3.1.

ii) For $f *_D g$ in $L_k^2(\mathbf{R}^d)$ the relation (4.17) can be deduced from Theorem 3.3. For the other case the two members of the relation (4.17) are infinite.

5. Inversion formula for the Dunkl intertwining operator and its dual. In the following sections we suppose that $x = 0$ is the only zero of the function $\omega_q(x)$. In this section we show that the Dunkl intertwining operator and its dual are bijective on spaces other than $\mathcal{E}(\mathbf{R}^d)$ and $\mathcal{D}(\mathbf{R}^d)$, and we give inversion formulas for these operators.

We consider the operators $\mathcal{K}$ and $\mathcal{K}_D$ defined respectively on $\mathcal{S}_0(\mathbf{R}^d)$ and $\mathcal{S}_0^0(\mathbf{R}^d)$ by, for all $x \in \mathbf{R}^d$,

$$\mathcal{K}(f)(x) = \mathcal{F}^{-1}[M_k \omega_k \mathcal{F}(f)](x), \tag{5.1}$$

$$\mathcal{K}_D(f)(x) = \mathcal{F}_D^{-1}[M_k \omega_k \mathcal{F}_D(f)](x), \tag{5.2}$$

where

$$M_k = \frac{(2\pi)^d C_k^2}{2^{2\gamma+d}}. \tag{5.3}$$

Proposition 5.1. i) *The operator $\mathcal{K}$, respectively $\mathcal{K}_D$, is a topological automorphism of $\mathcal{S}_0(\mathbf{R}^d)$, respectively $\mathcal{S}_0^0(\mathbf{R}^d)$.*

ii) *For all f in $\mathcal{S}_0^0(\mathbf{R}^d)$, we have*

$$\mathcal{K}_D(f) = {}^tV_k^{-1} \circ \mathcal{K} \circ {}^tV_k(f). \tag{5.4}$$

Proof. i) The mapping $f \rightarrow M_k \omega_k f$ is a topological automorphism of $\mathcal{S}^0(\mathbf{R}^d)$. Its inverse is given by $f \rightarrow (1/M_k)\omega_k^{-1} f$. We deduce the result from Theorem 3.1 and the fact that the transform $\mathcal{F}$ is a topological isomorphism from $\mathcal{S}_0(\mathbf{R}^d)$ onto $\mathcal{S}^0(\mathbf{R}^d)$.

ii) We obtain the result from the relations (5.2), (3.4) and Theorem 3.4.

Proposition 5.2. i) *For all f in $\mathcal{S}_0(\mathbf{R}^d)$ and for all g in $\mathcal{S}(\mathbf{R}^d)$, we have*

$$\mathcal{K}(f * g) = \mathcal{K}(f) * g. \tag{5.5}$$

ii) *For all f in $\mathcal{S}_0^0(\mathbf{R}^d)$ and for all g in $\mathcal{S}(\mathbf{R}^d)$, we have*

$$\mathcal{K}_D(f *_D g) = \mathcal{K}_D(f) *_D g. \tag{5.6}$$

Proof. We obtain these relations from (5.1), (5.2), Proposition 4.5, the relation (4.11), properties of the classical Fourier transform $\mathcal{F}$ and the classical convolution product on $\mathbf{R}^d$.

Theorem 5.1. *We have the following inversion formulas for the operators V_k and ${}^tV_k$. For f in $\mathcal{S}_0^0(\mathbf{R}^d)$,*

$$f = V_k \mathcal{K}^t V_k(f); \quad f = \mathcal{K}_D V_k {}^t V_k(f). \tag{5.7}$$

For f in $\mathcal{S}_0(\mathbf{R}^d)$,

$$f = {}^t V_k \mathcal{K}_D V_k(f); \quad f = \mathcal{K}^t V_k V_k(f). \tag{5.8}$$

Proof. Let f be in $\mathcal{S}_0^0(\mathbf{R}^d)$. Using the relations (3.6), (2.2), (4.11) and the inversion formula for the classical Fourier transform $\mathcal{F}$, we obtain for all $x \in \mathbf{R}^d$,

$$\begin{aligned} f(x) &= \frac{C_k^2}{2^{2\gamma+d}} \int_{\mathbf{R}^d} \mathcal{F}_D(f)(y) K(x, iy) \omega_k(y)\, dy, \\ &= \frac{C_k^2}{2^{2\gamma+d}} V_k \left[\int_{\mathbf{R}^d} \mathcal{F}_D(f)(y) e^{i\langle \cdot, y\rangle} \omega_k(y)\, dy \right](x), \\ &= V_k \left\{ \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} [M_k \omega_k(y) \mathcal{F} \circ {}^t V_k(f)(y)] e^{i\langle \cdot, y\rangle}\, dy \right\}(x), \end{aligned}$$

where M_k is the constant given by (5.3). Thus, for all $x \in \mathbf{R}^d$,

$$\begin{aligned} f(x) &= V_k \{\mathcal{F}^{-1}[M_k \omega_k \mathcal{F} \circ {}^t V_k(f)]\}(x), \\ f(x) &= V_k \mathcal{K}^t V_k(f)(x). \end{aligned} \tag{5.9}$$

On the other hand, from this relation and (4.11), we obtain for f in $\mathcal{S}_0(\mathbf{R}^d)$, the relation, for all $x \in \mathbf{R}^d$,

$$f(x) = {}^t V_k \mathcal{K}_D V_k(f)(x). \tag{5.10}$$

We obtain the other relations by writing (5.9) and (5.10), respectively, for the functions $V_k(f)$ and ${}^tV_k(f)$.

Corollary 5.1. *The operator V_k is a topological isomorphism from $\mathcal{S}_0(\mathbf{R}^d)$ onto $\mathcal{S}_0^0(\mathbf{R}^d)$.*

Proof. We deduce the result from Proposition 5.1 i), Theorem 3.4 and the relation (5.9).

Corollary 5.2. *For all f in $\mathcal{S}_0(\mathbf{R}^d)$ and g in $\mathcal{S}(\mathbf{R}^d)$, we have*

$$V_k(f * g) = V_k(f) *_D ({}^tV_k)^{-1}(g). \tag{5.11}$$

Proof. Using the relations (4.12), (5.9) and Proposition 5.2 i), we obtain

$$\begin{aligned} V_k^{-1}[V_k(f) *_D {}^tV_k{}^{-1}(g)] &= \mathcal{K}{}^tV_k[V_k(f) *_D ({}^tV_k)^{-1}(g)], \\ &= \mathcal{K}[{}^tV_kV_k(f) * g], \\ &= [\mathcal{K}{}^tV_kV_k(f)] * g. \end{aligned}$$

But from Theorem 5.1 we have

$$\mathcal{K}{}^tV_kV_k(f) = f.$$

Thus

$$V_k^{-1}[V_k(f) *_D ({}^tV_k)^{-1}(g)] = f * g.$$

We obtain the result from Corollary 5.1.

6. Dunkl wavelets.

6.1 *Classical wavelets on* $\mathbf{R}^d$. We say that a measurable function on $\mathbf{R}^d$ is a classical wavelet on $\mathbf{R}^d$ if it satisfies, for almost all $x \in \mathbf{R}^d$, the condition

$$0 < C_g^0 = \int_0^\infty |\mathcal{F}(g)(\lambda x)|^2 \frac{d\lambda}{\lambda} < +\infty, \tag{6.1}$$

where $\mathcal{F}$ is the classical Fourier transform given by the relation (3.5).

Let $a \in]0, +\infty[$ and g a classical wavelet on $\mathbf{R}^d$ in $L^2(\mathbf{R}^d)$ (the space of square integrable functions on $\mathbf{R}^d$ with respect to the Lebesgue measure). We consider the family $g_{a,x}$, $x \in \mathbf{R}^d$, of classical wavelets on $\mathbf{R}^d$ in $L^2(\mathbf{R}^d)$ defined by

$$g_{a,x}(y) = H_a(g)(x-y), \tag{6.2}$$

where H_a is the dilatation operator given by

$$H_a(g)(x) = \frac{1}{a^d} g\left(\frac{x}{a}\right), \quad x \in \mathbf{R}^d. \tag{6.3}$$

In the following we denote the function $H_a(g)$ by g_a^0.

We define for regular functions f on $\mathbf{R}^d$ the classical continuous wavelet transform S_g on $\mathbf{R}^d$ by

$$S_g(f)(a,x) = \int_{\mathbf{R}^d} f(y)\overline{g_{a,x}(y)}\, dy, \quad \text{for all } x \in \mathbf{R}^d. \tag{6.4}$$

This transform can also be written in the form

$$S_g(f)(a,x) = f * \overline{g_a^0}(x), \tag{6.5}$$

where $*$ is the classical convolution product given by (4.13).

The transform S_g has been studied in [**12**]. Several properties are given, in particular if we consider a classical wavelet g on $\mathbf{R}^d$ in $L^2(\mathbf{R}^d)$, we have the following results.

i) *Plancherel formula.* For all f in $L^2(\mathbf{R}^d)$, we have

$$\int_{\mathbf{R}^d} |f(x)|^2\, dx = \frac{1}{C_g^0} \int_0^\infty \int_{\mathbf{R}^d} |S_g(f)(a,x)|^2\, dx \frac{da}{a}. \tag{6.6}$$

ii) *Inversion formula.* For all f in $L^1(\mathbf{R}^d)$ (the space of integrable functions on $\mathbf{R}^d$ with respect to the Lebesgue measure) such that $\mathcal{F}(f)$ belongs to $L^1(\mathbf{R}^d)$, we have

$$f(x) = \frac{1}{C_g^0} \int_0^\infty \left(\int_{\mathbf{R}^d} S_g(f)(a,y) g_{a,x}(y)\, dy \right) \frac{da}{a}, \quad \text{a.e.}, \tag{6.7}$$

where for each $x \in \mathbf{R}^d$, both the inner integral and the outer integral are absolutely convergent but possibly not the double integral.

6.2 *Dunkl wavelets on* $\mathbf{R}^d$. Using the harmonic analysis associated with the Dunkl operators T_j, $j = 1, 2, \dots, d$, in particular, the Dunkl transform $\mathcal{F}_D$ and the Dunkl translation operators τ_x, $x \in \mathbf{R}^d$, we define and study in this subsection Dunkl wavelets on $\mathbf{R}^d$ and the Dunkl continuous wavelet transform on $\mathbf{R}^d$, and we prove Plancherel and inversion formulas for this transform.

Definition 6.1. A Dunkl wavelet on $\mathbf{R}^d$ is a measurable function g on $\mathbf{R}^d$ satisfying for almost all $x \in \mathbf{R}^d$ the condition

$$0 < C_g = \int_0^\infty |\mathcal{F}_D(g)(\lambda x)|^2 \frac{d\lambda}{\lambda} < +\infty. \tag{6.8}$$

Example. The function α_t, $t > 0$, defined by, for all $x \in \mathbf{R}^d$,

$$\alpha_t(x) = \frac{C_k}{(4t)^{\gamma+d/2}} e^{-\|x\|^2/4t}, \tag{6.9}$$

satisfies, for all $y \in \mathbf{R}^d$,

$$\mathcal{F}_D(\alpha_t)(y) = e^{-t\|y\|^2}, \tag{6.10}$$

(see [**5**, page 13] and [**20**, page 589]).

The function $g(x) = -(d/dt)\alpha_t(x)$ is a Dunkl wavelet on $\mathbf{R}^d$ in $\mathcal{S}(\mathbf{R}^d)$, and we have $C_g = (1/8t^2)$.

Proposition 6.1. *A function g is a Dunkl wavelet on $\mathbf{R}^d$ in $\mathcal{S}(\mathbf{R}^d)$, respectively $\mathcal{S}_0^0(\mathbf{R}^d)$, if and only if the function ${}^tV_k(g)$ is a classical wavelet on $\mathbf{R}^d$ in $\mathcal{S}(\mathbf{R}^d)$, respectively $\mathcal{S}_0(\mathbf{R}^d)$, and we have*

$$C_{{}^tV_k(g)} = C_g. \tag{6.11}$$

Proof. We deduce these results from Theorems 2.2, 3.4 and the relation (3.4).

Let $a \in]0, +\infty[$ and g a regular function on $\mathbf{R}^d$. We consider the function g_a defined by, for all $x \in \mathbf{R}^d$,

$$g_a(x) = \frac{1}{a^{2\gamma+d}} g\left(\frac{x}{a}\right). \tag{6.12}$$

It satisfies the following properties

i) For g in $L_k^2(\mathbf{R}^d)$ the function g_a belongs to $L_k^2(\mathbf{R}^d)$, and we have

$$\mathcal{F}_D(g_a)(y) = \mathcal{F}_D(g)(ay), \quad y \in \mathbf{R}^d. \tag{6.13}$$

ii) For g in $\mathcal{S}(\mathbf{R}^d)$, respectively $\mathcal{S}_0^0(\mathbf{R}^d)$, the function g_a belongs to $\mathcal{S}(\mathbf{R}^d)$, respectively $\mathcal{S}_0^0(\mathbf{R}^d)$, and we have

$$g_a = ({}^tV_k)^{-1} \circ H_a \circ {}^tV_k(g).$$

Let g be a Dunkl wavelet on $\mathbf{R}^d$ in $L_k^2(\mathbf{R}^d)$. We consider the family $g_{a,x}$, $x \in \mathbf{R}^d$, of Dunkl wavelets on $\mathbf{R}^d$ in $L_k^2(\mathbf{R}^d)$ defined by

$$g_{a,x}(y) = \tau_x g_a(-y), \quad y \in \mathbf{R}^d, \tag{6.14}$$

where τ_x, $x \in \mathbf{R}^d$, are the Dunkl translation operators given by (4.9).

Example. We consider the function α_t, $t > 0$, given by (6.9). From [**5**, page 13], (see also [**20**, page 589]), we have, for all $x, y \in \mathbf{R}^d$,

$$\tau_x(\alpha_t)(y) = \frac{C_k}{(4t)^{\gamma+d/2}} e^{\frac{-\|x\|^2+\|y\|^2}{4t}} K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right). \tag{6.15}$$

Using (6.10) we deduce that the family $g_{a,x}$, $x \in \mathbf{R}^d$, given by, for all $y \in \mathbf{R}^d$,

$$g_{a,x}(y) = -\tau_x\left(\frac{d}{dt}\alpha_t\right)(-y), \tag{6.16}$$

is a family of Dunkl wavelets on $\mathbf{R}^d$ in $\mathcal{S}(\mathbf{R}^d)$.

Definition 6.2 Let g be a Dunkl wavelet on $\mathbf{R}^d$ in $L^2_k(\mathbf{R}^d)$. The Dunkl continuous wavelet transform S^D_g on $\mathbf{R}^d$ is defined for regular functions f on $\mathbf{R}^d$ by

$$S^D_g(f)(a,x) = \int_{\mathbf{R}^d} f(y)\overline{g_{a,x}(y)}\omega_k(y)\,dy, \qquad a>0, \quad x\in\mathbf{R}^d. \tag{6.17}$$

This transform can also be written in the form

$$S^D_g(f)(a,x) = f *_D \overline{g_a}(x), \tag{6.18}$$

where $*_D$ is the Dunkl convolution product given by (4.10).

Theorem 6.1 (Plancherel formula for S^D_g). *Let g be a Dunkl wavelet on $\mathbf{R}^d$ in $L^2_k(\mathbf{R}^d)$. For all f in $L^2_k(\mathbf{R}^d)$, we have*

$$\int_{\mathbf{R}^d} |f(x)|^2\omega_k(x)\,dx = \frac{1}{C_g}\int_0^\infty \int_{\mathbf{R}^d} |S^D_g(f)(a,x)|^2\omega_k(x)\,dx\frac{da}{a}. \tag{6.19}$$

Proof. Using Fubini-Tonnelli's theorem, Proposition 4.5 ii) and the relations (6.18) and (6.13), we obtain

$$\begin{aligned}
&\frac{1}{C_g}\int_0^\infty \int_{\mathbf{R}^d} |S^D_g(f)(a,x)|^2\omega_k(x)\,dx\frac{da}{a}\\
&\qquad = \frac{1}{C_g}\int_0^\infty \left(\int_{\mathbf{R}^d} |f *_D \overline{g_a}(x)|^2\omega_k(x)\,dx\right)\frac{da}{a},\\
&\qquad = \frac{1}{C_g}\int_0^\infty \left(\int_{\mathbf{R}^d} |\mathcal{F}_D(f)(y)|^2|\mathcal{F}_D(\overline{g_a})(y)|^2\omega_k(y)\,dy\right)\frac{da}{a},\\
&\qquad = \int_{\mathbf{R}^d} |\mathcal{F}_D(f)(x)|^2\left(\frac{1}{C_g}\int_0^\infty |\mathcal{F}_D(g)(ay)|^2\frac{da}{a}\right)\omega_k(y)\,dy.
\end{aligned}$$

But, from Definition 6.1, we have for almost all $y\in\mathbf{R}^d$,

$$\frac{1}{C_g}\int_0^\infty |\mathcal{F}_D(g)(ay)|^2\frac{da}{a} = 1,$$

then

$$\frac{1}{C_g}\int_0^\infty \int_{\mathbf{R}^d} |S_g^D(f)(a,x)|^2\omega_k(x)\,dx\frac{da}{a} = \int_{\mathbf{R}^d} |\mathcal{F}_D(f)(y)|^2\omega_k(y)\,dy.$$

We deduce the relation (6.19) from Theorem 3.3.

The following theorem gives an inversion formula for the transform S_g^D.

Theorem 6.2. *Let g be a Dunkl wavelet on $\mathbf{R}^d$ in $L_k^2(\mathbf{R}^d)$. For f in $L_k^1(\mathbf{R}^d)$, respectively $L_k^2(\mathbf{R}^d)$, such that $\mathcal{F}_D(f)$ belongs to $L_k^1(\mathbf{R}^d)$, respectively $L_k^1(\mathbf{R}^d)\cap L_k^\infty(\mathbf{R}^d)$, we have*

$$(6.20)\quad f(x) = \frac{1}{C_g}\int_0^\infty \left(\int_{\mathbf{R}^d} S_g^D(f)(a,y)g_{a,x}(y)\omega_k(y)\,dy\right)\frac{da}{a}, \quad a.e.,$$

where for each $x\in\mathbf{R}^d$ both the inner integral and the outer integral are absolutely convergent, but possibly not the double integral.

Proof. We obtain (6.20) by using an analogous proof as for Theorem 6.III.3 of [**22**, page 199].

7. Inversion of the Dunkl intertwining operator and of its dual by using Dunkl wavelets. In this section we establish relations between the Dunkl continuous wavelet transform S_g^D on $\mathbf{R}^d$, and the classical continuous wavelet transform S_g on $\mathbf{R}^d$. Next, using the inversion formulas for the transforms S_g^D and S_g, we deduce relations which give the inverse operators of the Dunkl intertwining operator V_k and of its dual ${}^tV_k$.

Theorem 7.1. i) *Let g be a Dunkl wavelet on $\mathbf{R}^d$ in $\mathcal{D}(\mathbf{R}^d)$, respectively $\mathcal{S}(\mathbf{R}^d)$. Then, for all f in the same space as g, we have, for all $x\in\mathbf{R}^d$,*

$$(7.1)\qquad S_g^D(f)(a,x) = ({}^tV_k)^{-1}[S_{{}^tV_k(g)}({}^tV_k(f))(a,.)](x).$$

ii) *Let g be a Dunkl wavelet on $\mathbf{R}^d$ in $\mathcal{S}_0^0(\mathbf{R}^d)$. Then, for all f in $\mathcal{S}_0(\mathbf{R}^d)$, we have, for all $x \in \mathbf{R}^d$,*

$$S_{{}^tV_k(g)}(f)(a,x) = V_k^{-1}[S_g^D(V_k(f))(a,.)](x). \tag{7.2}$$

Proof. We deduce these results from the relations (6.5), (6.18) and properties of the Dunkl convolution product studied in the subsection 4.2.

Theorem 7.2. *Let g be a Dunkl wavelet on $\mathbf{R}^d$ in $\mathcal{S}_0^0(\mathbf{R}^d)$. Then*

i) *For all f in $\mathcal{S}_0^0(\mathbf{R}^d)$, we have, for all $x \in \mathbf{R}^d$,*

$$S_g^D(f)(a,x) = a^{-2\gamma} V_k[S_{\mathcal{K}({}^tV_k(g))}({}^tV_k(f))(a,.)](x). \tag{7.3}$$

ii) *For all f in $\mathcal{S}_0(\mathbf{R}^d)$, we have, for all $x \in \mathbf{R}^d$,*

$$S_{{}^tV_k(g)}(f)(a,x) = a^{-2\gamma\,t} V_k[S^D_{\mathcal{K}_D(g)}(V_k(f))(a,.)](x). \tag{7.4}$$

Proof. We obtain these relations from Theorems 7.1, 5.1 and the fact that

$$\mathcal{K}({}^tV_k(g)^0_a) = a^{-2\gamma}(\mathcal{K}({}^tV_k(g)))^0_a,$$

and

$$\mathcal{K}_D(g_a) = a^{-2\gamma}(\mathcal{K}_D(g))_a.$$

Theorem 7.3. *Let g be a Dunkl wavelet on $\mathbf{R}^d$ in $\mathcal{S}_0^0(\mathbf{R}^d)$. Then*

i) *For all f in $\mathcal{S}_0^0(\mathbf{R}^d)$, we have, for all $x \in \mathbf{R}^d$,*

$$\begin{aligned} &({}^tV_k)^{-1}(f)(x) \\ &= \frac{1}{C_g}\int_0^\infty \left(\int_{\mathbf{R}^d} V_k[S_{\mathcal{K}({}^tV_k(g))}(f)(a,.)](y) g_{a,x}(y)\omega_k(y)\,dy\right)\frac{da}{a^{2\gamma+1}}. \end{aligned} \tag{7.5}$$

ii) *For all f in $\mathcal{S}_0(\mathbf{R}^d)$, we have, for all $x \in \mathbf{R}^d$,*

$$(7.6) \quad V_k^{-1}(f)(x) = \frac{1}{C^0_{{}^tV_k(g)}} \int_0^\infty \left(\int_{\mathbf{R}^d} {}^tV_k[S^D_{\mathcal{K}_{D(g)}}(f)(a,.)](y) {}^tV_k(g)_{a,x}(y)\, dy \right) \frac{da}{a^{2\gamma+1}}.$$

Proof. We deduce (7.5) and (7.6) from Theorems 7.2, 6.2 and the relation (6.7).

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