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LEONARD PAIRS FROM 24 POINTS OF VIEW

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ABSTRACT. Let $\mathbf{K}$ denote a field and let V denote a vector space over $\mathbf{K}$ with finite positive dimension. We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy both conditions below:

(i) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal.

(ii) There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal.

We call such a pair a *Leonard pair* on V. Referring to the above Leonard pair, we investigate 24 bases for V on which the action of A and A^* take an attractive form. Our bases are described as follows. Let Ω denote the set consisting of four symbols $0, d, 0^*, d^*$. We identify the symmetric group S_4 with the set of all linear orderings of Ω. For each element g of S_4, we define an (ordered) basis for V, which we denote by $[g]$. The 24 resulting bases are related as follows. For all elements $wxyz$ in S_4, the transition matrix from the basis $[wxyz]$ to the basis $[xwyz]$, (respectively $[wyxz]$), is diagonal, (respectively lower triangular). The basis $[wxzy]$ is the basis $[wxyz]$ in inverted order. The transformations A and A^* act on the 24 bases as follows: For all $g \in S_4$, let A^g, (respectively A^{*g}), denote the matrix representing A, (respectively A^*), with respect to $[g]$. To describe A^g and A^{*g}, we refer to $0^*, d^*$ as the *starred* elements of Ω. Writing $g = wxyz$, if neither of y, z are starred then A^g is diagonal and A^{*g} is irreducible tridiagonal. If y is starred but z is not, then A^g is lower bidiagonal and A^{*g} is upper bidiagonal. If z is starred but y not, then A^g is upper bidiagonal and A^{*g} is lower bidiagonal. If both of y, z are starred, then A^g is irreducible tridiagonal and A^{*g} is diagonal.

We define a symmetric binary relation on S_4 called adjacency. An element $wxyz$ of S_4 is by definition adjacent to each of $xwyz$, $wyxz$, $wxzy$ and no other elements of S_4. For all ordered pairs of adjacent elements g, h in S_4, we find the entries of the transition matrix from the basis $[g]$ to the basis

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[h]. We express these entries in terms of the eigenvalues of A, the eigenvalues of A^*, and two sequences of parameters called the first split sequence and the second split sequence. For all $g \in S_4$, we compute the entries of A^g and A^{*g} in terms of the eigenvalues of A, the eigenvalues of A^*, the first split sequence and the second split sequence.

1. Leonard pairs. Throughout this paper, $\mathbf{K}$ will denote an arbitrary field, and $\tilde{\mathbf{K}}$ will denote the algebraic closure of $\mathbf{K}$.

We begin by recalling the notion of a Leonard pair.

Definition 1.1 [**44**]. Let V denote a vector space over $\mathbf{K}$ with finite positive dimension. By a Leonard pair on V, we mean an ordered pair A, A^*, where $A : V \to V$ and $A^* : V \to V$ are linear transformations that satisfy both (i) and (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal.

(ii) There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal.

(A tridiagonal matrix is said to be irreducible whenever all entries immediately above and below the main diagonal are nonzero.)

Note 1.2. According to a common notational convention, for a linear transformation A the conjugate-transpose of A is denoted A^*. We emphasize we are not using this convention. In a Leonard pair A, A^*, the linear transformations A and A^* are arbitrary subject to (i) and (ii) above.

Here is an example of a Leonard pair. Set $V = \mathbf{K}^4$ (column vectors), set

$$A = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}, \quad A^* = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix},$$

and view A and A^* as linear transformations from V to V. We assume the characteristic of $\mathbf{K}$ is not 2 or 3, to ensure A is irreducible. Then

A, A^* is a Leonard pair on V. Indeed, condition (ii) in Definition 1.1 is satisfied by the basis for V consisting of the columns of the 4 by 4 identity matrix. To verify condition (i), we display an invertible matrix P such that $P^{-1}AP$ is diagonal and $P^{-1}A^*P$ is irreducible tridiagonal. Set

$$P = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -3 & 3 & -1 \end{pmatrix}.$$

By matrix multiplication $P^2 = 8I$, where I denotes the identity, so P^{-1} exists. Also by matrix multiplication

$$AP = PA^*. \tag{1}$$

Apparently $P^{-1}AP$ equals A^* and is therefore diagonal. By (1), and since P^{-1} is a scalar multiple of P, we find $P^{-1}A^*P$ equals A and is therefore irreducible tridiagonal. Now condition (i) of Definition 1.1 is satisfied by the basis for V consisting of the columns of P.

The above example is a member of the following infinite family of Leonard pairs. For any nonnegative integer d, the pair

$$A = \begin{pmatrix} 0 & d & & & & \mathbf{0} \\ 1 & 0 & d-1 & & & \\ & 2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & 1 \\ \mathbf{0} & & & & d & 0 \end{pmatrix}, \tag{2}$$
$$A^* = \mathrm{diag}\,(d, d-2, d-4, \ldots, -d)$$

is a Leonard pair on the vector space $\mathbf{K}^{d+1}$ provided the characteristic of $\mathbf{K}$ is zero or an odd prime greater than d. This can be proved by modifying the proof for $d = 3$ given above. One shows $P^2 = 2^d I$ and $AP = PA^*$, where P denotes the matrix with ij entry

$$P_{ij} = \binom{d}{j}\, {}_2F_1\left(\left. \begin{matrix} -i, -j \\ -d \end{matrix} \right| 2 \right), \quad 0 \le i, j \le d. \tag{3}$$

We follow the standard notation for hypergeometric series [**10**]. The details of the above calculations are given in Section 16 below.

To motivate our results we mention some background on Leonard pairs. There is a connection between Leonard pairs and certain orthogonal polynomials contained in the Askey scheme [**26**]. Observe the ${}_2F_1$ that appears in (3) is a Krawtchouk polynomial [**26**]. There exist families of Leonard pairs similar to the one above in which the Krawtchouk polynomial is replaced by one of the following:

type	polynomial
${}_4F_3$	Racah
${}_3F_2$	Hahn, dual Hahn
${}_2F_1$	Krawtchouk
${}_4\phi_3$	q-Racah
${}_3\phi_2$	q-Hahn, dual q-Hahn
${}_2\phi_1$	q-Krawtchouk (classical, affine, quantum, dual)

The above polynomials are defined in Koekoek and Swarttouw [**26**], and the connection to Leonard pairs is given in [**44**, Chapter 15] and [**4**, page 260]. This connection is also discussed in Section 16 below.

Leonard pairs play a role in representation theory. For instance, Leonard pairs arise naturally in the representation theory of the Lie algebra sl_2 [**25**], the quantum algebra $U_q(sl_2)$ [**27**], [**28**], [**29**], [**30**], [**31**], [**35**, Chapter 4], [**42**], [**43**], the Askey-Wilson algebra [**12**], [**13**], [**14**], [**15**], [**16**], [**45**], [**46**], [**47**] and the tridiagonal algebra [**25**], [**43**], [**44**].

Leonard pairs play a role in combinatorics. For instance, there is a combinatorial object called a P- and Q-polynomial association scheme [**4**], [**5**], [**33**], [**37**], [**41**]. Leonard pairs have been used to describe certain irreducible modules for the subconstituent algebra of these schemes [**38**], [**39**], [**40**], (see [**6**], [**8**], [**11**], [**24**], [**25**], [**36**] for more information on Leonard pairs and association schemes).

Leonard pairs are closely related to the work of Grunbaum and Haine on the “bispectral problem” [**19, 20**]. See [**17, 18, 21**–**23**] for related work.

We now give an overview of the present paper. Let V denote a vector space over $\mathbf{K}$ with finite positive dimension, and let A, A^* denote a Leonard pair on V. Using this pair we define 24 bases for V which we find attractive. In our study of these 24 bases, we will be concerned

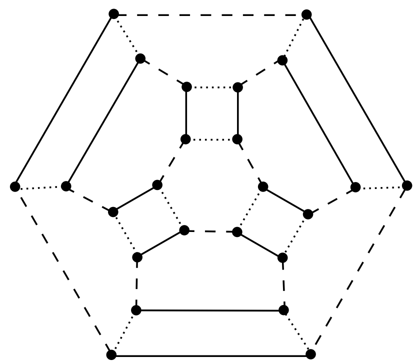

FIGURE 1. How the 24 bases are related. Each vertex represents one of the 24 bases. Solid arc: transition matrix is diagonal. Dashed arc: transition matrix is lower triangular. Dotted arc: inversion.

with (i) how these bases are related to each other and (ii) for each basis, the matrices that represent A and A^*. We will elaborate on these two points below, but first we sharpen our notation. By a *basis* for V, we mean a sequence of vectors in V that are linearly independent and span V. We emphasize the ordering is important. Let $v_0, v_1, \ldots, v_d$ denote a basis for V. Then the sequence $v_d, v_{d-1}, \ldots, v_0$ is a basis for V which we call the *inversion* of $v_0, v_1, \ldots, v_d$.

When we define our 24 bases, we will find they are related to each other according to the diagram in Figure 1. In that diagram each vertex represents one of the 24 bases. For each pair of bases in the diagram that are connected by an arc, consider the transition matrix from one of these bases to the other. The shading on the arc indicates the nature of this transition matrix. If the arc is solid, the transition matrix is diagonal. If the arc is dashed, the transition matrix is lower triangular. If the arc is dotted, the two bases are the inversion of one another.

The reader might observe the above diagram is a Cayley graph for the symmetric group S_4. Apparently, there is a connection between our 24 bases and S_4. We now make this connection explicit.

Let Ω denote the set consisting of four symbols $0, d, 0^*, d^*$. We identify the symmetric group S_4 with the set of all linear orderings of Ω. For $i = 1, 2, 3$ we define a symmetric binary relation on S_4

which we call i-*adjacency*. Each element $wxyz$ of S_4 is by definition 1-adjacent, (respectively 2-adjacent), (respectively 3-adjacent), to $xwyz$, (respectively $wyxz$), (respectively $wxzy$) and no other elements of S_4. Two elements in S_4 will be called *adjacent* whenever they are i-adjacent for some i, $(1 \leq i \leq 3)$. If we draw a diagram in which we represent the elements of S_4 by vertices and, for $i = 1, 2, 3$, we represent i-adjacency by solid, dashed, and dotted arcs, respectively, we get the diagram in Figure 1.

For each element g of S_4 we will define a certain basis for V which we denote by $[g]$. We will find that for all pairs g, h of adjacent elements in S_4,

(i) if g, h are 1-adjacent the transition matrix from $[g]$ to $[h]$ is diagonal,

(ii) if g, h are 2-adjacent the transition matrix from $[g]$ to $[h]$ is lower triangular,

(iii) if g, h are 3-adjacent then $[g]$ is the inversion of $[h]$.

When we define our 24 bases, we will find that A and A^* act on them as follows. For all $g \in S_4$, let A^g, (respectively A^{*g}), denote the matrix representing A, (respectively A^*), with respect to $[g]$. To describe A^g and A^{*g}, we refer to $0^*, d^*$ as the *starred* elements of Ω. Writing $g = wxyz$, we will find

(i) if neither of y, z are starred then A^g is diagonal and A^{*g} is irreducible tridiagonal.

(ii) if y is starred, but z is not, then A^g is lower bidiagonal and A^{*g} is upper bidiagonal.

(iii) if z is starred but y is not, then A^g is upper bidiagonal and A^{*g} is lower bidiagonal.

(iv) if both of y, z are starred, then A^g is irreducible tridiagonal and A^{*g} is diagonal.

(A square matrix is said to be *lower bidiagonal* whenever all nonzero entries lie either on or immediately below the main diagonal. A matrix is said to be *upper bidiagonal* whenever the transpose is lower bidiagonal.)

For all ordered pairs g, h of adjacent elements in S_4, we find the entries of the transition matrix from the basis $[g]$ to the basis $[h]$. We express these entries in terms of the eigenvalues of A, the eigenvalues of A^* and two sequences of scalars called the first split sequence and the second split sequence. For all $g \in S_4$, we compute the entries of A^g and A^{*g} in terms of the eigenvalues of A, the eigenvalues of A^*, the first split sequence and the second split sequence.

2. Leonard systems. When working with a Leonard pair, it is often convenient to consider a closely related and somewhat more abstract object, which we call a *Leonard system.* In order to define this, we first make an observation about Leonard pairs.

Lemma 2.1 [**44**]. *Let V denote a vector space over* $\mathbf{K}$ *with finite positive dimension, and let A, A^* denote a Leonard pair on V. Then the eigenvalues of A are distinct and contained in* $\mathbf{K}$. *Moreover, the eigenvalues of A^* are distinct and contained in* $\mathbf{K}$.

To prepare for our definition of a Leonard system, we recall a few concepts from elementary linear algebra. Let d denote a nonnegative integer, and let $\mathrm{Mat}_{d+1}(\mathbf{K})$ denote the $\mathbf{K}$-algebra consisting of all $d+1$ by $d+1$ matrices with entries in $\mathbf{K}$. We index the rows and columns by $0, 1, \ldots, d$. Let $\mathcal{A}$ denote a $\mathbf{K}$-algebra isomorphic to $\mathrm{Mat}_{d+1}(\mathbf{K})$. Let A denote an element of $\mathcal{A}$. By an *eigenvalue* of A, we mean a root of the minimal polynomial of A. The eigenvalues of A are contained in the algebraic closure of $\mathbf{K}$. The element A will be called *multiplicity-free* whenever it has $d+1$ distinct eigenvalues, all of which are in $\mathbf{K}$. Let A denote a multiplicity-free element of $\mathcal{A}$. Let $\theta_0, \theta_1, \ldots, \theta_d$ denote an ordering of the eigenvalues of A, and for $0 \le i \le d$, put

$$E_i = \prod_{\substack{0 \le j \le d \\ j \ne i}} \frac{A - \theta_j I}{\theta_i - \theta_j}, \tag{4}$$

where I denotes the identity of $\mathcal{A}$. By elementary linear algebra,

$$AE_i = E_i A = \theta_i E_i, \quad 0 \le i \le d, \tag{5}$$

$$E_i E_j = \delta_{ij} E_i, \quad 0 \le i, j \le d, \tag{6}$$

$$\sum_{i=0}^{d} E_i = I. \tag{7}$$

From this, one finds $E_0, E_1, \ldots, E_d$ is a basis for the subalgebra of $\mathcal{A}$ generated by A. We refer to E_i as the *primitive idempotent* of A associated with θ_i. It is helpful to think of these primitive idempotents as follows. Let V denote the irreducible left $\mathcal{A}$-module. Then

$$V = E_0V + E_1V + \cdots + E_dV, \quad \text{direct sum.} \tag{8}$$

For $0 \le i \le d$, E_iV is the (one-dimensional) eigenspace of A in V associated with the eigenvalue θ_i, and E_i acts on V as the projection onto this eigenspace.

Definition 2.2 [**44**]. Let d denote a nonnegative integer, let $\mathbf{K}$ denote a field, and let $\mathcal{A}$ denote a $\mathbf{K}$-algebra isomorphic to $\mathrm{Mat}_{d+1}(\mathbf{K})$. By a Leonard system in $\mathcal{A}$, we mean a sequence

$$\Phi = (A; E_0, E_1, \ldots, E_d; A^*; E^*_0, E^*_1, \ldots, E^*_d) \tag{9}$$

that satisfies (i)–(v) below.

(i) A, A^* are both multiplicity-free elements in $\mathcal{A}$.

(ii) $E_0, E_1, \ldots, E_d$ is an ordering of the primitive idempotents of A.

(iii) $E^*_0, E^*_1, \ldots, E^*_d$ is an ordering of the primitive idempotents of A^*.

(iv) $E_iA^*E_j = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1; \end{cases} \quad 0 \le i, j \le d.$

(v) $E^*_iAE^*_j = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \neq 0 & \text{if } |i-j| = 1; \end{cases} \quad 0 \le i, j \le d.$

We refer to d as the diameter of Φ and say Φ is over $\mathbf{K}$. We sometimes write $\mathcal{A} = \mathcal{A}(\Phi)$, $\mathbf{K} = \mathbf{K}(\Phi)$. For notational convenience, we set $E_{-1} = 0$, $E_{d+1} = 0$, $E^*_{-1} = 0$, $E^*_{d+1} = 0$.

In the two lemmas below, we explain the relationship between the notions of Leonard pair and Leonard system. We will use the following notation. Let V denote a vector space over $\mathbf{K}$ with finite positive dimension. We let $\mathrm{End}\,(V)$ denote the $\mathbf{K}$-algebra consisting of all linear transformations from V to V. We recall $\mathrm{End}\,(V)$ is $\mathbf{K}$-algebra isomorphic to $\mathrm{Mat}_{d+1}(\mathbf{K})$, where $d+1 = \dim V$.

Lemma 2.3. *Let V denote a vector space over $\mathbf{K}$ with finite positive dimension. Let A, A^* denote a Leonard pair on V, and observe each* *of* A, A^* *is multiplicity-free by Lemma* 2.1. *Let* $v_0, v_1, \ldots, v_d$ *denote a basis for* V *that satisfies Definition* 1.1 (i). *For* $0 \leq i \leq d$*, observe* v_i *is an eigenvector for* A; *let* θ_i *denote the corresponding eigenvalue, and let* E_i *denote the primitive idempotent of* A *associated with* θ_i. *Similarly, let* $v_0^*, v_1^*, \ldots, v_d^*$ *denote a basis for* V *that satisfies Definition* 1.1 (ii). *For* $0 \leq i \leq d$*, observe* v_i^* *is an eigenvector for* A^*; *let* θ_i^* *denote the corresponding eigenvalue, and let* E_i^* *denote the primitive idempotent of* A^* *associated with* θ_i^*. *Then the sequence*

$$(A; E_0, E_1, \ldots, E_d; A^*; E_0^*, E_1^*, \ldots, E_d^*)$$

is a Leonard system in $\mathrm{End}(V)$.

Proof. We verify the conditions (i)–(v) of Definition 2.2. Condition (i) is immediate from Lemma 2.1 and the definition of multiplicity-free. Conditions (ii) and (iii) are immediate from the construction. Condition (iv) holds, since by Definition 1.1(i) the matrix representing A^* with respect to the basis $v_0, v_1, \ldots, v_d$ is irreducible tridiagonal. Condition (v) holds, since by Definition 1.1(ii) the matrix representing A with respect to the basis $v_0^*, v_1^*, \ldots, v_d^*$ is irreducible tridiagonal. □

Lemma 2.4. *Let* Φ *denote the Leonard system in* (9)*, and let* V *denote the irreducible left* $\mathcal{A}$*-module. For* $0 \leq i \leq d$*, let* v_i *denote a nonzero vector in* E_iV. *Then* $v_0, v_1, \ldots, v_d$ *is a basis for* V *with respect to which the matrix representing* A *is diagonal and the matrix representing* A^* *is irreducible tridiagonal. For* $0 \leq i \leq d$*, let* v_i^* *denote a nonzero vector in* E_i^*V. *Then* $v_0^*, v_1^*, \ldots, v_d^*$ *is a basis for* V *with respect to which the matrix representing* A^* *is diagonal and the matrix representing* A *is irreducible tridiagonal. Moreover the pair* A, A^* *is a Leonard pair on* V.

Proof. Routine. □

We mention a few basics concerning Leonard systems.

Let Φ denote the Leonard system in (9), and let $\sigma : \mathcal{A} \to \mathcal{A}'$ denote an isomorphism of **K**-algebras. We write

$$\Phi^\sigma := (A^\sigma; E_0^\sigma, E_1^\sigma, \ldots, E_d^\sigma; A^{*\sigma}; E_0^{*\sigma}, E_1^{*\sigma}, \ldots, E_d^{*\sigma}), \tag{10}$$

and observe Φ^σ is a Leonard system in $\mathcal{A}'$.

Definition 2.5 [**44**]. Let Φ and Φ' denote Leonard systems over $\mathbf{K}$. By an isomorphism of Leonard systems from Φ to Φ', we mean an isomorphism of $\mathbf{K}$-algebras $\sigma : \mathcal{A}(\Phi) \to \mathcal{A}(\Phi')$ such that $\Phi^\sigma = \Phi'$. The Leonard systems Φ, Φ' are said to be isomorphic whenever there exists an isomorphism of Leonard systems from Φ to Φ'.

We finish this section with a remark. Let d denote a nonnegative integer, and let $\mathcal{A}$ denote a $\mathbf{K}$-algebra isomorphic to $\mathrm{Mat}_{d+1}(\mathbf{K})$. Let $\sigma : \mathcal{A} \to \mathcal{A}$ denote any map. Then by the Skolem-Noether theorem [**9**], σ is an isomorphism of $\mathbf{K}$-algebras if and only if there exists an invertible $S \in \mathcal{A}$ such that $X^\sigma = SXS^{-1}$ for all $X \in \mathcal{A}$.

3. The structure of a Leonard system.

Let Φ denote the Leonard system in (9). In this section we show there does not exist an isomorphism of Leonard systems from Φ to itself, other than the identity map. We begin with a lemma.

Lemma 3.1. *Let Φ denote the Leonard system in* (9)*. Then the elements*

$$A^r E_0^* A^s, \quad 0 \le r, s \le d \tag{11}$$

form a basis for $\mathcal{A}$.

Proof. The number of elements in (11) equals $(d+1)^2$, and this number is the dimension of $\mathcal{A}$. Therefore it suffices to show the elements in (11) are linearly independent. To do this we represent the elements in (11) by matrices. Let V denote the irreducible left $\mathcal{A}$-module. For $0 \le i \le d$, let v_i denote a nonzero vector in E_i^*V and observe $v_0, v_1, \ldots, v_d$ is a basis for V. For the purposes of this proof, let us identify each element of $\mathcal{A}$ with the matrix in $\mathrm{Mat}_{d+1}(\mathbf{K})$ that represents it with respect to the basis $v_0, v_1, \ldots, v_d$. Adopting this point of view, A is irreducible tridiagonal and A^* is diagonal. For $0 \le r,\, s \le d$, we show the entries of $A^r E_0^* A^s$ satisfy

$$(A^r E_0^* A^s)_{ij} = \begin{cases} 0 & \text{if } i > r \text{ or } j > s \\ \\ \neq 0 & \text{if } i = r \text{ and } j = s, \end{cases} \qquad 0 \le i,\, j \le d\,. \tag{12}$$

Observe that for $0 \leq i,\ j \leq d$, the ij^{th} entry of E_0^* is one if both $i = 0, j = 0$ and zero otherwise. From this we find

$$(A^r E_0^* A^s)_{ij} = A^r_{i0} A^s_{0j}, \quad 0 \leq i, j \leq d. \tag{13}$$

Since A is irreducible tridiagonal, we find that for $0 \leq i \leq d$, the $i0$th entry of A^r is zero if $i > r$ and nonzero if $i = r$. Similarly for $0 \leq j \leq d$, the $0j$th entry of A^s is zero if $j > s$ and nonzero if $j = s$. Combining these facts with (13) we routinely obtain (12) and it follows that the elements (11) are linearly independent. Apparently the elements (11) form a basis for $\mathcal{A}$, as desired. □

Corollary 3.2. *Let Φ denote the Leonard system in* (9)*. Then the elements A, E_0^* together generate $\mathcal{A}$. Moreover, the elements A, A^* together generate $\mathcal{A}$.*

Proof. The first assertion is immediate from Lemma 3.1. The second assertion follows from the first and the observation that E_0^* is a polynomial in A^*. □

We mention a useful consequence of Corollary 3.2.

Corollary 3.3. *Let Φ denote the Leonard system* (9)*, and let X denote an element in $\mathcal{A}$ that commutes with both A and A^*. Then X is a scalar multiple of the identity. Put another way, there does not exist an isomorphism of Leonard system from Φ to itself, other than the identity map.*

Proof. Since A, A^* together generate $\mathcal{A}$, we find X commutes with everything in $\mathcal{A}$. Now X is a scalar multiple of the identity by elementary linear algebra. The last assertion follows in view of our remark at the end of Section 2. □

We mention an implication of Lemma 3.1 that will be useful later in the paper.

Lemma 3.4. *Let Φ denote the Leonard system in* (9)*. Let $\mathcal{D}$ denote the subalgebra of $\mathcal{A}$ generated by A, and observe $\mathcal{D}$ has dimension $d+1$*

since A is multiplicity-free. Let $X_0, X_1, \dots, X_d$ denote a basis for $\mathcal{D}$. Then the elements

$$X_r E_0^* X_s, \quad 0 \le r, s \le d \tag{14}$$

form a basis for $\mathcal{A}$.

Proof. The number of elements in (14) is $(d+1)^2$, and this number is the dimension of $\mathcal{A}$. Therefore it suffices to show the elements (14) span $\mathcal{A}$. But this is immediate from Lemma 3.1 and, since each element in (11) is contained in the span of the elements (14). □

Corollary 3.5. *Let Φ denote the Leonard system in* (9). *Then the elements*

$$E_r E_0^* E_s, \quad 0 \le r, s \le d \tag{15}$$

form a basis for $\mathcal{A}$.

Proof. Immediate from Lemma 3.4, with $X_i = E_i$ for $0 \le i \le d$. □

4. The relatives of a Leonard system. A given Leonard system can be modified in several ways to get a new Leonard system. For instance, let Φ denote the Leonard system in (9). Then each of the following three sequences is a Leonard system in $\mathcal{A}$.

$$\Phi^* := (A^*; E_0^*, E_1^*, \dots, E_d^*; A; E_0, E_1, \dots, E_d), \tag{16}$$

$$\Phi^{\downarrow} := (A; E_0, E_1, \dots, E_d; A^*; E_d^*, E_{d-1}^*, \dots, E_0^*), \tag{17}$$

$$\Phi^{\Downarrow} := (A; E_d, E_{d-1}, \dots, E_0; A^*; E_0^*, E_1^*, \dots, E_d^*). \tag{18}$$

We refer to Φ^*, (respectively $\Phi^{\downarrow}$), (respectively $\Phi^{\Downarrow}$), as the *dual*, (respectively *first inversion*), (respectively *second inversion*), of Φ. Viewing $*, \downarrow, \Downarrow$ as permutations on the set of all Leonard systems,

$$*^* = \downarrow^2 = \Downarrow^2 = 1, \tag{19}$$

$$\Downarrow * = * \downarrow, \quad \downarrow * = * \Downarrow, \quad \downarrow\Downarrow = \Downarrow\downarrow . \tag{20}$$

The group generated by symbols $*, \downarrow, \Downarrow$ subject to the relations (19) and (20) is the dihedral group D_4. We recall D_4 is the group of symmetries of a square and has 8 elements. Apparently $*, \downarrow, \Downarrow$ induce an action of D_4 on the set of all Leonard systems. Two Leonard systems will be called *relatives* whenever they are in the same orbit of this D_4 action. The relatives of Φ are as follows:

name	relative
Φ	$(A; E_0, E_1, \ldots, E_d; A^*; E^*_0, E^*_1, \ldots, E^*_d)$
$\Phi^{\downarrow}$	$(A; E_0, E_1, \ldots, E_d; A^*; E^*_d, E^*_{d-1}, \ldots, E^*_0)$
$\Phi^{\Downarrow}$	$(A; E_d, E_{d-1}, \ldots, E_0; A^*; E^*_0, E^*_1, \ldots, E^*_d)$
$\Phi^{\downarrow\Downarrow}$	$(A; E_d, E_{d-1}, \ldots, E_0; A^*; E^*_d, E^*_{d-1}, \ldots, E^*_0)$
Φ^*	$(A^*; E^*_0, E^*_1, \ldots, E^*_d; A; E_0, E_1, \ldots, E_d)$
$\Phi^{\downarrow *}$	$(A^*; E^*_d, E^*_{d-1}, \ldots, E^*_0; A; E_0, E_1, \ldots, E_d)$
$\Phi^{\Downarrow *}$	$(A^*; E^*_0, E^*_1, \ldots, E^*_d; A; E_d, E_{d-1}, \ldots, E_0)$
$\Phi^{\downarrow\Downarrow *}$	$(A^*; E^*_d, E^*_{d-1}, \ldots, E^*_0; A; E_d, E_{d-1}, \ldots, E_0)$

We remark that there may be some isomorphisms among the above Leonard systems.

We finish this section by recalling some parameters that will help us describe a given Leonard system.

Definition 4.1 [**44**]. Let Φ denote the Leonard system in (9). For $0 \leq i \leq d$, we let θ_i, (respectively θ^*_i), denote the eigenvalue of A, (respectively A^*), associated with E_i, (respectively E^*_i). We refer to $\theta_0, \theta_1, \ldots, \theta_d$ as the eigenvalue sequence of Φ. We refer to $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ as the dual eigenvalue sequence of Φ. We observe $\theta_0, \theta_1, \ldots, \theta_d$ are mutually distinct and contained in $\mathbf{K}$. Similarly, $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ are mutually distinct and contained in $\mathbf{K}$.

5. The standard basis and the split basis. Let Φ denote the Leonard system in (9), and let V denote the irreducible left $\mathcal{A}$-module. As we mentioned earlier, we will obtain 24 bases for V. One way to view our construction is as follows. Using Φ we define three bases for V, called the Φ-*standard basis*, the Φ-*split basis* and the Φ-*inverted split basis*. In each of the three cases, the basis is defined up to multiplication of each element by the same nonzero scalar in $\mathbf{K}$. Our set of 24 bases will consist of a Ψ-standard basis, a Ψ-split basis and a Ψ-inverted split basis for each relative Ψ of Φ.

We now define the notion of a *standard basis*.

Lemma 5.1. *Let Φ denote the Leonard system in* (9) *and let V denote the irreducible left $\mathcal{A}$-module. Let u denote a nonzero element of E_0^*V. Then for $0 \leq i \leq d$, the element E_iu is nonzero and hence a basis for E_iV. Moreover the sequence*

$$E_0u, E_1u, \ldots, E_du \tag{21}$$

is a basis for V.

Proof. Let the integer i be given. Recall E_0^*V has dimension 1 and u is a nonzero vector in E_0^*V, so u spans E_0^*V. Apparently E_iu spans $E_iE_0^*V$. Observe $E_iE_0^*$ is nonzero by Corollary 3.5 so $E_iE_0^*V$ is nonzero. Apparently E_iu is nonzero and is therefore a basis for E_iV as desired. The sequence (21) is a basis for V in view of (8). □

Definition 5.2. Let Φ denote the Leonard system in (9) and let V denote the irreducible left $\mathcal{A}$-module. By a Φ-standard basis for V, we mean a sequence (21) where u is a nonzero vector in E_0^*V. When the identity of Φ is clear, we will occasionally speak of a standard basis instead of a Φ-standard basis.

Let Φ denote the Leonard system in (9), and let V denote the irreducible left $\mathcal{A}$-module. With respect to any Φ-standard basis for V, the matrix representing A is

$$\mathrm{diag}\,(\theta_0, \theta_1, \ldots, \theta_d),$$

where the θ_i are from Definition 4.1. Moreover, by Lemma 2.4, the matrix representing A^* is irreducible tridiagonal. We will work out the entries of this tridiagonal matrix in due course, but it is convenient to wait until after we have introduced some more bases. For those who wish to skip ahead, the entries of this tridiagonal matrix can be found in the second table of Theorem 11.2, row 1.

We now define the notion of a *split basis*. In the process we will recall two sequences of scalars which we will find useful. These sequences are called the *first split sequence* of Φ and the *second split sequence* of Φ.

In order to define a split basis, we review some results of [**25**], [**44**]. Let Φ denote the Leonard system in (9), and let V denote the irreducible left $\mathcal{A}$-module. For $0 \leq i \leq d$, we define

$$U_i = (E_0^*V + E_1^*V + \cdots + E_i^*V) \cap (E_iV + E_{i+1}V + \cdots + E_dV). \tag{22}$$

We showed in [**25**] that each of $U_0, U_1, \ldots, U_d$ has dimension 1 and that

$$V = U_0 + U_1 + \cdots + U_d, \quad \text{direct sum.} \tag{23}$$

Moreover,

$$U_0 + U_1 + \cdots + U_i = E_0^*V + E_1^*V + \cdots + E_i^*V, \tag{24}$$

$$U_i + U_{i+1} + \cdots + U_d = E_iV + E_{i+1}V + \cdots + E_dV \tag{25}$$

for $0 \leq i \leq d$. The elements A and A^* act on the U_i as follows. We showed in [**44**] that

$$(A - \theta_i I)U_i = U_{i+1}, \quad 0 \leq i \leq d-1, \quad (A - \theta_d I)U_d = 0, \tag{26}$$

$$(A^* - \theta_i^* I)U_i = U_{i-1}, \quad 1 \leq i \leq d, \quad (A^* - \theta_0^* I)U_0 = 0, \tag{27}$$

where the θ_i, θ_i^* are from Definition 4.1. Pick an integer i, $(1 \leq i \leq d)$. By (27) we find $(A^* - \theta_i^* I)U_i = U_{i-1}$ and by (26) we find $(A - \theta_{i-1} I)U_{i-1} = U_i$. Apparently U_i is an eigenspace for $(A - \theta_{i-1}I)(A^* - \theta_i^* I)$ and the corresponding eigenvalue is a nonzero element of $\mathbf{K}$. We denote this eigenvalue by φ_i. We refer to the sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$ as the *first split sequence* of Φ. We let $\phi_1, \phi_2, \ldots, \phi_d$ denote the first split sequence for $\Phi^{\Downarrow}$ and call this the *second split sequence* of Φ. For notational convenience, we define $\varphi_0 = 0$, $\varphi_{d+1} = 0$, $\phi_0 = 0$, $\phi_{d+1} = 0$.

We obtain our split basis as follows. Setting $i = 0$ in (24) we find $U_0 = E_0^*V$. Combining this with (26), we find

$$U_i = (A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I)E_0^*V, \quad 0 \leq i \leq d. \tag{28}$$

Let u denote a nonzero vector in E_0^*V. From (28) we find that for $0 \leq i \leq d$, the vector $(A - \theta_0 I) \cdots (A - \theta_{i-1} I)u$ is a basis for U_i. From this and (23) we find the sequence

$$(A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{i-1} I)u, \quad 0 \leq i \leq d \tag{29}$$

is a basis for V.

Definition 5.3. Let Φ denote the Leonard system in (9), and let V denote the irreducible left $\mathcal{A}$-module. By a Φ-split basis for V, we mean a sequence (29) where u is a nonzero vector in E_0^*V. When the identity of Φ is clear, we will occasionally speak of a split basis instead of a Φ-split basis.

Let Φ denote the Leonard system in (9), and let V denote the irreducible left $\mathcal{A}$-module. From (29) and the lines below (27), we find that with respect to any Φ-split basis for V, the matrices representing A and A^* are

$$(30)\qquad \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \qquad \begin{pmatrix} \theta_0^* & \varphi_1 & & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & & \\ & & \theta_2^* & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta_d^* \end{pmatrix},$$

respectively.

We now define the notion of an *inverted split basis*. As its name implies, an inverted split basis is nothing but the inversion of a split basis. To be concrete, we make the following definition.

Definition 5.4. Let Φ denote the Leonard system in (9), and let V denote the irreducible left $\mathcal{A}$-module. By a Φ-inverted split basis for V, we mean a sequence

$$(31)\qquad (A-\theta_0 I)(A-\theta_1 I)\cdots(A-\theta_{d-i-1}I)u, \quad 0 \le i \le d,$$

where u is a nonzero vector in E_0^*V. When the identity of Φ is clear, we will occasionally speak of an inverted split basis instead of a Φ-inverted split basis.

Let Φ denote the Leonard system in (9), and let V denote the irreducible left $\mathcal{A}$-module. Combining (30) with Definition 5.4, we find that with respect to any Φ-inverted split basis for V, the matrices representing A and A^* are

$$\begin{pmatrix} \theta_d & 1 & & & & \mathbf{0} \\ & \theta_{d-1} & 1 & & & \\ & & \theta_{d-2} & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & 1 \\ \mathbf{0} & & & & & \theta_0 \end{pmatrix}, \qquad \begin{pmatrix} \theta^*_d & & & & & \mathbf{0} \\ \varphi_d & \theta^*_{d-1} & & & & \\ & \varphi_{d-1} & \theta^*_{d-2} & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & \varphi_1 & \theta^*_0 \end{pmatrix}, \tag{32}$$

respectively.

6. A classification of Leonard systems. In the preceding section we defined the first and second split sequence of a Leonard system. The scalars involved in these sequences are related by many equations. To describe these relationships we recall our classification of Leonard systems.

Theorem 6.1 [**44**]. *Let d denote a nonnegative integer, and let*

$$\theta_0, \theta_1, \ldots, \theta_d; \quad \theta^*_0, \theta^*_1, \ldots, \theta^*_d; \tag{33}$$

$$\varphi_1, \varphi_2, \ldots, \varphi_d; \quad \phi_1, \phi_2, \ldots, \phi_d \tag{34}$$

denote scalars in $\mathbf{K}$. *Then there exists a Leonard system Φ over $\mathbf{K}$ with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$, first split sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$, and second split sequence $\phi_1, \phi_2, \ldots, \phi_d$ if and only if* (i)–(v) *hold below.*

(i) $\varphi_i \neq 0, \quad \phi_i \neq 0, \quad 1 \leq i \leq d,$

(ii) $\theta_i \neq \theta_j$, $\theta^*_i \neq \theta^*_j$ *if* $i \neq j$, $0 \leq i, j \leq d,$

(iii) $\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{i-1} - \theta_d)$, $1 \leq i \leq d,$

(iv) $\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0)$, $1 \leq i \leq d,$

(v) *The expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \quad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \tag{35}$$

are equal and independent of i for $2 \leq i \leq d-1$.

Moreover, if (i)–(v) *hold above, then* Φ *is unique up to isomorphism of Leonard systems.*

We view Theorem 6.1 as a linear algebraic version of a theorem of Leonard [**32**], [**4**, page 260]. This is discussed in [**44**].

One nice feature of the parameter sequences (33) and (34) is that they are modified in a simple way as one passes from a given Leonard system to a relative. Our result is the following

Theorem 6.2 [**44**]. *Let* Φ *denote a Leonard system with eigenvalue sequence* $\theta_0, \theta_1, \dots, \theta_d$, *dual eigenvalue sequence* $\theta^*_0, \theta^*_1, \dots, \theta^*_d$, *first split sequence* $\varphi_1, \varphi_2, \dots, \varphi_d$ *and second split sequence* $\phi_1, \phi_2, \dots, \phi_d$. *Then* (i)–(iii) *hold below.*

(i) *The eigenvalue and dual eigenvalue sequences of* Φ^* *are given by* $\theta^*_0, \theta^*_1, \dots, \theta^*_d$ *and* $\theta_0, \theta_1, \dots, \theta_d$, *respectively. The first and second split sequences of* Φ^* *are given by* $\varphi_1, \varphi_2, \dots, \varphi_d$ *and* $\phi_d, \phi_{d-1}, \dots, \phi_1$, *respectively.*

(ii) *The eigenvalue and dual eigenvalue sequences of* $\Phi^{\downarrow}$ *are given by* $\theta_0, \theta_1, \dots, \theta_d$ *and* $\theta^*_d, \theta^*_{d-1}, \dots, \theta^*_0$, *respectively. The first and second split sequences of* $\Phi^{\downarrow}$ *are given by* $\phi_d, \phi_{d-1}, \dots, \phi_1$ *and* $\varphi_d, \varphi_{d-1}, \dots, \varphi_1$, *respectively.*

(iii) *The eigenvalue and dual eigenvalue sequences of* $\Phi^{\Downarrow}$ *are given by* $\theta_d, \theta_{d-1}, \dots, \theta_0$ *and* $\theta^*_0, \theta^*_1, \dots, \theta^*_d$, *respectively. The first and second split sequences of* $\Phi^{\Downarrow}$ *are given by* $\phi_1, \phi_2, \dots, \phi_d$ *and* $\varphi_1, \varphi_2, \dots, \varphi_d$, *respectively.*

7. Four flags for V**.** Let Φ denote the Leonard system in (9), and let V denote the irreducible left $\mathcal{A}$-module. We mentioned earlier we will obtain 24 bases for V. In Section 5 we described these bases to some extent, but we stopped short of displaying them. The reason is we wish to first introduce our labeling scheme. As we indicated in Section 1, it is appropriate to label our bases with elements of S_4. We begin with a definition.

Definition 7.1. Let Ω denote the set consisting of four symbols $0, d, 0^*, d^*$. We identify the symmetric group S_4 with the set of all linear orderings of Ω. For $i = 1, 2, 3$, we define a symmetric binary relation on S_4 which we call i-adjacency. An element $wxyz$ of S_4 is by definition 1-adjacent, (respectively 2-adjacent), (respectively 3-adjacent), to $xwyz$, (respectively $wyxz$), (respectively $wxzy$), and no other elements of S_4. Two elements in S_4 will be called adjacent whenever they are i-adjacent for some i, $(1 \leq i \leq 3)$.

Let Φ denote the Leonard system in (9), and let V denote the irreducible left $\mathcal{A}$-module. We recall the notion of a flag on V. By a *flag* on V, we mean a sequence $F_0, F_1, \ldots, F_d$ consisting of subspaces of V such that $F_{i-1} \subseteq F_i$ for $1 \leq i \leq d$ and such that F_i has dimension $i+1$ for $0 \leq i \leq d$. We refer to F_i as the ith *component* of the flag.

The following construction yields a flag on V. To explain the construction we make a definition. By a *decomposition* of V, we mean a sequence $L_0, L_1, \ldots, L_d$ consisting of one-dimensional subspaces of V such that

$$V = L_0 + L_1 + \cdots + L_d, \quad \text{(direct sum)}. \tag{36}$$

Let $L_0, L_1, \ldots, L_d$ denote a decomposition of V, and set

$$F_i = L_0 + L_1 + \cdots + L_i$$

for $0 \leq i \leq d$. Then the sequence $F_0, F_1, \ldots, F_d$ is a flag on V.

We will be concerned with the following four flags on V.

Definition 7.2. Let Φ denote the Leonard system in (9), and let V denote the irreducible left $\mathcal{A}$-module. Let the set Ω be as in Definition 7.1. For each element $z \in \Omega$, we define a flag on V which we denote by $[z]$. To define this flag, we display its ith component for $0 \leq i \leq d$.

z	ith component of the flag $[z]$
0	$E_0V + E_1V + \cdots + E_iV$
d	$E_dV + E_{d-1}V + \cdots + E_{d-i}V$
0^*	$E^*_0V + E^*_1V + \cdots + E^*_iV$
d^*	$E^*_dV + E^*_{d-1}V + \cdots + E^*_{d-i}V$

Let Φ denote the Leonard system in (9), and let V denote the irreducible left $\mathcal{A}$-module. We recall what it means for two flags on

V to be opposite. Suppose we are given two flags on V, denoted $F_0, F_1, \ldots, F_d$ and $G_0, G_1, \ldots, G_d$. These flags are said to be *opposite* whenever

$$F_i \cap G_j = 0 \quad \text{if } i + j < d, \quad 0 \leq i, j \leq d. \tag{37}$$

Given a decomposition of V, the following construction yields an ordered pair of opposite flags on V. Let $L_0, L_1, \ldots, L_d$ denote a decomposition of V, and set

$$\begin{aligned} F_i &= L_0 + L_1 + \cdots + L_i, \\ G_i &= L_d + L_{d-1} + \cdots + L_{d-i} \end{aligned} \tag{38}$$

for $0 \leq i \leq d$. Then the sequences $F_0, F_1, \ldots, F_d$ and $G_0, G_1, \ldots, G_d$ are opposite flags on V.

We now turn things around. Given an ordered pair of opposite flags on V, the following construction yields a decomposition of V. Suppose we are given an ordered pair of opposite flags on V, denoted $F_0, F_1, \ldots, F_d$ and $G_0, G_1, \ldots, G_d$. Set

$$L_i = F_i \cap G_{d-i}, \quad 0 \leq i \leq d. \tag{39}$$

Then the sequence $L_0, L_1, \ldots, L_d$ is a decomposition of V.

Let D denote the set consisting of all decompositions of V, and let F denote the set consisting of all ordered pairs of opposite flags on V. In the previous two paragraphs, we defined a map from D to F and a map from F to D. It is routine to show that these maps are inverses of one another [**34**]. In particular, each of these maps is a bijection.

We now return to the Leonard system Φ.

Theorem 7.3. *The four flags in Definition* 7.2 *are mutually opposite.*

Proof. It is immediate from the construction that flags $[0], [d]$ are opposite, and that flags $[0^*], [d^*]$ are opposite. We now show the flags $[0^*], [d]$ are opposite. For $0 \leq i \leq d$, let U_i denote the subspace of V from (22). By the two lines following (22), we find the sequence $U_0, U_1, \ldots, U_d$ is a decomposition of V. By (24), (25) and the line following (38), we find the flags $[0^*], [d]$ are opposite. Applying this fact to the relatives of Φ, we see that the remaining pairs of flags in Definition 7.2 are opposite. □

8. Twelve decompositions of V. Let Φ denote the Leonard system in (9), let V denote the irreducible left $\mathcal{A}$-module, and let the set Ω be as in Definition 7.1. In this section we obtain for each ordered pair yz of distinct elements in Ω, a decomposition of V which we denote by $[yz]$.

Definition 8.1. Let Φ denote the Leonard system in (9), let V denote the irreducible left $\mathcal{A}$-module and let the set Ω be as in Definition 7.1. Let yz denote an ordered pair of distinct elements in Ω. Set

$$L_i = F_i \cap G_{d-i}, \quad 0 \le i \le d,$$

where F_j, (respectively G_j), denotes the jth component of the flag $[y]$, (respectively $[z]$), for $0 \le j \le d$. Recall $[y]$ and $[z]$ are opposite, so the sequence $L_0, L_1, \ldots, L_d$ is a decomposition of V. We denote this decomposition by $[yz]$.

With reference to Definition 8.1, we remark on the difference between $[yz]$ and $[zy]$. To do this we use the following notation. Let $L_0, L_1, \ldots, L_d$ denote a decomposition of V. Then the sequence $L_d, L_{d-1}, \ldots, L_0$ is a decomposition of V, which we call the *inversion* of $L_0, L_1, \ldots, L_d$.

Lemma 8.2. *Let Φ denote the Leonard system in* (9)*, let V denote the irreducible left $\mathcal{A}$-module, and let the set Ω be as in Definition* 7.1*. Let y, z denote distinct elements in Ω. Then each of the decompositions $[yz], [zy]$ is the inversion of the other.*

Proof. Immediate from Definition 8.1 and the definition of inversion. □

Let Φ denote the Leonard system in (9), let V denote the irreducible left $\mathcal{A}$-module, and let the set Ω be as in Definition 7.1. In Definition 8.1

we obtained for each ordered pair yz of distinct elements in Ω, a decomposition of V denoted $[yz]$. This gives 12 decompositions of V. By Lemma 8.2 these consist of six pairs of inverse decompositions. To be concrete, we now display these decompositions.

Theorem 8.3. *Let Φ denote the Leonard system in* (9)*, let V denote the irreducible left $\mathcal{A}$-module, and let the set Ω be as in Definition* 7.1*. Let yz denote an ordered pair of distinct elements in Ω, and consider the corresponding decomposition $[yz]$ of V from Definition* 8.1*. For $0 \leq i \leq d$, the ith subspace of $[yz]$ is given in the following table.*

yz	*ith subspace of decomposition $[yz]$*
0^*d	$(E^*_0V + \cdots + E^*_iV) \cap (E_iV + \cdots + E_dV)$
$d0^*$	$(E^*_0V + \cdots + E^*_{d-i}V) \cap (E_{d-i}V + \cdots + E_dV)$
$0d^*$	$(E_0V + \cdots + E_iV) \cap (E^*_iV + \cdots + E^*_dV)$
d^*0	$(E_0V + \cdots + E_{d-i}V) \cap (E^*_{d-i}V + \cdots + E^*_dV)$
00^*	$(E_0V + \cdots + E_iV) \cap (E^*_{d-i}V + \cdots + E^*_0V)$
0^*0	$(E_0V + \cdots + E_{d-i}V) \cap (E^*_iV + \cdots + E^*_0V)$
dd^*	$(E_dV + \cdots + E_{d-i}V) \cap (E^*_iV + \cdots + E^*_dV)$
d^*d	$(E_dV + \cdots + E_iV) \cap (E^*_{d-i}V + \cdots + E^*_dV)$
$0d$	E_iV
$d0$	$E_{d-i}V$
0^*d^*	E^*_iV
d^*0^*	$E^*_{d-i}V$

Describing our 12 decompositions from another point of view, we have the following.

Theorem 8.4. *Let Φ denote the Leonard system in* (9)*, let V denote the irreducible left $\mathcal{A}$-module, and let the set Ω be as in Definition* 7.1*. Let yz denote an ordered pair of distinct elements in Ω, and consider the corresponding decomposition $[yz]$ from Definition* 8.1*. Let us denote this decomposition by $L_0, L_1, \ldots, L_d$. Then for $0 \leq i \leq d$, the sums $L_0 + L_1 + \cdots + L_i$ and $L_i + L_{i+1} + \cdots + L_d$ are given as follows.*

yz	$L_0+\cdots+L_i$	$L_i+\cdots+L_d$
0^*d	$E^*_0V+\cdots+E^*_iV$	$E_iV+\cdots+E_dV$
$d0^*$	$E_dV+\cdots+E_{d-i}V$	$E^*_{d-i}V+\cdots+E^*_0V$
$0d^*$	$E_0V+\cdots+E_iV$	$E^*_iV+\cdots+E^*_dV$
d^*0	$E^*_dV+\cdots+E^*_{d-i}V$	$E_{d-i}V+\cdots+E_0V$
00^*	$E_0V+\cdots+E_iV$	$E^*_{d-i}V+\cdots+E^*_0V$
0^*0	$E^*_0V+\cdots+E^*_iV$	$E_{d-i}V+\cdots+E_0V$
dd^*	$E_dV+\cdots+E_{d-i}V$	$E^*_iV+\cdots+E^*_dV$
d^*d	$E^*_dV+\cdots+E^*_{d-i}V$	$E_iV+\cdots+E_dV$
$0d$	$E_0V+\cdots+E_iV$	$E_iV+\cdots+E_dV$
$d0$	$E_dV+\cdots+E_{d-i}V$	$E_{d-i}V+\cdots+E_0V$
0^*d^*	$E^*_0V+\cdots+E^*_iV$	$E^*_iV+\cdots+E^*_dV$
d^*0^*	$E^*_dV+\cdots+E^*_{d-i}V$	$E^*_{d-i}V+\cdots+E^*_0V$

9. 24 bases for V. Let Φ denote the Leonard system in (9), and let V denote the irreducible left $\mathcal{A}$-module. For each element $g \in S_4$, we display a basis for V, denoted $[g]$. To describe our procedure we use the following notation.

Let $u_0, u_1, \dots, u_d$ denote a basis for V, and set $L_i = \mathrm{Span}\,(u_i)$ for $0 \le i \le d$. Observe the sequence $L_0, L_1, \dots, L_d$ is a decomposition of V. We say this decomposition is *induced* by $u_0, u_1, \dots, u_d$.

Let the set Ω be as in Definition 7.1, and let yz denote an ordered pair of distinct elements of Ω. Consider the corresponding decomposition of V, denoted $[yz]$. We define two bases for V, both of which induce $[yz]$. We denote these bases by $[wxyz]$ and $[xwyz]$ where w and x denote the elements in Ω other than y, z. Apparently this procedure yields, for each $g \in S_4$, a basis $[g]$ for V. These 24 bases are displayed below.

Theorem 9.1. *Let Φ denote the Leonard system in* (9)*, and let V denote the irreducible left $\mathcal{A}$-module. Let $\eta_0, \eta_d, \eta^*_0, \eta^*_d$ denote nonzero vectors in V such that*

$$\eta_0 \in E_0V, \quad \eta_d \in E_dV, \quad \eta^*_0 \in E^*_0V, \quad \eta^*_d \in E^*_dV. \tag{40}$$

With reference to Definition 7.1*, let g denote an element of S_4 and consider row g of the table below. For $0 \le i \le d$, the vector v_i given* *in that row is a basis for the subspace given to its right. Moreover, the sequence* $v_0, v_1, \ldots, v_d$ *is a basis for* V. *We denote this basis by* $[g]$.

g	v_i	v_i is basis for
d^*00^*d	$(A-\theta_0)\cdots(A-\theta_{i-1})\eta^*_0$	$(E^*_0V+\cdots+E^*_iV)\cap(E_iV+\cdots+E_dV)$
$0d^*0^*d$	$(A^*-\theta^*_d)\cdots(A^*-\theta^*_{i+1})\eta_d$	$(E^*_0V+\cdots+E^*_iV)\cap(E_iV+\cdots+E_dV)$
d^*0d0^*	$(A-\theta_0)\cdots(A-\theta_{d-i-1})\eta^*_0$	$(E^*_0V+\cdots+E^*_{d-i}V)\cap(E_{d-i}V+\cdots+E_dV)$
$0d^*d0^*$	$(A^*-\theta^*_d)\cdots(A^*-\theta^*_{d-i+1})\eta_d$	$(E^*_0V+\cdots+E^*_{d-i}V)\cap(E_{d-i}V+\cdots+E_dV)$
$d0^*0d^*$	$(A^*-\theta^*_0)\cdots(A^*-\theta^*_{i-1})\eta_0$	$(E_0V+\cdots+E_iV)\cap(E^*_iV+\cdots+E^*_dV)$
0^*d0d^*	$(A-\theta_d)\cdots(A-\theta_{i+1})\eta^*_d$	$(E_0V+\cdots+E_iV)\cap(E^*_iV+\cdots+E^*_dV)$
$d0^*d^*0$	$(A^*-\theta^*_0)\cdots(A^*-\theta^*_{d-i-1})\eta_0$	$(E_0V+\cdots+E_{d-i}V)\cap(E^*_{d-i}V+\cdots+E^*_dV)$
0^*dd^*0	$(A-\theta_d)\cdots(A-\theta_{d-i+1})\eta^*_d$	$(E_0V+\cdots+E_{d-i}V)\cap(E^*_{d-i}V+\cdots+E^*_dV)$
dd^*00^*	$(A^*-\theta^*_d)\cdots(A^*-\theta^*_{d-i+1})\eta_0$	$(E_0V+\cdots+E_iV)\cap(E^*_{d-i}V+\cdots+E^*_0V)$
d^*d00^*	$(A-\theta_d)\cdots(A-\theta_{i+1})\eta^*_0$	$(E_0V+\cdots+E_iV)\cap(E^*_{d-i}V+\cdots+E^*_0V)$
dd^*0^*0	$(A^*-\theta^*_d)\cdots(A^*-\theta^*_{i+1})\eta_0$	$(E_0V+\cdots+E_{d-i}V)\cap(E^*_iV+\cdots+E^*_0V)$
d^*d0^*0	$(A-\theta_d)\cdots(A-\theta_{d-i+1})\eta^*_0$	$(E_0V+\cdots+E_{d-i}V)\cap(E^*_iV+\cdots+E^*_0V)$
00^*dd^*	$(A^*-\theta^*_0)\cdots(A^*-\theta^*_{i-1})\eta_d$	$(E_dV+\cdots+E_{d-i}V)\cap(E^*_iV+\cdots+E^*_dV)$
0^*0dd^*	$(A-\theta_0)\cdots(A-\theta_{d-i-1})\eta^*_d$	$(E_dV+\cdots+E_{d-i}V)\cap(E^*_iV+\cdots+E^*_dV)$
00^*d^*d	$(A^*-\theta^*_0)\cdots(A^*-\theta^*_{d-i-1})\eta_d$	$(E_dV+\cdots+E_iV)\cap(E^*_{d-i}V+\cdots+E^*_dV)$
0^*0d^*d	$(A-\theta_0)\cdots(A-\theta_{i-1})\eta^*_d$	$(E_dV+\cdots+E_iV)\cap(E^*_{d-i}V+\cdots+E^*_dV)$
d^*0^*0d	$E_i\eta^*_0$	E_iV
0^*d^*0d	$E_i\eta^*_d$	E_iV
d^*0^*d0	$E_{d-i}\eta^*_0$	$E_{d-i}V$
0^*d^*d0	$E_{d-i}\eta^*_d$	$E_{d-i}V$
$d00^*d^*$	$E^*_i\eta_0$	E^*_iV
$0d0^*d^*$	$E^*_i\eta_d$	E^*_iV
$d0d^*0^*$	$E^*_{d-i}\eta_0$	$E^*_{d-i}V$
$0dd^*0^*$	$E^*_{d-i}\eta_d$	$E^*_{d-i}V$

Proof. Concerning the first row of the above table, our assertions follow from the lines preceding (29). Concerning the third row of the above table, our assertions follow upon replacing i by $d-i$ in the first row. We have now proved our assertions for the first and third rows of the table. Applying these assertions to the relatives of Φ, we obtain the first 16 rows of the table. Consider the next remaining row, where g equals d^*0^*0d. For this row, our assertions are immediate from Lemma 5.1. Applying this result to the relatives of Φ, we obtain the remaining rows of the table. □

We record a few observations.

Lemma 9.2. *Referring to Theorem* 9.1*, for all elements* $wxyz$ *in* S_4*, the basis* $[wxyz]$ *from Theorem* 9.1 *induces the decomposition* $[yz]$ *of* V *from Definition* 8.1.

Proof. Compare the data in Theorem 9.1 with the data in Theorem 8.3. ◻

Lemma 9.3. *Let* Φ *denote the Leonard system in* (9)*, and let* V *denote the irreducible left* $\mathcal{A}$*-module. In the table below each basis for* V *contained the first column, (respectively second column), (respectively third column), is a* Ψ*-standard basis, (respectively* Ψ*-split basis), (respectively* Ψ*-inverted split basis), where* Ψ *is the relative of* Φ *given to the left of this basis.*

Ψ	Ψ-*standard basis*	Ψ-*split basis*	Ψ-*inv. split basis*
Φ	$[d^*0^*0d]$	$[d^*00^*d]$	$[d^*0d0^*]$
$\Phi^{\downarrow}$	$[0^*d^*0d]$	$[0^*0d^*d]$	$[0^*0dd^*]$
$\Phi^{\Downarrow}$	$[d^*0^*d0]$	$[d^*d0^*0]$	$[d^*d00^*]$
$\Phi^{\downarrow\Downarrow}$	$[0^*d^*d0]$	$[0^*dd^*0]$	$[0^*d0d^*]$
Φ^*	$[d00^*d^*]$	$[d0^*0d^*]$	$[d0^*d^*0]$
$\Phi^{\downarrow *}$	$[d0d^*0^*]$	$[dd^*00^*]$	$[dd^*0^*0]$
$\Phi^{\Downarrow *}$	$[0d0^*d^*]$	$[00^*dd^*]$	$[00^*d^*d]$
$\Phi^{\downarrow\Downarrow *}$	$[0dd^*0^*]$	$[0d^*d0^*]$	$[0d^*0^*d]$

Proof. Immediate from inspecting the table in Theorem 9.1. ◻

Later in the paper we will compute, for each ordered pair g, h of adjacent elements in S_4, the entries in the transition matrix from the basis $[g]$ to the basis $[h]$. Before going that far, we say something about the general nature of these transition matrices. First we recall our terms.

Let Φ denote the Leonard system in (9), and let V denote the irreducible left $\mathcal{A}$-module. Suppose we are given two bases for V, written $u_0, u_1, \ldots, u_d$ and $v_0, v_1, \ldots, v_d$. By the *transition matrix* from $u_0, u_1, \ldots, u_d$ to $v_0, v_1, \ldots, v_d$, we mean the matrix T in $\mathrm{Mat}_{d+1}(\mathbf{K})$ satisfying

$$v_j = \sum_{i=0}^{d} T_{ij} u_i, \quad 0 \leq j \leq d. \tag{41}$$

We recall a few properties of transition matrices. Let T denote the transition matrix from $u_0, u_1, \ldots, u_d$ to $v_0, v_1, \ldots, v_d$. Then T^{-1} exists and equals the transition matrix from $v_0, v_1, \ldots, v_d$ to $u_0, u_1, \ldots, u_d$. Let $w_0, w_1, \ldots, w_d$ denote a basis for V, and let S denote the transition matrix from $v_0, v_1, \ldots, v_d$ to $w_0, w_1, \ldots, w_d$. Then TS is the transition matrix from $u_0, u_1, \ldots, u_d$ to $w_0, w_1, \ldots, w_d$.

Lemma 9.4. *Let Φ denote the Leonard system in* (9)*, and let V denote the irreducible left $\mathcal{A}$-module. With reference to Definition* 7.1*, let g, h denote adjacent elements in S_4, and consider the corresponding bases $[g], [h]$ for V given in Theorem* 9.1*. Then* (i)–(iii) *hold below.*

(i) *Suppose g, h are* 1*-adjacent. Then the transition matrix from $[g]$ to $[h]$ is diagonal.*

(ii) *Suppose g, h are* 2*-adjacent. Then the transition matrix from $[g]$ to $[h]$ is lower triangular.*

(iii) *Suppose g, h are* 3*-adjacent. Then $[g]$ is the inversion of $[h]$.*

Proof. For notational convenience we write $g = wxyz$.

(i) In this case $h = xwyz$. Observe $[g]$ and $[h]$ both induce the decomposition $[yz]$ by Lemma 9.2, so the transition matrix from $[g]$ to $[h]$ is diagonal.

(ii) In this case $h = wyxz$. By Lemma 9.2 the bases $[g]$ and $[h]$ induce the decompositions $[yz]$ and $[xz]$, respectively. When we consider how the decompositions $[yz]$ and $[xz]$ are related, we find the transition matrix from $[g]$ to $[h]$ is lower triangular.

(iii) In this case $h = wxzy$. In the table of Theorem 9.1, for each block we compare rows 1,3 and rows 2,4. We find in all cases $[g]$ is the inversion of $[h]$. $\square$

10. Some scalars. Our next goal is to compute the matrices representing A and A^* with respect to each of the bases in Theorem 9.1. To describe the entries of these matrices, we will use the following parameters.

Definition 10.1. Let Φ denote the Leonard system in (9). We define

$$a_i = \operatorname{tr} AE_i^*, \quad a_i^* = \operatorname{tr} A^*E_i, \quad 0 \le i \le d, \tag{42}$$

where tr means trace.

The scalars a_i, a_i^* have the following interpretation.

Lemma 10.2. *With reference to Definition* 10.1*,*

$$E_i^*AE_i^* = a_iE_i^*, \quad 0 \le i \le d, \tag{43}$$

$$E_iA^*E_i = a_i^*E_i, \quad 0 \le i \le d. \tag{44}$$

Proof. Concerning (43), let i be given. Since E_i^* is a rank 1 idempotent, a scalar $\alpha_i \in \mathbf{K}$ exists such that

$$E_i^*AE_i^* = \alpha_iE_i^*. \tag{45}$$

Taking the trace of both sides of (45), and recalling XY, YX have the same trace, we routinely find $\alpha_i = a_i$. We have now proved (43). Applying this to Φ^*, we obtain (44). □

Lemma 10.3. *Let Φ denote the Leonard system in* (9)*. Then for $0 \le i \le d$, the scalar a_i equals both*

$$\theta_i + \frac{\varphi_i}{\theta_i^* - \theta_{i-1}^*} + \frac{\varphi_{i+1}}{\theta_i^* - \theta_{i+1}^*}, \quad \theta_{d-i} + \frac{\phi_i}{\theta_i^* - \theta_{i-1}^*} + \frac{\phi_{i+1}}{\theta_i^* - \theta_{i+1}^*}, \tag{46}$$

where $\theta_{-1}^*, \theta_{d+1}^*$ denote indeterminants. Moreover, the scalar a_i^* equals both

$$\theta_i^* + \frac{\varphi_i}{\theta_i - \theta_{i-1}} + \frac{\varphi_{i+1}}{\theta_i - \theta_{i+1}}, \quad \theta_{d-i}^* + \frac{\phi_{d-i+1}}{\theta_i - \theta_{i-1}} + \frac{\phi_{d-i}}{\theta_i - \theta_{i+1}}, \tag{47}$$

where $\theta_{-1}, \theta_{d+1}$ denote indeterminants.

Proof. Let the integer i be given. The scalar a_i equals the expression on the left in (46) by [**44**, Lemma 5.1]. Applying this fact to $\Phi^{\Downarrow}$, and using Theorem 6.2(iii), we find a_i equals the expression on the right in (46). We have now shown a_i equals the two expressions in (46). Applying this to Φ^* and using Theorem 6.2(i), we find a_i^* equals the two expressions in (47). □

11. The 24 bases; matrices representing A and A^*. In this section we return to the 24 bases in Theorem 9.1. For each $g \in S_4$, we compute the matrices representing A and A^* with respect to the basis $[g]$.

We use the following notation.

Definition 11.1. Let Φ denote the Leonard system in (9), and let V denote the irreducible left $\mathcal{A}$-module. With reference to Definition 7.1, let g denote an element in S_4. For all $X \in \mathcal{A}$, we let X^g denote the matrix in $\mathrm{Mat}_{d+1}(\mathbf{K})$ that represents X with respect to the basis $[g]$, where $[g]$ is from Theorem 9.1. Denoting this basis by $v_0, v_1, \ldots, v_d$, we have

$$Xv_j = \sum_{i=0}^{d} X^g_{ij} v_i, \quad 0 \le j \le d.$$

We observe the map $X \to X^g$ is a $\mathbf{K}$-algebra isomorphism from $\mathcal{A}$ to $\mathrm{Mat}_{d+1}(\mathbf{K})$.

Theorem 11.2. *Let g denote an element of S_4. With reference to Definition* 11.1*, the entries of A^g and A^{*g} are given in the tables below. Any entry not displayed is zero.*

g	$A^g_{i,i-1}$	A^g_{ii}	$A^g_{i-1,i}$	$A^{*g}_{i,i-1}$	A^{*g}_{ii}	$A^{*g}_{i-1,i}$
d^*00^*d	1	θ_i	0	0	θ^*_i	φ_i
$0d^*0^*d$	φ_i	θ_i	0	0	θ^*_i	1
d^*0d0^*	0	θ_{d-i}	1	φ_{d-i+1}	θ^*_{d-i}	0
$0d^*d0^*$	0	θ_{d-i}	φ_{d-i+1}	1	θ^*_{d-i}	0
$d0^*0d^*$	0	θ_i	φ_i	1	θ^*_i	0
0^*d0d^*	0	θ_i	1	φ_i	θ^*_i	0
$d0^*d^*0$	φ_{d-i+1}	θ_{d-i}	0	0	θ^*_{d-i}	1
0^*dd^*0	1	θ_{d-i}	0	0	θ^*_{d-i}	φ_{d-i+1}
dd^*00^*	0	θ_i	ϕ_{d-i+1}	1	θ^*_{d-i}	0
d^*d00^*	0	θ_i	1	ϕ_{d-i+1}	θ^*_{d-i}	0
dd^*0^*0	ϕ_i	θ_{d-i}	0	0	θ^*_i	1
d^*d0^*0	1	θ_{d-i}	0	0	θ^*_i	ϕ_i
00^*dd^*	0	θ_{d-i}	ϕ_i	1	θ^*_i	0
0^*0dd^*	0	θ_{d-i}	1	ϕ_i	θ^*_i	0
00^*d^*d	ϕ_{d-i+1}	θ_i	0	0	θ^*_{d-i}	1
0^*0d^*d	1	θ_i	0	0	θ^*_{d-i}	ϕ_{d-i+1}

g	A^g_{ii}	$A^{*g}_{i,i-1}$
d^*0^*0d	θ_i	$\phi_{d-i+1}\frac{(\theta_i-\theta_d)\cdots(\theta_i-\theta_{i+1})}{(\theta_{i-1}-\theta_d)\cdots(\theta_{i-1}-\theta_i)}$
0^*d^*0d	θ_i	$\varphi_i\frac{(\theta_i-\theta_d)\cdots(\theta_i-\theta_{i+1})}{(\theta_{i-1}-\theta_d)\cdots(\theta_{i-1}-\theta_i)}$
d^*0^*d0	θ_{d-i}	$\varphi_{d-i+1}\frac{(\theta_{d-i}-\theta_0)\cdots(\theta_{d-i}-\theta_{d-i-1})}{(\theta_{d-i+1}-\theta_0)\cdots(\theta_{d-i+1}-\theta_{d-i})}$
0^*d^*d0	θ_{d-i}	$\phi_i\frac{(\theta_{d-i}-\theta_0)\cdots(\theta_{d-i}-\theta_{d-i-1})}{(\theta_{d-i+1}-\theta_0)\cdots(\theta_{d-i+1}-\theta_{d-i})}$

A^{*g}_{ii}	$A^{*g}_{i-1,i}$
a^*_i	$\varphi_i \frac{(\theta_{i-1}-\theta_0)\cdots(\theta_{i-1}-\theta_{i-2})}{(\theta_i-\theta_0)\cdots(\theta_i-\theta_{i-1})}$
a^*_i	$\phi_{d-i+1} \frac{(\theta_{i-1}-\theta_0)\cdots(\theta_{i-1}-\theta_{i-2})}{(\theta_i-\theta_0)\cdots(\theta_i-\theta_{i-1})}$
a^*_{d-i}	$\phi_i \frac{(\theta_{d-i+1}-\theta_d)\cdots(\theta_{d-i+1}-\theta_{d-i+2})}{(\theta_{d-i}-\theta_d)\cdots(\theta_{d-i}-\theta_{d-i+1})}$
a^*_{d-i}	$\varphi_{d-i+1} \frac{(\theta_{d-i+1}-\theta_d)\cdots(\theta_{d-i+1}-\theta_{d-i+2})}{(\theta_{d-i}-\theta_d)\cdots(\theta_{d-i}-\theta_{d-i+1})}$

g	$A^g_{i,i-1}$	A^g_{ii}
$d00^*d^*$	$\phi_i \frac{(\theta^*_i-\theta^*_d)\cdots(\theta^*_i-\theta^*_{i+1})}{(\theta^*_{i-1}-\theta^*_d)\cdots(\theta^*_{i-1}-\theta^*_i)}$	a_i
$0d0^*d^*$	$\varphi_i \frac{(\theta^*_i-\theta^*_d)\cdots(\theta^*_i-\theta^*_{i+1})}{(\theta^*_{i-1}-\theta^*_d)\cdots(\theta^*_{i-1}-\theta^*_i)}$	a_i
$d0d^*0^*$	$\varphi_{d-i+1} \frac{(\theta^*_{d-i}-\theta^*_0)\cdots(\theta^*_{d-i}-\theta^*_{d-i-1})}{(\theta^*_{d-i+1}-\theta^*_0)\cdots(\theta^*_{d-i+1}-\theta^*_{d-i})}$	a_{d-i}
$0dd^*0^*$	$\phi_{d-i+1} \frac{(\theta^*_{d-i}-\theta^*_0)\cdots(\theta^*_{d-i}-\theta^*_{d-i-1})}{(\theta^*_{d-i+1}-\theta^*_0)\cdots(\theta^*_{d-i+1}-\theta^*_{d-i})}$	a_{d-i}

$A^g_{i-1,i}$	A^{*g}_{ii}
$\varphi_i \frac{(\theta^*_{i-1}-\theta^*_0)\cdots(\theta^*_{i-1}-\theta^*_{i-2})}{(\theta^*_i-\theta^*_0)\cdots(\theta^*_i-\theta^*_{i-1})}$	θ^*_i
$\phi_i \frac{(\theta^*_{i-1}-\theta^*_0)\cdots(\theta^*_{i-1}-\theta^*_{i-2})}{(\theta^*_i-\theta^*_0)\cdots(\theta^*_i-\theta^*_{i-1})}$	θ^*_i
$\phi_{d-i+1} \frac{(\theta^*_{d-i+1}-\theta^*_d)\cdots(\theta^*_{d-i+1}-\theta^*_{d-i+2})}{(\theta^*_{d-i}-\theta^*_d)\cdots(\theta^*_{d-i}-\theta^*_{d-i+1})}$	θ^*_{d-i}
$\varphi_{d-i+1} \frac{(\theta^*_{d-i+1}-\theta^*_d)\cdots(\theta^*_{d-i+1}-\theta^*_{d-i+2})}{(\theta^*_{d-i}-\theta^*_d)\cdots(\theta^*_{d-i}-\theta^*_{d-i+1})}$	θ^*_{d-i}

Proof. Consider the first row of the first table where g equals d^*00^*d. As indicated in the table of Lemma 9.3, row 1, column 2, the basis $[g]$ is a Φ-split basis. From the line above (30), we find A^g, (respectively A^{*g}), is given on the left, (respectively right), in (30). From this we obtain our results for the first row of the first table. Now consider the third row of the first table, where g equals d^*0d0^*. From the table of Lemma 9.3, row 1, column 3, the basis $[d^*0d0^*]$ is a Φ-inverted split basis. From the line above (32) we find A^g, (respectively A^{*g}), is given on the left, (respectively right), in (32). From this we obtain our results for the third row of the first table. We have now proved our assertions for rows 1 and 3 of the first table. Applying this result to the relatives of Φ, and using Theorem 6.2 we obtain the remaining rows of the first table. Consider the first row of the second table, where g equals d^*0^*0d. From the table of Theorem 9.1, row 17, we find the corresponding basis $[g]$ is

$$E_0\eta_0^*, E_1\eta_0^*, \ldots, E_d\eta_0^*. \tag{48}$$

For $0 \leq i \leq d$, the vector $E_i\eta_0^*$ is an eigenvector for A, with eigenvalue θ_i. Therefore,

$$A^g = \mathrm{diag}\,(\theta_0, \theta_1, \ldots, \theta_d). \tag{49}$$

We now find A^{*g}. From the construction and since A, A^* is a Leonard pair, the matrix A^{*g} is irreducible tridiagonal. From (44) we find the diagonal entries $A^{*g}_{ii} = a^*_i$ for $0 \leq i \leq d$. We show

$$A^{*g}_{i-1,i} = \varphi_i \frac{(\theta_{i-1}-\theta_0)(\theta_{i-1}-\theta_1)\cdots(\theta_{i-1}-\theta_{i-2})}{(\theta_i-\theta_0)(\theta_i-\theta_1)\cdots(\theta_i-\theta_{i-1})} \tag{50}$$

for $1 \leq i \leq d$. To see (50), we momentarily return to the basis $[d^*00^*d]$. From the table of Theorem 9.1, row 1, we find that for $0 \leq j \leq d$ the j^{th} vector in the basis $[d^*00^*d]$ is given by

$$(A-\theta_0 I)(A-\theta_1 I)\cdots(A-\theta_{j-1}I)\eta_0^*. \tag{51}$$

We write (51) in terms of (48). Recall the sum $E_0+E_1+\cdots+E_d$ equals the identity I. Applying this sum to the vector (51) and simplifying the result using (5), we find the vector (51) equals

$$\sum_{i=0}^{d}(\theta_i-\theta_0)(\theta_i-\theta_1)\cdots(\theta_i-\theta_{j-1})E_i\eta_0^*. \tag{52}$$

Let L denote the matrix in $\mathrm{Mat}_{d+1}(\mathbf{K})$ with ij^{th} entry $(\theta_i-\theta_0)\cdots(\theta_i-\theta_{j-1})$ for $0 \leq i,j \leq d$. Apparently L is the transition matrix from the basis $[d^*0^*0d]$ to the basis $[d^*00^*d]$. By linear algebra, we obtain

$$A^{*g}L = LA^{*h}, \tag{53}$$

where we recall $g = d^*0^*0d$ and we abbreviate $h = d^*00^*d$. For $1 \leq i \leq d$, we compute the $i-1, i$ entry in (53). Since A^{*g} is tridiagonal, and since L is lower triangular, we find the $i-1, i$ entry of $A^{*g}L$ equals $A^{*g}_{i-1,i}L_{ii}$ or in other words

$$A^{*g}_{i-1,i}(\theta_i-\theta_0)(\theta_i-\theta_1)\cdots(\theta_i-\theta_{i-1}). \tag{54}$$

We mentioned above the matrix A^{*h} is given on the right in (30). Since A^{*h} is upper bidiagonal and since L is lower triangular, we find the $i-1, i$ entry of LA^{*h} equals $L_{i-1,i-1}A^{*h}_{i-1,i}$ or in other words

$$(\theta_{i-1}-\theta_0)(\theta_{i-1}-\theta_1)\cdots(\theta_{i-1}-\theta_{i-2})\varphi_i. \tag{55}$$

Equating (54) and (55), we obtain (50). Applying (50) to $\Phi^{\Downarrow}$ and using Theorem 6.2, we routinely find

$$A^{*g}_{i,i-1} = \phi_{d-i+1}\frac{(\theta_i-\theta_d)(\theta_i-\theta_{d-1})\cdots(\theta_i-\theta_{i+1})}{(\theta_{i-1}-\theta_d)(\theta_{i-1}-\theta_{d-1})\cdots(\theta_{i-1}-\theta_i)}$$

for $1 \leq i \leq d$. We have now proved our assertions for the first row of the second table. Applying these facts to the relatives of Φ and using Theorem 6.2, we obtain the remaining rows of the second table and all rows of the third table. $\square$

Summarizing the data from Theorem 11.2, we have the following.

Lemma 11.3. *Referring to Theorem* 11.2*, pick any* $g \in S_4$ *and consider the form of* A^g *and* A^{*g}*. Writing* $g = wxyz$*, this form is given as follows.*

$y \in \{0^*, d^*\}$	$z \in \{0^*, d^*\}$	A^g	A^{*g}
No	No	diagonal	irred. tridiagonal
Yes	No	lower bidiagonal	upper bidiagonal
No	Yes	upper bidiagonal	lower bidiagonal
Yes	Yes	irred. tridiagonal	diagonal

We remark the number of elements in S_4 satisfying each of the above four cases is $4, 8, 8, 4$*, respectively.*

Proof. Follows from the data in Theorem 11.2. $\square$

12. The eigenvalues and dual eigenvalues. Our next goal is to compute, for each ordered pair g, h of adjacent elements in S_4, the entries in the transition matrix from the basis $[g]$ to the basis $[h]$. In order to describe these entries, we make some comments about eigenvalues and define some expressions. In this section we focus on eigenvalues.

Let β denote a scalar in $\mathbf{K}$. Let d denote a nonnegative integer, and let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote a sequence of scalars taken from $\mathbf{K}$. We say this sequence is *β-recurrent* whenever $\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1}$ is independent of i for $1 \leq i \leq d-1$. Let Φ denote the Leonard system in Theorem 6.1. Then by condition (v) of that theorem, the eigenvalue sequence and the dual eigenvalue sequence of Φ are β-recurrent, where $\beta + 1$ is the common value of (35). These two sequences are the ones we wish to discuss in this section, but since what we have to say about them applies to all β-recurrent sequences, we keep things general.

We begin by mentioning some well-known formula concerning β-recurrent sequences. Recall $\tilde{\mathbf{K}}$ denotes the algebraic closure of the field $\mathbf{K}$.

Lemma 12.1. *Let d denote a nonnegative integer, and let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote a sequence of scalars taken from $\mathbf{K}$. Let β denote a scalar in $\mathbf{K}$, and assume $\sigma_0, \sigma_1, \ldots, \sigma_d$ is β-recurrent. Let q denote a nonzero scalar in $\tilde{\mathbf{K}}$ such that $q + q^{-1} = \beta$.*

(i) *Suppose $q \neq 1$, $q \neq -1$. Then there exist scalars a, b, c in $\tilde{\mathbf{K}}$ such that*

$$\sigma_i = a + bq^i + cq^{-i}, \quad 0 \leq i \leq d. \tag{56}$$

(ii) *Suppose $q = 1$. Then there exist scalars a, b, c in $\mathbf{K}$ such that*

$$\sigma_i = a + bi + ci(i-1)/2, \quad 0 \leq i \leq d.$$

(iii) *Suppose $q=-1$, and that the characteristic of $\mathbf{K}$ is not 2. Then there exist scalars a, b, c in $\mathbf{K}$ such that*

$$\sigma_i = a + b(-1)^i + ci(-1)^i, \quad 0 \le i \le d.$$

Referring to case (ii) above, if $\mathbf{K}$ has characteristic 2, we interpret the expression $i(i-1)/2$ as 0 if $i=0$ or $i=1 \pmod 4$, and as 1 if $i=2$ or $i=3 \pmod 4$.

Definition 12.2. Let q denote a nonzero scalar in $\tilde{\mathbf{K}}$, and let n denote an integer. We let $[n]_q$ denote the following scalar in $\tilde{\mathbf{K}}$.

First assume n is odd. In this case we define

$$[n]_q = \begin{cases} \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}} & \text{if } q \neq 1; \\ n & \text{if } q = 1. \end{cases} \tag{57}$$

We observe

$$[n]_q = q^{(n-1)/2} + q^{(n-3)/2} + \cdots + q^{(3-n)/2} + q^{(1-n)/2}, \quad \text{if } n > 0$$

and that $[-n]_q = -[n]_q$. For example,

$$[-5]_q = -q^2 - q - 1 - q^{-1} - q^{-2}, \quad [-3]_q = -q - 1 - q^{-1}, \quad [-1]_q = -1,$$
$$[1]_q = 1, \quad [3]_q = q + 1 + q^{-1}, \quad [5]_q = q^2 + q + 1 + q^{-1} + q^{-2}.$$

Next assume n is even. In this case we define

$$[n]_q = \begin{cases} \frac{q^{n/2}-q^{-n/2}}{q-q^{-1}} & \text{if } q \neq 1,\ q \neq -1; \\ n/2 & \text{if } q = 1; \\ (-1)^{n/2-1}n/2 & \text{if } q = -1. \end{cases} \tag{58}$$

We observe

$$[n]_q = q^{n/2-1} + q^{n/2-3} + \cdots + q^{3-n/2} + q^{1-n/2}, \quad \text{if } n \ge 0$$

and that $[-n]_q = -[n]_q$. For example,

$$[-6]_q = -q^2 - 1 - q^{-2}, \quad [-4]_q = -q - q^{-1}, \quad [-2]_q = -1, \quad [0]_q = 0,$$
$$[2]_q = 1, \quad [4]_q = q + q^{-1}, \quad [6]_q = q^2 + 1 + q^{-2}.$$

Referring to the cases $q = 1$, $q = -1$ of (58), if $\mathbf{K}$ has characteristic 2, we interpret $n/2$ as 1 if $n = 2 \pmod 4$ and as 0 if $n = 0 \pmod 4$.

We mention a handy recursion.

Lemma 12.3. *Let q denote a nonzero scalar in $\tilde{\mathbf{K}}$. Then for all integers n,*

$$(q + q^{-1})[n]_q = [n + 2]_q + [n - 2]_q. \tag{59}$$

Proof. Routine calculation using (57) and (58). □

Corollary 12.4. *Let q denote a nonzero element of $\tilde{\mathbf{K}}$ such that $q + q^{-1} \in \mathbf{K}$. Then $[n]_q \in \mathbf{K}$ for all integers n.*

Proof. The scalars $[0]_q$ and $[2]_q$ are contained in $\mathbf{K}$ since these equal 0 and 1, respectively. By this and a routine induction using Lemma 12.3, we find $[n]_q$ is contained in $\mathbf{K}$ for all even integers n. The scalars $[-1]_q$ and $[1]_q$ are contained in $\mathbf{K}$ since these equal -1 and 1, respectively. By this and a routine induction using Lemma 12.3, we find $[n]_q$ is contained in $\mathbf{K}$ for all odd integers n. □

Lemma 12.5. *Let d denote a nonnegative integer, and let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote a sequence of scalars taken from $\mathbf{K}$. Let β denote a scalar in $\mathbf{K}$, and assume $\sigma_0, \sigma_1, \ldots, \sigma_d$ is β-recurrent. Let q denote a nonzero scalar in $\tilde{\mathbf{K}}$ such that $q + q^{-1} = \beta$. Then for $0 \leq i, j, r, s \leq d$, we have*

$$[r - s]_q(\sigma_i - \sigma_j) = [i - j]_q(\sigma_r - \sigma_s), \tag{60}$$

provided $i + j = r + s$.

Proof. Let the integers i, j, r, s be given and assume $i+j = r+s$. First suppose $q \neq 1$, $q \neq -1$. Let n denote the common value of $i + j$, $r + s$, and for convenience set $e = q^{1/2} - q^{-1/2}$, (if n is odd), and $e = q - q^{-1}$ (if n is even). Observe $r - s$ and $r + s = n$ have the same parity, so by Definition 12.2,

$$\begin{aligned} [r - s]_q &= (q^{(r-s)/2} - q^{(s-r)/2})e^{-1} \\ &= (q^r - q^s)e^{-1}q^{-n/2}. \end{aligned} \tag{61}$$

Similarly

$$[i-j]_q = (q^i - q^j)e^{-1}q^{-n/2}. \tag{62}$$

By Lemma 12.1(i) there exist scalars a, b, c in $\tilde{\mathbf{K}}$ such that $\sigma_0, \sigma_1, \ldots, \sigma_d$ are given by (56). Observe

$$\begin{aligned} \sigma_i - \sigma_j &= b(q^i - q^j) + c(q^{-i} - q^{-j}) \\ &= (q^i - q^j)(b - cq^{-n}). \end{aligned} \tag{63}$$

Similarly

$$\sigma_r - \sigma_s = (q^r - q^s)(b - cq^{-n}). \tag{64}$$

Combining (61)–(64) we obtain (60). We have now proved the lemma for the case $q \neq 1$, $q \neq -1$. The proof for the cases $q = 1$, $q = -1$ is similar and omitted. $\square$

Let q denote a nonzero scalar in $\tilde{\mathbf{K}}$, and let r, s, t denote nonnegative integers. A bit later in the paper, we will define some expressions $[r, s, t]_q$ that make sense under the assumption $[i]_q \neq 0$ for $1 \leq i \leq r + s + t$. We comment on this assumption. First observe $[1]_q$ and $[2]_q$ are nonzero, since these scalars both equal 1. For $i \geq 3$, it could happen that $[i]_q = 0$; we explain how in the next result.

Lemma 12.6. *Let q denote a nonzero scalar in $\tilde{\mathbf{K}}$, and let i denote a positive integer. Then* (i)–(vi) *hold below.*

(i) *Assume $q \neq 1$, $q \neq -1$. Then $[i]_q = 0$ if and only if $q^i = 1$.*

(ii) *Assume $q = 1$ and that $\mathbf{K}$ has characteristic 0. Then $[i]_q \neq 0$.*

(iii) *Assume $q = 1$ and that $\mathbf{K}$ has characteristic p, $p \geq 3$. Then $[i]_q = 0$ if and only if p divides i.*

(iv) *Assume $q = -1$ and that $\mathbf{K}$ has characteristic 0. Then $[i]_q \neq 0$.*

(v) *Assume $q = -1$ and that $\mathbf{K}$ has characteristic p, $p \geq 3$. Then $[i]_q = 0$ if and only if $2p$ divides i.*

(vi) *Assume $q = 1$ and that $\mathbf{K}$ has characteristic 2. Then $[i]_q = 0$ if and only if 4 divides i.*

Proof. First assume $q \neq 1$, $q \neq -1$. Then $[i]_q$ is a nonzero scalar multiple of $q^i - 1$ by Definition 12.2 and assertion (i) follows. Next assume $q = 1$ and that the characteristic of $\mathbf{K}$ is not 2. Then the sequence $[1]_q, [2]_q, \ldots$ is given by $1, 1, 3, 2, 5, 3, 7, 4 \ldots$ and assertions (ii) and (iii) follow. Next assume $q = -1$ and that the characteristic of $\mathbf{K}$ is not 2. Then the sequence $[1]_q, [2]_q, \ldots$ is given by $1, 1, -1, -2, 1, 3, -1, -4, \ldots$ and assertions (iv) and (v) follow. Now assume $q = 1$ and that $\mathbf{K}$ has characteristic 2. Then the sequence $[1]_q, [2]_q, \ldots$ is given by $1, 1, 1, 0, 1, 1, 1, 0, \ldots$ and assertion (vi) follows. $\square$

Lemma 12.7. *Let d denote an integer at least 3. Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote a sequence of distinct scalars taken from $\mathbf{K}$, and assume*

$$\frac{\sigma_{i-2} - \sigma_{i+1}}{\sigma_{i-1} - \sigma_i} \tag{65}$$

is independent of i for $2 \leq i \leq d-1$. Let q denote a nonzero scalar in $\tilde{\mathbf{K}}$ such that $q + q^{-1} + 1$ equals the common value of (65). *Then $[i]_q \neq 0$ for $1 \leq i \leq d$.*

Proof. Abbreviate $\beta = q + q^{-1}$, and observe $\sigma_0, \sigma_1, \ldots, \sigma_d$ is β-recurrent. First suppose $q \neq 1$ and $q \neq -1$. Then for $1 \leq i \leq d$ we have $q^i \neq 1$; otherwise $\sigma_i = \sigma_0$ by Lemma 12.1(i). The result now follows by Lemma 12.6(i). Next suppose $q = 1$ and that $\mathbf{K}$ has characteristic 0. Then the result holds by Lemma 12.6(ii). Next suppose $q = 1$ and that $\mathbf{K}$ has characteristic $p, p \geq 3$. Then $d < p$; otherwise $\sigma_p = \sigma_0$ in view of Lemma 12.1(ii). The result now follows by Lemma 12.6(iii). Next suppose $q = -1$ and that $\mathbf{K}$ has characteristic 0. Then the result holds by Lemma 12.6(iv). Next suppose $q = -1$ and that $\mathbf{K}$ has characteristic $p, p \geq 3$. Then $d < 2p$; otherwise, $\sigma_{2p} = \sigma_0$ in view of Lemma 12.1(iii). The result now follows by Lemma 12.6(v). Now suppose $q = 1$ and that $\mathbf{K}$ has characteristic 2. Then $d \leq 3$; otherwise $\sigma_4 = \sigma_0$ by Lemma 12.1(iii) and the comment at the end of that lemma. The result now follows by Lemma 12.6(vi). $\square$

Corollary 12.8. *Let Φ denote the Leonard system in* (9) *and assume $d \geq 3$. Let q denote a nonzero scalar in $\tilde{\mathbf{K}}$ such that $q + q^{-1} + 1$ equals the common value of* (35). *Then $[i]_q \neq 0$ for $1 \leq i \leq d$.*

Proof. Apply Lemma 12.7 to the eigenvalue sequence of Φ. $\square$

We finish this section with a definition.

Definition 12.9. Let q denote a nonzero scalar in $\tilde{\mathbf{K}}$. For each nonnegative integer n we define

$$[n]!_q = [1]_q[2]_q \cdots [n]_q. \tag{66}$$

We interpret $[0]!_q = 1$.

13. The scalars $[r,s,t]_q$. A bit later in the paper we will compute, for each ordered pair g, h of adjacent elements in S_4, the entries in the transition matrix from the basis $[g]$ to the basis $[h]$. Among the entries in these transition matrices, we will encounter an expression that occurs so often we will give it a name. The details are in the following definition.

Definition 13.1. Let q denote a nonzero scalar in $\tilde{\mathbf{K}}$ and let r, s, t denote nonnegative integers. We define the expressions $(r,s,t)_q$ and $[r,s,t]_q$ as follows. We set

$$(r,s,t)_q = \begin{cases} q + q^{-1} + 2 & \text{if each of } r,s,t \text{ is odd;} \\ 1 & \text{if at least one of } r,s,t \text{ is even.} \end{cases} \tag{67}$$

Next assume that $[i]_q \neq 0$ for $1 \leq i \leq r+s+t$. Then we set

$$[r,s,t]_q = \frac{[r+s]!_q[r+t]!_q[s+t]!_q(r,s,t)_q}{[r]!_q[s]!_q[t]!_q[r+s+t]!_q}. \tag{68}$$

We remark $[r,s,t]_q \in \mathbf{K}$ provided $q + q^{-1} \in \mathbf{K}$. Moreover, $[r,s,t]_q = 1$ if at least one of r,s,t equals 0.

Referring to the above definition, to get a better appreciation for $[r,s,t]_q$ we now evaluate the expression on the right in (68) using Definition 12.2. To express our results, we use the following notation. For all $a, q \in \tilde{\mathbf{K}}$, we define

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n = 0,1,2,\ldots$$

and interpret $(a;q)_0 = 1$.

Lemma 13.2. *Let q denote a nonzero scalar in $\tilde{\mathbf{K}}$, let r, s, t denote nonnegative integers, and assume $[i]_q \neq 0$ for $1 \leq i \leq r+s+t$.*

(i) *Suppose $q \neq 1$, $q \neq -1$. Then*

$$[r,s,t]_q = \frac{(q;q)_{r+s}(q;q)_{r+t}(q;q)_{s+t}}{(q;q)_r(q;q)_s(q;q)_t(q;q)_{r+s+t}}. \tag{69}$$

(ii) *Suppose $q = 1$ and that the characteristic of $\mathbf{K}$ is not 2. Then*

$$[r,s,t]_q = \frac{(r+s)!(r+t)!(s+t)!}{r!s!t!(r+s+t)!}. \tag{70}$$

(iii) *Suppose $q = -1$ and that the characteristic of $\mathbf{K}$ is not 2. If each of r, s, t is odd, then $[r,s,t]_q = 0$. If at least one of r, s, t is even, then*

$$[r,s,t]_q = \frac{\lfloor (r+s)/2 \rfloor ! \lfloor (r+t)/2 \rfloor ! \lfloor (s+t)/2 \rfloor !}{\lfloor r/2 \rfloor ! \lfloor s/2 \rfloor ! \lfloor t/2 \rfloor ! \lfloor (r+s+t)/2 \rfloor !}. \tag{71}$$

The expression $\lfloor n \rfloor$ denotes the greatest integer less than or equal to n.

(iv) *Suppose $q = 1$ and that $\mathbf{K}$ has characteristic 2. Recall in this case $r+s+t \leq 3$ by Lemma 12.6(vi). If each of r, s, t equals 1, then $[r,s,t]_q = 0$. If at least one of r, s, t equals 0 then $[r,s,t]_q = 1$.*

Concerning the expressions on the right in (69), (70), (71), the denominator is nonzero by Lemma 12.6.

Proof. Evaluate (68) using Definition 12.2, (66) and (67). □

We will need the following identity.

Lemma 13.3. *Let q denote a nonzero scalar in $\tilde{\mathbf{K}}$, and let r, s, t denote positive integers. Assume $[i]_q \neq 0$ for $1 \leq i < r+s+t$. Then with reference to Definition 13.1 we have*

$$[r-t]_q[r+t]_q^{-1}[r,s-1,t]_q = [r-1,s,t]_q - [r,s,t-1]_q. \tag{72}$$

Proof. First assume $q \neq 1$ and $q \neq -1$. By Definition 12.2 and since the integers $r+t, r-t$ have the same parity, we find

$$\frac{[r-t]_q}{[r+t]_q} = \frac{q^{(r-t)/2} - q^{(t-r)/2}}{q^{(r+t)/2} - q^{-(r+t)/2}} = \frac{q^t - q^r}{1 - q^{r+t}}. \tag{73}$$

Using (69), we obtain

$$[r-1, s, t]_q = x(1-q^{s+t})(1-q^r), \tag{74}$$

$$[r, s-1, t]_q = x(1-q^{r+t})(1-q^s), \tag{75}$$

$$[r, s, t-1]_q = x(1-q^{r+s})(1-q^t), \tag{76}$$

where

$$x = \frac{(q;q)_{r+s-1}(q;q)_{r+t-1}(q;q)_{s+t-1}}{(q;q)_r(q;q)_s(q;q)_t(q;q)_{r+s+t-1}}.$$

One readily verifies

$$(q^t - q^r)(1-q^s) = (1-q^{s+t})(1-q^r) - (1-q^{r+s})(1-q^t). \tag{77}$$

Multiplying both sides of (77) by x and evaluating the result using (73)–(76), we routinely obtain (72). We have now proved the result for the case $q \neq 1$, $q \neq -1$. The proof for the cases $q = 1$, $q = -1$ are similar and omitted. □

14. The scalars $\varepsilon_0, \varepsilon_d, \varepsilon_0^*, \varepsilon_d^*$. In the next section we will compute, for each ordered pair g, h of adjacent elements in S_4, the entries in the transition matrix from the basis $[g]$ to the basis $[h]$. Recall our 24 bases are constructed using four vectors $\eta_0, \eta_d, \eta_0^*, \eta_d^*$, and each of these vectors is determined only up to multiplication by a nonzero scalar. To account for this we introduce four scalars $\varepsilon_0, \varepsilon_d, \varepsilon_0^*, \varepsilon_d^*$.

For convenience, we make the following definition.

Definition 14.1. Let Φ denote the Leonard system in (9). We define

$$\tilde{E}_0 = (A - \theta_1 I)(A - \theta_2 I) \cdots (A - \theta_d I), \tag{78}$$

$$\tilde{E}_d = (A - \theta_0 I)(A - \theta_1 I) \cdots (A - \theta_{d-1} I), \tag{79}$$

$$\tilde{E}_0^* = (A^* - \theta_1^* I)(A^* - \theta_2^* I) \cdots (A^* - \theta_d^* I), \tag{80}$$

$$\tilde{E}_d^* = (A^* - \theta_0^* I)(A^* - \theta_1^* I) \cdots (A^* - \theta_{d-1}^* I), \tag{81}$$

where the θ_i, θ^*_i are from Definition 4.1.

Lemma 14.2. *Let Φ denote the Leonard system in* (9). *Then with reference to Definition* 14.1*,*

(i) $\tilde{E}_0 = E_0(\theta_0 - \theta_1)(\theta_0 - \theta_2)\cdots(\theta_0 - \theta_d)$,

(ii) $\tilde{E}_d = E_d(\theta_d - \theta_0)(\theta_d - \theta_1)\cdots(\theta_d - \theta_{d-1})$,

(iii) $\tilde{E}^*_0 = E^*_0(\theta^*_0 - \theta^*_1)(\theta^*_0 - \theta^*_2)\cdots(\theta^*_0 - \theta^*_d)$,

(iv) $\tilde{E}^*_d = E^*_d(\theta^*_d - \theta^*_0)(\theta^*_d - \theta^*_1)\cdots(\theta^*_d - \theta^*_{d-1})$.

Proof. To get (i) set $i = 0$ in (4) and compare the result with (78). Assertions (ii)–(iv) are similarly proved. □

Lemma 14.3. *Let Φ denote the Leonard system in* (9)*. Let g denote the element d^*00^*d of S_4 and recall by Lemma* 9.3 *that $[g]$ is a Φ-split basis. For $0 \leq i, j \leq d$, the ij^{th} entry of the matrices $\tilde{E}^g_0, \tilde{E}^g_d, \tilde{E}^{*g}_0, \tilde{E}^{*g}_d$ are given as follows.*

(i) *The ij^{th} entry of $\tilde{E}^g_0$ is*

$$(\theta_0 - \theta_{i+1})(\theta_0 - \theta_{i+2})\cdots(\theta_0 - \theta_d)$$

if $j = 0$ and 0 if $j \neq 0$.

(ii) *The ij^{th} entry of $\tilde{E}^g_d$ is*

$$(\theta_d - \theta_0)(\theta_d - \theta_1)\cdots(\theta_d - \theta_{j-1})$$

if $i = d$ and 0 if $i \neq d$.

(iii) *The ij^{th} entry of $\tilde{E}^{*g}_0$ is*

$$(\theta^*_0 - \theta^*_{j+1})(\theta^*_0 - \theta^*_{j+2})\cdots(\theta^*_0 - \theta^*_d)\varphi_1\varphi_2\cdots\varphi_j$$

if $i = 0$ and 0 if $i \neq 0$.

(iv) *The ij^{th} entry of $\tilde{E}^{*g}_d$ is*

$$(\theta^*_d - \theta^*_0)(\theta^*_d - \theta^*_1)\cdots(\theta^*_d - \theta^*_{i-1})\varphi_{i+1}\varphi_{i+2}\cdots\varphi_d$$

if $j = d$ *and* 0 *if* $j \neq d$.

Proof. The entries of $E_0^g, E_d^g, E_0^{*g}, E_d^{*g}$ are given in [**44**, Theorem 4.8]. Using these entries and Lemma 14.2, we routinely obtain the assertions of the present lemma. □

For notational convenience, we introduce the following notation.

Definition 14.4. Let Φ denote the Leonard system in (9). We set

$$\varphi := \varphi_1 \varphi_2 \cdots \varphi_d, \quad \phi := \phi_1 \phi_2 \cdots \phi_d, \tag{82}$$

where $\varphi_1, \varphi_2, \ldots, \varphi_d$ denotes the first split sequence of Φ and where $\phi_1, \phi_2, \ldots, \phi_d$ denotes the second split sequence of Φ. We observe by Theorem 6.1(i) that $\varphi \neq 0$, $\phi \neq 0$.

Lemma 14.5. *Let* Φ *denote the Leonard system in* (9). *Then with reference to Definition* 14.1, *the trace of each of* $\tilde{E}_d \tilde{E}_0^*$, $\tilde{E}_0 \tilde{E}_d^*$ *equals* φ. *Moreover, the trace of each* $\tilde{E}_0 \tilde{E}_0^*$, $\tilde{E}_d \tilde{E}_d^*$ *equals* ϕ.

Proof. Using the data in Lemma 14.3, we routinely find the trace of $\tilde{E}_d \tilde{E}_0^*$ equals φ. To obtain the remaining assertions, apply this result to the relatives of Φ and use Theorem 6.2. □

Lemma 14.6. *Let* Φ *denote the Leonard system in (9). Then with reference to Definition* 14.1,

$$\tilde{E}_0^* \tilde{E}_d \tilde{E}_0^* = \varphi \tilde{E}_0^*, \quad \tilde{E}_d \tilde{E}_0^* \tilde{E}_d = \varphi \tilde{E}_d, \tag{83}$$

$$\tilde{E}_0 \tilde{E}_d^* \tilde{E}_0 = \varphi \tilde{E}_0, \quad \tilde{E}_d^* \tilde{E}_0 \tilde{E}_d^* = \varphi \tilde{E}_d^*, \tag{84}$$

$$\tilde{E}_0 \tilde{E}_0^* \tilde{E}_0 = \phi \tilde{E}_0, \quad \tilde{E}_0^* \tilde{E}_0 \tilde{E}_0^* = \phi \tilde{E}_0^*, \tag{85}$$

$$\tilde{E}_d \tilde{E}_d^* \tilde{E}_d = \phi \tilde{E}_d, \quad \tilde{E}_d^* \tilde{E}_d \tilde{E}_d^* = \phi \tilde{E}_d^*. \tag{86}$$

Proof. We first prove the equation on the left in (83). Since E_0^* is a rank one idempotent, and since $\tilde{E}_0^*$ is a nonzero scalar multiple of E_0^*, a scalar $\alpha \in \mathbf{K}$ exists such that $\tilde{E}^*_0 \tilde{E}_d \tilde{E}^*_0 = \alpha \tilde{E}^*_0$. We show $\alpha = \varphi$. We mentioned $\tilde{E}^*_0$ is a nonzero scalar multiple of E^*_0 so

$$E^*_0 \tilde{E}_d \tilde{E}^*_0 = \alpha E^*_0. \tag{87}$$

We take the trace of each side of (87). Observe the trace of E^*_0 equals 1, so the trace of the right side of (87) equals α. Since XY and YX have the same trace and, using $\tilde{E}^*_0 E^*_0 = \tilde{E}^*_0$, we find in view of Lemma 14.5 that the trace of the left side of (87) equals φ. Apparently $\alpha = \varphi$ and this implies the equation on the left in (83). Applying this result to the relatives of Φ, we obtain the remaining assertions. □

Lemma 14.7. *Let Φ denote the Leonard system in* (9)*. Then with reference to Definition* 14.1*, we have the following:*

$$\tilde{E}^*_d \tilde{E}_0 \tilde{E}^*_0 = \tilde{E}^*_d \tilde{E}_d \tilde{E}^*_0, \quad \tilde{E}_d \tilde{E}^*_0 \tilde{E}_0 = \tilde{E}_d \tilde{E}^*_d \tilde{E}_0, \tag{88}$$

$$\tilde{E}_0 \tilde{E}^*_0 \tilde{E}_d = \tilde{E}_0 \tilde{E}^*_d \tilde{E}_d, \quad \tilde{E}^*_0 \tilde{E}_0 \tilde{E}^*_d = \tilde{E}^*_0 \tilde{E}_d \tilde{E}^*_d. \tag{89}$$

Proof. The equation on the left in (88) is readily obtained using the matrix representations given in Lemma 14.3. Applying this equation to the relatives of Φ, we obtain the remaining equations in (88), (89). □

Lemma 14.8. *Let Φ denote the Leonard system in* (9)*, and let V denote the irreducible left $\mathcal{A}$-module. Let $\eta_0, \eta_d, \eta^*_0, \eta^*_d$ denote nonzero vectors in V that satisfy* (40)*. Then there exist nonzero scalars $\varepsilon_0, \varepsilon_d, \varepsilon^*_0, \varepsilon^*_d$ in $\mathbf{K}$ such that*

$$\tilde{E}_d \eta^*_0 / \varepsilon^*_0 = \eta_d / \varepsilon_d, \quad \tilde{E}_d \eta^*_d / \varepsilon^*_d = \eta_d / \varepsilon_d, \tag{90}$$

$$\tilde{E}^*_0 \eta_0 / \varepsilon_0 = \eta^*_0 / \varepsilon^*_0, \quad \tilde{E}^*_0 \eta_d / \varepsilon_d = \varphi \eta^*_0 / \varepsilon^*_0, \tag{91}$$

$$\tilde{E}^*_d \eta_0 / \varepsilon_0 = \eta^*_d / \varepsilon^*_d, \quad \tilde{E}^*_d \eta_d / \varepsilon_d = \phi \eta^*_d / \varepsilon^*_d, \tag{92}$$

$$\tilde{E}_0 \eta^*_0 / \varepsilon^*_0 = \phi \eta_0 / \varepsilon_0, \quad \tilde{E}_0 \eta^*_d / \varepsilon^*_d = \varphi \eta_0 / \varepsilon_0. \tag{93}$$

Proof. Let ε_0^* denote an arbitrary nonzero scalar in $\mathbf{K}$. To obtain ε_0, consider the basis $[d^*d00^*]$ from the table of Theorem 9.1, row 10. Using (78), we recognize the vector $\tilde{E}_0\eta_0^*$ is the 0^{th} vector in this basis. By Theorem 9.1, we find $\tilde{E}_0\eta_0^*$ is a basis for E_0V. By the construction η_0 is a basis for E_0V so $\tilde{E}_0\eta_0^*$ is a nonzero scalar multiple of η_0. Apparently, a nonzero scalar $\varepsilon_0 \in \mathbf{K}$ exists that satisfies the equation on the left in (93). Similarly, nonzero scalars $\varepsilon_d, \varepsilon_d^*$ in $\mathbf{K}$ exist that satisfy the equations on the left in (90), (92), respectively. To obtain the equation on the right in (92), apply the equation on the left in (88) to η_0^*/ε_0^* and evaluate the result using $E_0^*\eta_0^* = \eta_0^*$, Lemma 14.2(iii), and the equations on the left in (90), (92) and (93). To obtain the equation on the left in (91), apply the equation on the right in (85) to η_0^*/ε_0^*, and evaluate the result using $E_0^*\eta_0^* = \eta_0^*$, Lemma 14.2(iii) and the equation on the left in (93). The equations on the right in (90), (91) and (93) are similarly obtained. □

Note 14.9. The scalars $\varepsilon_0, \varepsilon_d, \varepsilon_0^*, \varepsilon_d^*$ from Lemma 14.8 are "free" in the following sense. Let Φ denote the Leonard system in (9), and let V denote the irreducible left $\mathcal{A}$-module. Let $\varepsilon_0, \varepsilon_d, \varepsilon_0^*, \varepsilon_d^*$ denote arbitrary nonzero scalars in $\mathbf{K}$. Then nonzero vectors $\eta_0, \eta_d, \eta_0^*, \eta_d^*$ exist in V that satisfy (40) and (90)–(93).

Note 14.10. The reader may notice a certain lack of symmetry in the definition of $\varepsilon_0, \varepsilon_d, \varepsilon_0^*, \varepsilon_d^*$. We accept this asymmetry to avoid introducing the square roots of φ and ϕ. We remark that these square roots may not be in $\mathbf{K}$. To display the underlying symmetry in (90)–(93) make the following change of variables:

$$\varepsilon_0 = \varepsilon_0', \quad \varepsilon_d = \varepsilon_d'\varphi^{-1/2}\phi^{-1/2} \quad \varepsilon_0^* = \varepsilon_0^{*\prime}\phi^{-1/2}, \quad \varepsilon_d^* = \varepsilon_d^{*\prime}\varphi^{-1/2}.$$

The following equations will be useful.

Lemma 14.11. *Let Φ denote the Leonard system in* (9)*, and let V denote the irreducible left $\mathcal{A}$-module. Let $\eta_0, \eta_d, \eta_0^*, \eta_d^*$ denote nonzero vectors in V that satisfy* (40)*. Let the scalars $\varepsilon_0, \varepsilon_d, \varepsilon_0^*, \varepsilon_d^*$ be as in*

Lemma 14.8. *Then*

$$E_d\eta_d^*/\varepsilon_d^* = E_d\eta_0^*/\varepsilon_0^*, \quad E_0^*\eta_d/\varepsilon_d = \varphi E_0^*\eta_0/\varepsilon_0, \tag{94}$$

$$E_d^*\eta_d/\varepsilon_d = \phi E_d^*\eta_0/\varepsilon_0, \quad E_0\eta_d^*/\varepsilon_d^* = \varphi/\phi E_0\eta_0^*/\varepsilon_0^*. \tag{95}$$

Proof. First consider the equation on the left in (94). Comparing the two equations in (90), we find $\tilde{E}_d\eta_d^*/\varepsilon_d^* = \tilde{E}_d\eta_0^*/\varepsilon_0^*$. Recall E_d is a scalar multiple of $\tilde{E}_d$ so $E_d\eta_d^*/\varepsilon_d^* = E_d\eta_0^*/\varepsilon_0^*$. We now have the equation on the left in (94). The remaining equations in (94) and (95) are similarly proved. □

15. The 24 bases; transition matrices. Let Φ denote the Leonard system in (9), and let V denote the irreducible left $\mathcal{A}$-module. For each element $g \in S_4$, we displayed in Theorem 9.1 a basis for V, denoted $[g]$. In this section we compute, for each ordered pair g, h of adjacent elements of S_4, the entries in the transition matrix from the basis $[g]$ to the basis $[h]$.

We mention a few points from linear algebra. In line (41) we recalled the notion of a transition matrix. We now recall the closely related concept of an intertwining matrix. Let g, h denote elements of S_4, and consider the corresponding bases $[g], [h]$ of V. By an *intertwining matrix* from $[g]$ to $[h]$, we mean a nonzero matrix $S \in \mathrm{Mat}_{d+1}(\mathbf{K})$ satisfying

$$X^g S = S X^h, \quad \forall X \in \mathcal{A}.$$

We observe a matrix in $\mathrm{Mat}_{d+1}(\mathbf{K})$ is an intertwining matrix from $[g]$ to $[h]$ if and only if it is a nonzero scalar multiple of the transition matrix from $[g]$ to $[h]$.

The following matrix will play a role in our discussion. We let Z denote the matrix in $\mathrm{Mat}_{d+1}(\mathbf{K})$ with entries

$$Z_{ij} = \begin{cases} 1 & \text{if } i+j=d; \\ 0 & \text{if } i+j \neq d, \end{cases} \quad 0 \leq i, j \leq d. \tag{96}$$

We observe $Z^2 = I$.

Lemma 15.1. *Let Φ denote the Leonard system in* (9)*, and let g, h denote elements in S_4. Then for all $S \in \mathrm{Mat}_{d+1}(\mathbf{K})$, the following are equivalent.*

(i) *S is an intertwining matrix from $[g]$ to $[h]$.*

(ii) *S is nonzero and both*

$$A^g S = S A^h, \quad A^{*g} S = S A^{*h}. \tag{97}$$

Proof. The implication (i) $\rightarrow$ (ii) is clear, so consider the implication (ii) $\rightarrow$ (i). Let T denote the transition matrix from $[g]$ to $[h]$. We show S is a nonzero scalar multiple of T. Since T is the transition matrix from $[g]$ to $[h]$, it is an intertwining matrix from $[g]$ to $[h]$. Therefore

$$A^g T = T A^h, \quad A^{*g} T = T A^{*h}. \tag{98}$$

Combining (97) and (98), we find ST^{-1} commutes with both A^g and A^{*g}. We mentioned the map $X \rightarrow X^g$ from $\mathcal{A}$ to $\mathrm{Mat}_{d+1}(\mathbf{K})$ is an isomorphism of $\mathbf{K}$-algebras. Combining this with our previous comment and using Corollary 3.3, we see ST^{-1} is a scalar multiple of the identity. Denoting this scalar by α we have $S = \alpha T$. We observe $\alpha \neq 0$ since $S \neq 0$. Apparently S is a nonzero scalar multiple of T, so S is an intertwining matrix from $[g]$ to $[h]$. □

Theorem 15.2. *Let Φ denote the Leonard system in* (9)*. With reference to Definition* 7.1*, let $wxyz$ denote an element of S_4, and consider the transition matrices from the basis $[wxyz]$ to the bases*

$$[xwyz], \quad [wyxz], \quad [wxzy]. \tag{99}$$

The first and second transition matrices are diagonal and lower triangular, respectively, and their entries are given in the following tables. The third transition matrix is the matrix Z from (96)*. In the tables below, $\theta_0, \theta_1, \ldots, \theta_d$ (respectively $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$) denotes the eigenvalue sequence (respectively dual eigenvalue sequence) for Φ. Moreover, $\varphi_1, \varphi_2, \ldots, \varphi_d$ (respectively $\phi_1, \phi_2, \ldots, \phi_d$) denotes the first split* *sequence (respectively second split sequence) for* Φ. *The scalars* φ, ϕ *are from* (82), *and the scalars* $\varepsilon_0, \varepsilon_d, \varepsilon_0^*, \varepsilon_d^*$ *are from Lemma* 14.8.

$wxyz$	$[wxyz] \to [xwyz]$ ii entry	$[wxyz] \to [wyxz]$ ij entry $(i \geq j)$
d^*00^*d	$\frac{1}{\varphi_1 \cdots \varphi_i} \frac{\varepsilon_d \varphi}{\varepsilon_0^*}$	$\frac{1}{(\theta_j - \theta_0) \cdots (\theta_j - \theta_{j-1})} \frac{1}{(\theta_j - \theta_{j+1}) \cdots (\theta_j - \theta_i)}$
$0d^*0^*d$	$\varphi_1 \cdots \varphi_i \frac{\varepsilon_0^*}{\varepsilon_d \varphi}$	$(\theta_d^* - \theta_0^*) \cdots (\theta_d^* - \theta_{i-j-1}^*)[j, i-j, d-i]_q$
d^*0d0^*	$\varphi_d \cdots \varphi_{d-i+1} \frac{\varepsilon_d}{\varepsilon_0^*}$	$(\theta_0 - \theta_d) \cdots (\theta_0 - \theta_{d-i+j+1})[j, i-j, d-i]_q$
$0d^*d0^*$	$\frac{1}{\varphi_d \cdots \varphi_{d-i+1}} \frac{\varepsilon_0^*}{\varepsilon_d}$	$\frac{1}{(\theta_{d-j}^* - \theta_d^*) \cdots (\theta_{d-j}^* - \theta_{d-j+1}^*)} \frac{1}{(\theta_{d-j}^* - \theta_{d-j-1}^*) \cdots (\theta_{d-j}^* - \theta_{d-i}^*)}$
$d0^*0d^*$	$\frac{1}{\varphi_1 \cdots \varphi_i} \frac{\varepsilon_d^* \varphi}{\varepsilon_0}$	$\frac{1}{(\theta_j^* - \theta_0^*) \cdots (\theta_j^* - \theta_{j-1}^*)} \frac{1}{(\theta_j^* - \theta_{j+1}^*) \cdots (\theta_j^* - \theta_i^*)}$
0^*d0d^*	$\varphi_1 \cdots \varphi_i \frac{\varepsilon_0}{\varepsilon_d^* \varphi}$	$(\theta_d - \theta_0) \cdots (\theta_d - \theta_{i-j-1})[j, i-j, d-i]_q$
$d0^*d^*0$	$\varphi_d \cdots \varphi_{d-i+1} \frac{\varepsilon_d^*}{\varepsilon_0}$	$(\theta_0^* - \theta_d^*) \cdots (\theta_0^* - \theta_{d-i+j+1}^*)[j, i-j, d-i]_q$
0^*dd^*0	$\frac{1}{\varphi_d \cdots \varphi_{d-i+1}} \frac{\varepsilon_0}{\varepsilon_d^*}$	$\frac{1}{(\theta_{d-j} - \theta_d) \cdots (\theta_{d-j} - \theta_{d-j+1})} \frac{1}{(\theta_{d-j} - \theta_{d-j-1}) \cdots (\theta_{d-j} - \theta_{d-i})}$
dd^*00^*	$\frac{1}{\phi_d \cdots \phi_{d-i+1}} \frac{\varepsilon_0^* \phi}{\varepsilon_0}$	$\frac{1}{(\theta_{d-j}^* - \theta_d^*) \cdots (\theta_{d-j}^* - \theta_{d-j+1}^*)} \frac{1}{(\theta_{d-j}^* - \theta_{d-j-1}^*) \cdots (\theta_{d-j}^* - \theta_{d-i}^*)}$
d^*d00^*	$\phi_d \cdots \phi_{d-i+1} \frac{\varepsilon_0}{\varepsilon_0^* \phi}$	$(\theta_d - \theta_0) \cdots (\theta_d - \theta_{i-j-1})[j, i-j, d-i]_q$
dd^*0^*0	$\phi_1 \cdots \phi_i \frac{\varepsilon_0^*}{\varepsilon_0}$	$(\theta_d^* - \theta_0^*) \cdots (\theta_d^* - \theta_{i-j-1}^*)[j, i-j, d-i]_q$
d^*d0^*0	$\frac{1}{\phi_1 \cdots \phi_i} \frac{\varepsilon_0}{\varepsilon_0^*}$	$\frac{1}{(\theta_{d-j} - \theta_d) \cdots (\theta_{d-j} - \theta_{d-j+1})} \frac{1}{(\theta_{d-j} - \theta_{d-j-1}) \cdots (\theta_{d-j} - \theta_{d-i})}$
00^*dd^*	$\frac{1}{\phi_1 \cdots \phi_i} \frac{\varepsilon_d^*}{\varepsilon_d}$	$\frac{1}{(\theta_j^* - \theta_0^*) \cdots (\theta_j^* - \theta_{j-1}^*)} \frac{1}{(\theta_j^* - \theta_{j+1}^*) \cdots (\theta_j^* - \theta_i^*)}$
0^*0dd^*	$\phi_1 \cdots \phi_i \frac{\varepsilon_d}{\varepsilon_d^*}$	$(\theta_0 - \theta_d) \cdots (\theta_0 - \theta_{d-i+j+1})[j, i-j, d-i]_q$
00^*d^*d	$\phi_d \cdots \phi_{d-i+1} \frac{\varepsilon_d^*}{\varepsilon_d \phi}$	$(\theta_0^* - \theta_d^*) \cdots (\theta_0^* - \theta_{d-i+j+1}^*)[j, i-j, d-i]_q$
0^*0d^*d	$\frac{1}{\phi_d \cdots \phi_{d-i+1}} \frac{\varepsilon_d \phi}{\varepsilon_d^*}$	$\frac{1}{(\theta_j - \theta_0) \cdots (\theta_j - \theta_{j-1})} \frac{1}{(\theta_j - \theta_{j+1}) \cdots (\theta_j - \theta_i)}$

In the above table, q denotes a scalar in the algebraic closure of $\mathbf{K}$ such that $q + q^{-1} + 1$ is the common value of (35).

$wxyz$	$[wxyz] \to [xwyz]$ ii entry	$[wxyz] \to [wyxz]$ ij entry $(i \geq j)$
d^*0^*0d	$\frac{\phi_d \cdots \phi_{d-i+1}}{\varphi_1 \cdots \varphi_i} \frac{\varepsilon^*_d \varphi}{\varepsilon^*_0 \phi}$	$(\theta_i - \theta_0) \cdots (\theta_i - \theta_{j-1})$
0^*d^*0d	$\frac{\varphi_1 \cdots \varphi_i}{\phi_d \cdots \phi_{d-i+1}} \frac{\varepsilon^*_0 \phi}{\varepsilon^*_d \varphi}$	$(\theta_i - \theta_0) \cdots (\theta_i - \theta_{j-1})$
d^*0^*d0	$\frac{\varphi_d \cdots \varphi_{d-i+1}}{\phi_1 \cdots \phi_i} \frac{\varepsilon^*_d}{\varepsilon^*_0}$	$(\theta_{d-i} - \theta_d) \cdots (\theta_{d-i} - \theta_{d-j+1})$
0^*d^*d0	$\frac{\phi_1 \cdots \phi_i}{\varphi_d \cdots \varphi_{d-i+1}} \frac{\varepsilon^*_0}{\varepsilon^*_d}$	$(\theta_{d-i} - \theta_d) \cdots (\theta_{d-i} - \theta_{d-j+1})$
$d00^*d^*$	$\frac{\phi_1 \cdots \phi_i}{\varphi_1 \cdots \varphi_i} \frac{\varepsilon_d \varphi}{\varepsilon_0}$	$(\theta^*_i - \theta^*_0) \cdots (\theta^*_i - \theta^*_{j-1})$
$0d0^*d^*$	$\frac{\varphi_1 \cdots \varphi_i}{\phi_1 \cdots \phi_i} \frac{\varepsilon_0}{\varepsilon_d \varphi}$	$(\theta^*_i - \theta^*_0) \cdots (\theta^*_i - \theta^*_{j-1})$
$d0d^*0^*$	$\frac{\varphi_d \cdots \varphi_{d-i+1}}{\phi_d \cdots \phi_{d-i+1}} \frac{\varepsilon_d \phi}{\varepsilon_0}$	$(\theta^*_{d-i} - \theta^*_d) \cdots (\theta^*_{d-i} - \theta^*_{d-j+1})$
$0dd^*0^*$	$\frac{\phi_d \cdots \phi_{d-i+1}}{\varphi_d \cdots \varphi_{d-i+1}} \frac{\varepsilon_0}{\varepsilon_d \phi}$	$(\theta^*_{d-i} - \theta^*_d) \cdots (\theta^*_{d-i} - \theta^*_{d-j+1})$

Proof. The basis $[wxzy]$, which is on the right in (99), is the inversion of $[wxyz]$ by Lemma 9.4(iii). Apparently Z is the transition matrix from $[wxyz]$ to $[wxzy]$. We now consider the other two bases in (99). For these we prove our assertions case by case. We begin with the first row of the first table, where $wxyz$ equals d^*00^*d. We consider the transition matrix from $[d^*00^*d]$ to $[0d^*0^*d]$. We denote this matrix by T and let D denote the diagonal matrix in $\mathrm{Mat}_{d+1}(\mathbf{K})$ with iith entry

$$D_{ii} = \frac{1}{\varphi_1 \varphi_2 \cdots \varphi_i} \frac{\varepsilon_d \varphi}{\varepsilon^*_0}, \quad 0 \leq i \leq d. \tag{100}$$

We show $D = T$. Recall $\varepsilon_d \neq 0$ by Lemma 14.8 and $\varphi \neq 0$ by Definition 14.4, so $D \neq 0$. Using the data in the first table in Theorem 11.2, rows 1 and 2, we routinely find $A^g D = D A^h$ and $A^{*g} D = D A^{*h}$, where we abbreviate g for d^*00^*d and h for $0d^*0^*d$. Applying Lemma 15.1, we find D is an intertwining matrix from $[d^*00^*d]$ to $[0d^*0^*d]$. Therefore D is a scalar multiple of T. We show this scalar is 1. To do this, we compare the ddth entry of D and T. Setting $i = d$ in (100) and recalling $\varphi = \varphi_1 \varphi_2 \cdots \varphi_d$, we find the ddth entry of D equals $\varepsilon_d / \varepsilon^*_0$. We now find the ddth entry of T. From the table in Theorem 9.1, row 1, we find the dth vector in the basis $[d^*00^*d]$ is $\tilde{E}_d \eta^*_0$. From the same table, row 2, we find the dth vector in the basis $[0d^*0^*d]$ is η_d. From the equation on the left in (90), we find $\eta_d = \varepsilon_d / \varepsilon^*_0 \tilde{E}_d \eta^*_0$, and it follows that the ddth entry of T is $\varepsilon_d / \varepsilon^*_0$. We now see D and T have the same ddth entry, so $D = T$. In particular, D is the transition matrix from $[d^*00^*d]$ to $[0d^*0^*d]$.

We now consider the transition matrix from $[d^*00^*d]$ to $[d^*0^*0d]$. We found the transition matrix from $[d^*0^*0d]$ to $[d^*00^*d]$ in the proof of Theorem 11.2. To summarize, let L denote the matrix in $\mathrm{Mat}_{d+1}(\mathbf{K})$ with ij^{th} entry

$$L_{ij} = (\theta_i - \theta_0)(\theta_i - \theta_1)\cdots(\theta_i - \theta_{j-1}), \quad 0 \leq i, j \leq d. \tag{101}$$

Then L is the transition matrix from $[d^*0^*0d]$ to $[d^*00^*d]$. To get the transition matrix from $[d^*00^*d]$ to $[d^*0^*0d]$, we find the inverse of L. Observe L is lower triangular. Let K denote the lower triangular matrix in $\mathrm{Mat}_{d+1}(\mathbf{K})$ with ij^{th} entry

$$K_{ij} = \frac{1}{(\theta_j - \theta_0)\cdots(\theta_j - \theta_{j-1})}\,\frac{1}{(\theta_j - \theta_{j+1})\cdots(\theta_j - \theta_i)} \tag{102}$$

for $0 \leq j \leq i \leq d$. We recall $\theta_0, \theta_1, \ldots, \theta_d$ are mutually distinct, so the denominator in (102) is nonzero. We claim K is the inverse of L. To prove this, we show $LK = I$. The matrices L and K are both lower triangular, so LK is lower triangular. By (101) and (102), we find that for $0 \leq i \leq d$,

$$K_{ii} = \frac{1}{(\theta_i - \theta_0)\cdots(\theta_i - \theta_{i-1})} = L_{ii}^{-1}$$

so $(LK)_{ii} = 1$. We now show $(LK)_{ij} = 0$ for $0 \leq j < i \leq d$. Let i, j be given. It suffices to show $(\theta_i - \theta_j)(LK)_{ij} = 0$, since $\theta_0, \theta_1, \ldots, \theta_d$ are mutually distinct. Observe

$$\begin{aligned}
(\theta_i - \theta_j)(LK)_{ij} &= (\theta_i - \theta_j)\sum_{h=0}^{d} L_{ih}K_{hj} \\
&= (\theta_i - \theta_j)\sum_{h=j}^{i} L_{ih}K_{hj} \\
&= \sum_{h=j}^{i} L_{ih}K_{hj}(\theta_i - \theta_h + \theta_h - \theta_j) \\
&= \sum_{h=j}^{i-1} L_{ih}(\theta_i - \theta_h)K_{hj} - \sum_{h=j+1}^{i} L_{ih}K_{hj}(\theta_j - \theta_h) \\
&= \sum_{h=j}^{i-1} L_{i,h+1}K_{hj} - \sum_{h=j+1}^{i} L_{ih}K_{h-1,j} \qquad (103) \\
&= 0
\end{aligned}$$

since the two sums in (103) are one and the same. We have now shown $(LK)_{ij} = 0$ for $0 \leq j < i \leq d$. Combining our above arguments, we find $LK = I$ so K is the inverse of L. Now apparently K is the transition matrix from $[d^*00^*d]$ to $[d^*0^*0d]$.

We have now proved our assertions concerning the first row of the first table. Applying these assertions to the relatives of Φ, and using both Theorem 6.2 and Note 14.10, we obtain our assertions concerning the first and fourth rows of each block of the first table.

We now consider the second row of the first table, where $wxyz$ equals $0d^*0^*d$. We find the transition matrix from $[0d^*0^*d]$ to $[d^*00^*d]$. Referring to the diagonal matrix D from (100) we showed D is the transition matrix from $[d^*00^*d]$ to $[0d^*0^*d]$. Therefore, D^{-1} is the transition matrix from $[0d^*0^*d]$ to $[d^*00^*d]$.

We now consider the transition matrix from $[0d^*0^*d]$ to $[00^*d^*d]$. Let q denote a nonzero scalar in $\tilde{\mathbf{K}}$ such that $q + q^{-1} + 1$ is the common value of (35). Let H denote the lower triangular matrix in $\mathrm{Mat}_{d+1}(\mathbf{K})$ with ij^{th} entry

$$H_{ij} = (\theta^*_d - \theta^*_0)(\theta^*_d - \theta^*_1) \cdots (\theta^*_d - \theta^*_{i-j-1})[j, i-j, d-i]_q \tag{104}$$

for $0 \leq j \leq i \leq d$. The expression $[j, i-j, d-i]_q$ is given in (68). We remark each of $[1]_q, [2]_q, \ldots, [d]_q$ is nonzero by Corollary 12.8 so the denominator in $[j, i-j, d-i]_q$ is nonzero. We show H is the transition matrix from $[0d^*0^*d]$ to $[00^*d^*d]$. Observe $H_{ii} = 1$ for $0 \leq i \leq d$, so H is invertible. We show $A^{*g}H = HA^{*h}$, where we abbreviate g for $0d^*0^*d$ and h for 00^*d^*d. The entries of A^{*g} and A^{*h} are given in the first table of Theorem 11.2, rows 2 and 15. Using this information we find that, for $0 \leq i, j \leq d$, the ij^{th} entry of $A^{*g}H$ is given by

$$\theta^*_i H_{ij} + H_{i+1,j}, \tag{105}$$

where we interpret $H_{i+1,j} = 0$ if $i = d$. Similarly, the ij^{th} entry of HA^{*h} is given by

$$H_{i,j-1} + \theta^*_{d-j} H_{ij}, \tag{106}$$

where we interpret $H_{i,j-1} = 0$ if $j = 0$. We show (105) equals (106) or, in other words,

$$(\theta^*_i - \theta^*_{d-j}) H_{ij} = H_{i,j-1} - H_{i+1,j}. \tag{107}$$

To prove (107), first suppose $j-i>1$. Then each of $H_{ij}, H_{i,j-1}, H_{i+1,j}$ is zero since H is lower triangular, so both sides of (107) are zero. Next suppose $j-i=1$. Then $H_{ij}=0$ since H is lower triangular. Moreover, $H_{i,j-1}=H_{ii}=1$ and $H_{i+1,j}=H_{jj}=1$, so both sides of (107) are zero. Next suppose $i=d$ and $j=0$. Then both sides of (107) are zero. Next suppose $i=d$ and $1 \leq j \leq d$. Then using (104) we find both sides of (107) equal $(\theta^*_d-\theta^*_0)(\theta^*_d-\theta^*_1)\cdots(\theta^*_d-\theta^*_{d-j})$. Next suppose $0 \leq i < d$ and $j=0$. Then using (104) we find both sides of (107) equal the opposite of $(\theta^*_d-\theta^*_0)(\theta^*_d-\theta^*_1)\cdots(\theta^*_d-\theta^*_i)$. Finally suppose $1 \leq j \leq i \leq d-1$. To verify (107) in this case, we use Lemma 13.3. Set $r=j$, $s=i-j+1$, $t=d-i$, and observe each of r,s,t is positive. Since $r+s+t=d+1$, and since each of $[1]_q, [2]_q, \ldots [d]_q$ is nonzero, we find $[h]_q \neq 0$ for $1 \leq h < r+s+t$. Apparently our choice of r,s,t satisfy the conditions of Lemma 13.3. Applying that lemma we find

$$\begin{aligned}(108)\quad & \frac{[i-d+j]_q}{[d-i+j]_q}\,[j,i-j,d-i]_q \\ &\qquad = [j-1,i-j+1,d-i]_q - [j,i-j+1,d-i-1]_q.\end{aligned}$$

Applying Lemma 12.5 to the sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ and recalling each of $[1]_q, [2]_q, \ldots, [d]_q$ is nonzero, we find

$$\frac{\theta^*_i-\theta^*_{d-j}}{\theta^*_d-\theta^*_{i-j}} = \frac{[i-d+j]_q}{[d-i+j]_q}. \tag{109}$$

Combining (108) and (109), we obtain

$$\begin{aligned}(110)\quad & \frac{\theta^*_i-\theta^*_{d-j}}{\theta^*_d-\theta^*_{i-j}}\,[j,i-j,d-i]_q \\ &\qquad = [j-1,i-j+1,d-i]_q - [j,i-j+1,d-i-1]_q.\end{aligned}$$

Multiplying both sides of (110) by $(\theta^*_d-\theta^*_0)(\theta^*_d-\theta^*_1)\cdots(\theta^*_d-\theta^*_{i-j})$, and evaluating the result using (104), we routinely obtain (107). We have now shown (107) holds for $0 \leq i,j \leq d$, and it follows $A^{*g}H = HA^{*h}$. Recall we are trying to show H is the transition matrix from $[0d^*0^*d]$ to $[00^*d^*d]$. Let N denote this transition matrix. To show $H=N$, we proceed in two steps. We first show H is a scalar multiple of N. We then show this scalar equals 1. Proceeding with the first step, we define $S := NH^{-1}$ and show S is a scalar multiple of the identity. By Lemma

9.4(ii), we find N is lower triangular. Recall H is lower triangular, so S is lower triangular. Since N is the transition matrix from $[0d^*0^*d]$ to $[00^*d^*d]$ we find N is an intertwining matrix from $[0d^*0^*d]$ to $[00^*d^*d]$. Therefore, $A^{*g}N = NA^{*h}$. Combining this with $A^{*g}H = HA^{*h}$, we find $SA^{*g} = A^{*g}S$. We claim S is diagonal. Suppose not. Then there exists a pair of integers i, j, $(0 \leq j < i \leq d)$, such that $S_{ij} \neq 0$. Of all such pairs i, j, pick one with $i - j$ maximal. We compute the ij^{th} entry in $SA^{*g} = A^{*g}S$. Observe the ij^{th} entry of SA^{*g} is $S_{ij}\theta^*_j$ and that of $A^{*g}S$ is $\theta^*_i S_{ij}$, so $(\theta^*_i - \theta^*_j)S_{ij} = 0$. Observe $\theta^*_i \neq \theta^*_j$, so $S_{ij} = 0$, a contradiction. We have now shown S is diagonal. Computing entries just above the main diagonal in $SA^{*g} = A^{*g}S$, we find S is a scalar multiple of the identity. Apparently H is a scalar multiple of N. We now show this scalar equals 1. To do this, we compare the ddth entry of H and N. We saw above that the ddth entry of H equals 1. We find the ddth entry of N. From the table in Theorem 9.1, row 2, we find the dth vector in the basis $[0d^*0^*d]$ is η_d. From the same table, row 15, we find the dth vector in the basis $[00^*d^*d]$ is η_d. Apparently the ddth entry of N equals 1. We now see H and N have the same ddth entry, so $H = N$. In particular, H is the transition matrix from $[0d^*0^*d]$ to $[00^*d^*d]$.

We have now proved our assertions concerning the second row of the first table. Applying these assertions to the relatives of Φ, and using both Theorem 6.2 and Note 14.10, we obtain our assertions concerning the second and third rows of each block of the first table. We have now verified all our assertions concerning the first table.

Consider the first row of the second table, where $wxyz$ equals d^*0^*0d. We find the transition matrix from $[d^*0^*0d]$ to $[0^*d^*0d]$. Let P denote this matrix and let F denote the diagonal matrix in $\mathrm{Mat}_{d+1}(\mathbf{K})$ with diagonal entries

$$F_{ii} = \frac{\phi_d\phi_{d-1}\cdots\phi_{d-i+1}}{\varphi_1\varphi_2\cdots\varphi_i}\frac{\varepsilon^*_d\varphi}{\varepsilon^*_0\phi}, \quad 0 \leq i \leq d. \tag{111}$$

We show $F = P$. To do this, we first show F is an intertwining matrix from $[d^*0^*0d]$ to $[0^*d^*0d]$. Clearly $F \neq 0$. We show $A^gF = FA^h$, $A^{*g}F = FA^{*h}$, where we abbreviate g for d^*0^*0d and h for 0^*d^*0d. The matrices representing A and A^* with respect to $[d^*0^*0d]$ and $[0^*d^*0d]$ are given in the second table of Theorem 11.2, rows 1 and 2. Using the data in these rows, we routinely find $A^gF = FA^h$,

$A^{*g}F = FA^{*h}$. Applying Lemma 15.1, we find F is an intertwining matrix from $[d^*0^*0d]$ to $[0^*d^*0d]$. Now apparently F is a scalar multiple of P. We show this scalar equals 1. To do this, we compare the ddth entry of F and P. Setting $i = d$ in (111) and, recalling $\varphi = \varphi_1\varphi_2 \cdots \varphi_d$, $\phi = \phi_1\phi_2 \cdots \phi_d$, we find the ddth entry of F equals $\varepsilon^*_d/\varepsilon^*_0$. We now find the ddth entry of P. From the table in Theorem 9.1, row 17, we find the dth vector in the basis $[d^*0^*0d]$ is $E_d\eta^*_0$. From the same table, row 18, we find the dth vector in the basis $[0^*d^*0d]$ is $E_d\eta^*_d$. From the equation on the left in (94), we find $E_d\eta^*_d = \varepsilon^*_d/\varepsilon^*_0 E_d\eta^*_0$. Apparently, the ddth entry of P equals $\varepsilon^*_d/\varepsilon^*_0$. We now see F and P have the same ddth entry, so $F = P$. In particular, F is the transition matrix from $[d^*0^*0d]$ to $[0^*d^*0d]$.

We already found the transition matrix from $[d^*0^*0d]$ to $[d^*00^*d]$. This is the matrix L from (101).

We have now obtained our assertions concerning the first row of the second table. Applying these assertions to the relatives of Φ, and using both Theorem 6.2 and Note 14.10, we obtain all our assertions concerning the second table. This completes the proof. □

We finish this section with some comments on the transition matrices. Let Φ denote the Leonard system in (9), and let g, h denote elements in S_4. Consider the transition matrix from the basis $[g]$ to the basis $[h]$. If g and h are adjacent in the sense of Definition 7.1, then this transition matrix is given in Theorem 15.2. If the above restriction on g, h is removed, then this transition matrix can be computed as follows. To explain the idea, we use the following notation. By an *edge* in S_4, we mean an ordered pair consisting of adjacent elements of S_4. Let r denote a nonnegative integer. By a *walk of length* r in S_4, we mean a sequence $g_0, g_1, \ldots, g_r$ of element of S_4 such that g_{i-1}, g_i is an edge for $1 \leq i \leq r$. The above walk is said to be *from* g_0 *to* g_r. Let gh denote an edge in S_4. By the *weight* of that edge, we mean the transition matrix from $[g]$ to $[h]$. Let $g_0, g_1, \ldots, g_r$ denote a walk in S_4. By the *weight* of this walk, we mean the product $W_1W_2 \cdots W_r$ where W_i is the weight of the edge g_{i-1}, g_i for $1 \leq i \leq r$. Let g, h denote elements in S_4. Then the transition matrix from $[g]$ to $[h]$ is given by the weight of any walk from g to h.

16. Remarks. In the introduction to this paper, we mentioned that Leonard pairs are related to certain orthogonal polynomials contained in the Askey scheme. One significance of the polynomials is that they give the entries in the transition matrices relating certain pairs of bases among our set of 24. In this section we illustrate what is going on with some examples. For related work, see [**12**], [**13**], [**15**], [**45**] and [**27**], [**28**], [**29**], [**30**], [**31**], [**35**, Chapter 4].

Throughout this section we let Φ denote the Leonard system in (9) with eigenvalue sequence $\theta_0, \theta_1, \ldots, \theta_d$, dual eigenvalue sequence $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$, first split sequence $\varphi_1, \varphi_2, \ldots, \varphi_d$ and second split sequence $\phi_1, \phi_2, \ldots, \phi_d$. For $0 \leq i, j \leq d$, we define

$$\mathcal{P}_{ij} = \sum_{n=0}^{d} \frac{(\theta_i-\theta_0)(\theta_i-\theta_1)\cdots(\theta_i-\theta_{n-1})(\theta^*_j-\theta^*_0)(\theta^*_j-\theta^*_1)\cdots(\theta^*_j-\theta^*_{n-1})}{\varphi_1\varphi_2\cdots\varphi_n}. \tag{112}$$

We observe $\mathcal{P}_{ij}$ is a polynomial of degree j in θ_i and a polynomial of degree i in θ^*_j. These are the polynomials of interest.

The $\mathcal{P}_{ij}$ arise in the following context. Let V denote the irreducible left $\mathcal{A}$-module. In Theorem 9.1 we presented 24 bases for V. Of these, we focus on the following two:

$$[d^*0^*0d]: \quad E_0\eta^*_0, E_1\eta^*_0, \ldots, E_d\eta^*_0, \tag{113}$$

$$[d00^*d^*]: \quad E^*_0\eta_0, E^*_1\eta_0, \ldots, E^*_d\eta_0. \tag{114}$$

We recall the basis (113) is a Φ-standard basis. With respect to this basis, the matrix representing A is diagonal, and the matrix representing A^* is irreducible tridiagonal. We denote these matrices by H and B^*, respectively. Their entries are given in the second table of Theorem 11.2, row 1. The basis (114) is a Φ^*-standard basis. With respect to this basis, the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal. We denote these matrices by H^* and B, respectively. Their entries are given in the third table of Theorem 11.2, row 1. Let P denote the transition matrix from (113) to (114), with the vectors η_0, η^*_0 chosen so that

$$\eta^*_0 = E^*_0\eta_0. \tag{115}$$

The effect of (115) is that $P_{i0} = 1$ for $0 \leq i \leq d$. We let P^* denote the transition matrix from (114) to (113), this time with the η_0, η^*_0 chosen so that

$$\eta_0 = E_0\eta^*_0. \tag{116}$$

As expected $P^*_{i0} = 1$ for $0 \le i \le d$. From the construction of P and P^* we find there exists a nonzero scalar $\nu \in \mathbf{K}$ such that

$$PP^* = \nu I. \tag{117}$$

Moreover, by Lemma 15.1, we have

$$B^*P = PH^*, \quad BP^* = P^*H. \tag{118}$$

We compute the entries of P. For this we use the method outlined in the last paragraph of the previous section. The following is a walk in S_4 from d^*0^*0d to $d00^*d^*$.

$$d^*0^*0d,\ d^*00^*d,\ 0d^*0^*d,\ 0d^*d0^*,\ 0dd^*0^*,\ d0d^*0^*,\ d00^*d^*. \tag{119}$$

Apparently P equals the weight of the walk (119). Computing this weight using the data in Theorem 15.2, we find

$$P_{ij} = k_j \mathcal{P}_{ij}, \quad 0 \le i, j \le d, \tag{120}$$

where $\mathcal{P}_{ij}$ is from (112), and where k_j equals

$$\frac{\varphi_1\varphi_2\cdots\varphi_j}{\phi_1\phi_2\cdots\phi_j} \tag{121}$$

times

$$\frac{(\theta^*_0-\theta^*_1)(\theta^*_0-\theta^*_2)\cdots(\theta^*_0-\theta^*_d)}{(\theta^*_j-\theta^*_0)\cdots(\theta^*_j-\theta^*_{j-1})(\theta^*_j-\theta^*_{j+1})\cdots(\theta^*_j-\theta^*_d)} \tag{122}$$

for $0 \le j \le d$. We now compute P^*. Replacing Φ by Φ^* in the above discussion, and using Theorem 6.2, we routinely find

$$P^*_{ij} = k^*_j \mathcal{P}_{ji}, \quad 0 \le i, j \le d, \tag{123}$$

where $\mathcal{P}_{ji}$ is from (112), and where k^*_j equals

$$\frac{\varphi_1\varphi_2\cdots\varphi_j}{\phi_d\phi_{d-1}\cdots\phi_{d-j+1}} \tag{124}$$

times

$$\frac{(\theta_0-\theta_1)(\theta_0-\theta_2)\cdots(\theta_0-\theta_d)}{(\theta_j-\theta_0)\cdots(\theta_j-\theta_{j-1})(\theta_j-\theta_{j+1})\cdots(\theta_j-\theta_d)} \tag{125}$$

for $0 \le j \le d$. We now compute the scalar ν from (117). From the construction of P and P^* we routinely find $\nu E_0 E_0^* E_0 = E_0$. Taking the trace in this equation we find

$$\operatorname{trace} E_0 E_0^* = \nu^{-1}. \tag{126}$$

Evaluating the left side in (126) using Lemma 14.2 and Lemma 14.5, we routinely find

$$\nu = \frac{(\theta_0-\theta_1)(\theta_0-\theta_2)\cdots(\theta_0-\theta_d)(\theta_0^*-\theta_1^*)(\theta_0^*-\theta_2^*)\cdots(\theta_0^*-\theta_d^*)}{\phi_1\phi_2\cdots\phi_d}. \tag{127}$$

From (117) we obtain the following orthogonality relations for the $\mathcal{P}_{ij}$. Expanding the left side of $PP^* = \nu I$ using matrix multiplication, and evaluating the result using (120) and (123), we find

$$\sum_{n=0}^{d} \mathcal{P}_{in}\mathcal{P}_{jn}k_n = \delta_{ij}\nu k_j^{*-1}, \quad 0 \le i,j \le d. \tag{128}$$

Doing something similar with the equation $P^*P = \nu I$, we find

$$\sum_{n=0}^{d} \mathcal{P}_{ni}\mathcal{P}_{nj}k_n^* = \delta_{ij}\nu k_j^{-1}, \quad 0 \le i,j \le d. \tag{129}$$

We remark the equations (118) express several three-term recurrences satisfied by the $\mathcal{P}_{ij}$.

We now indicate how the $\mathcal{P}_{ij}$ fit into the Askey scheme. Instead of giving a complete treatment, we content ourselves with two examples.

Our first example is associated with the Leonard pair from (2). For this example the $\mathcal{P}_{ij}$ will turn out to be Krawtchouk polynomials. Let d denote a nonnegative integer, and consider the following elements of $\mathbf{K}$.

$$\theta_i = d-2i, \quad \theta_i^* = d-2i, \quad 0 \le i \le d, \tag{130}$$

$$\varphi_i = -2i(d-i+1), \quad \phi_i = 2i(d-i+1), \quad 1 \le i \le d. \tag{131}$$

To avoid degenerate situations we assume the characteristic of $\mathbf{K}$ is zero or an odd prime greater than d. It is routine to show that (130) and (131) satisfy the conditions (i)–(v) of Theorem 6.1. Let us assume that Φ is the corresponding Leonard system from that theorem. For this Φ, we routinely find B and B^* both equal the matrix on the left in (2). Moreover, H and H^* both equal the matrix on the right in (2). Pick any integers i, j, $(0 \leq i, j \leq d)$. Evaluating the right side of (112) using (130) and (131), we find $\mathcal{P}_{ij}$ equals

$$\sum_{n=0}^{d} \frac{(-i)_n(-j)_n 2^n}{(-d)_n n!}, \tag{132}$$

where

$$(a)_n := a(a+1)(a+2)\cdots(a+n-1), \quad n = 0, 1, 2, \ldots .$$

Hypergeometric series are defined in [**10**, page 3]. From this definition we find (132) is the hypergeometric series

$${}_2F_1\left(\begin{matrix} -i, -j \\ -d \end{matrix} \middle| 2 \right). \tag{133}$$

A definition of the Krawtchouk polynomials can be found in [**1**] or [**26**]. Comparing this definition with (133), we find $\mathcal{P}_{ij}$ is a Krawtchouk polynomial of degree j in θ_i and a Krawtchouk polynomial of degree i in θ^*_j. Pick an integer j, $(0 \leq j \leq d)$. Evaluating (121), (122) and (124), (125) using (130) and (131), we find k_j and k^*_j both equal the binomial coefficient

$$\binom{d}{j}.$$

Evaluating (127) using (130) and (131), we find $\nu = 2^d$. We comment that for this example $P = P^*$, so $P^2 = 2^d I$.

We now give our second example. For this example the $\mathcal{P}_{ij}$ will turn out to be q-Racah polynomials. To begin, let d denote a nonnegative integer, and consider the following elements in $\mathbf{K}$.

$$\theta_i = \theta_0 + h(1-q^i)(1-sq^{i+1})/q^i, \tag{134}$$

$$\theta^*_i = \theta^*_0 + h^*(1-q^i)(1-s^*q^{i+1})/q^i \tag{135}$$

for $0 \le i \le d$, and

$$\varphi_i = hh^* q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-r_1 q^i)(1-r_2 q^i), \tag{136}$$

$$\phi_i = hh^* q^{1-2i}(1-q^i)(1-q^{i-d-1})(r_1 - s^* q^i)(r_2 - s^* q^i)/s^* \tag{137}$$

for $1 \le i \le d$. We assume $q, h, h^*, s, s^*, r_1, r_2$ are nonzero scalars in the algebraic closure $\tilde{\mathbf{K}}$, and that $r_1 r_2 = ss^* q^{d+1}$. It is routine to show (134)–(137) give a parametric solution to Theorem 6.1 (iii)–(v). Let us assume conditions (i) and (ii) of Theorem 6.1 are satisfied as well, so that (134)–(137) correspond to a Leonard system. We assume Φ is the corresponding Leonard system from Theorem 6.1. For this Φ we find $B, B^*, \mathcal{P}_{ij}, k_j, k_j^*, \nu$. Recall the entries of B are given in the third table of Theorem 11.2, row 1. Evaluating these entries using (134)–(137), we find

$$\begin{aligned}
B_{01} &= \frac{h(1-q^{-d})(1-r_1 q)(1-r_2 q)}{1-s^* q^2}, \\
B_{i-1,i} &= \frac{h(1-q^{i-d-1})(1-s^* q^i)(1-r_1 q^i)(1-r_2 q^i)}{(1-s^* q^{2i-1})(1-s^* q^{2i})}, \quad 2 \le i \le d, \\
B_{i,i-1} &= \frac{h(1-q^i)(1-s^* q^{i+d+1})(r_1 - s^* q^i)(r_2 - s^* q^i)}{s^* q^d (1-s^* q^{2i})(1-s^* q^{2i+1})}, \quad 1 \le i \le d-1, \\
B_{d,d-1} &= \frac{h(1-q^d)(r_1 - s^* q^d)(r_2 - s^* q^d)}{s^* q^d (1-s^* q^{2d})}, \\
B_{ii} &= \theta_0 - B_{i,i-1} - B_{i,i+1}, \ (0 \le i \le d),
\end{aligned}$$

where we define $B_{0,-1} := 0$, $B_{d,d+1} := 0$. The entries of B^* are similarly obtained. To get the entries of B^*, in the above formulae exchange (θ_0, h, s) and (θ_0^*, h^*, s^*) and preserve (r_1, r_2, q). Pick integers i, j, $(0 \le i, j \le d)$. Evaluating the right side of (112) using (134)–(137), we find $\mathcal{P}_{ij}$ equals

$$\sum_{n=0}^{d} \frac{(q^{-i};q)_n (sq^{i+1};q)_n (q^{-j};q)_n (s^* q^{j+1};q)_n q^n}{(r_1 q;q)_n (r_2 q;q)_n (q^{-d};q)_n (q;q)_n}, \tag{138}$$

where

$$(a;q)_n := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}), \quad n = 0, 1, 2, \ldots.$$

Basic hypergeometric series are defined in [**10**, page 4]. From that definition we find (138) is the basic hypergeometric series

$$
{}_4\phi_3\left(\begin{matrix} q^{-i}, sq^{i+1}, q^{-j}, s^*q^{j+1} \\ r_1q, r_2q, q^{-d} \end{matrix}\,\middle|\, q, q\right). \tag{139}
$$

A definition of the q-Racah polynomials can be found in [**2**], [**3**] or [**26**]. Comparing this definition with (139), and recalling $r_1r_2 = ss^*q^{d+1}$, we find $\mathcal{P}_{ij}$ is a q-Racah polynomial of degree j in θ_i and a q-Racah polynomial of degree i in θ^*_j. Pick an integer j, $(0 \le j \le d)$. Evaluating (121) and (122) using (134)–(137), we find

$$
k_j = \frac{(r_1q;q)_j(r_2q;q)_j(q^{-d};q)_j(s^*q;q)_j(1-s^*q^{2j+1})}{s^jq^j(q;q)_j(s^*q/r_1;q)_j(s^*q/r_2;q)_j(s^*q^{d+2};q)_j(1-s^*q)}. \tag{140}
$$

The scalar k^*_j is similarly found. To get k^*_j, in (140), exchange s and s^* and preserve (r_1, r_2, q). Evaluating (127) using (134)–(137), we find

$$
\nu = \frac{(sq^2;q)_d(s^*q^2;q)_d}{r_1^dq^d(sq/r_1;q)_d(s^*q/r_1;q)_d}.
$$

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