ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 2, Summer 2002

MULTIPLE GAMMA FUNCTION, ITS q- AND ELLIPTIC ANALOGUE

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ABSTRACT. Vignéras's multiple gamma function is introduced as a function satisfying a generalization of the Bohr-Mollerup theorem. An infinite product representation and an asymptotic expansion of the function are given. Furthermore, its q- and elliptic analogue are introduced as relevant with the defining relations of q-gamma function and of elliptic gamma function.

1. Introduction. In 1904, Barnes [**4, 5, 6, 7**] introduced his multiple gamma function by using the multiple Hurwitz zeta functions. After his discovery, Hardy [**14, 15**] studied this function from his viewpoint of the theory of elliptic functions, and Shintani [**37, 38**] applied it to the study on the Kronecker limit formula for zeta functions attached to certain algebraic fields. At the end of the 70's, Vignéras [**45**] redefined the multiple gamma function as the function satisfying a generalization of the Bohr-Mollerup theorem. Vignéras [**45**], Voros [**46**], Vardi [**44**] and Kurokawa [**22**–**25**] showed that it is applicable to represent gamma factors of the Selberg zeta functions of compact Riemann surfaces and the determinants of the Laplacians on some Riemannian manifolds. Recently, multiple gamma functions have been applied to studies on the Kniznik-Zamolodchikov equation with $|q| = 1$ [**18, 26, 27**], on eigenfunctions of commuting difference operators [**36**] and on q-analysis with $|q| = 1$ [**29, 33, 40**].

In the theory of special functions, constructions of q-analogue and of elliptic analogue are known as generalizations in the other direction. q-Analogue of Euler's gamma function was introduced by Jackson [**16, 17**]. This function plays essential roles in q-analysis. One of the most significant features of the function was pointed out by Askey [**3**]. He remarked that the q-gamma function satisfies a q-analogue of the Bohr-Mollerup theorem. Considering Askey's result together with Vignéras' one, it would be natural to consider the construction of functions satisfying a q-analogue of the generalized Bohr-Mollerup theorem. On

Received by the editors on September 30, 2001.

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the other hand, in relation with elliptic solutions for difference systems arising from solvable lattice models and from quantum integral systems, "elliptic analogue" of special functions have been constructed, for example, Spiridonov and Zhedanov [**39**]. Recently, an elliptic analogue of Euler's gamma function was introduced by Ruijsenaars [**35**] and by Felder and Varchenko [**12**]. This function is applied to construct eignenfunctions of elliptic difference systems [**49**] and to represent Frenkel and Turaev's elliptic hypergeometric series [**13, 47**]. We can also generalize the multiple gamma function and introduce an elliptic analogue of multiple gamma function.

In this article, the author and Ueno's results [**29, 31, 32, 43**] on multiple gamma function and its generalization are reported. In Section 2 we consider some properties of Vignéras's multiple gamma function. We derive an infinite product representation and an asymptotic expansion of this function. The former is a generalization of the Weierstrass product representation of Euler's gamma function and the latter is of the Stirling formula. In Section 3 we construct a q-analogue of the multiple gamma function. This q-analogue is introduced as a function satisfying a q-analogue of a q-analogue of the generalized Bohr-Mollerup theorem and its classical limit coincides with the multiple gamma function. We can see that the q-analogue has some properties which generalize the corresponding properties of q-gamma function. In Section 4 we construct an elliptic analogue of the multiple gamma function inspired by Ruijsenaars's and Felder-Varchenko's result. We also show its uniqueness and study its properties.

2. Vignéras's multiple gamma function.

2.1 *Definition and uniqueness.* Vignéras [**45**] introduced a hierarchy of functions that she called "the multiple gamma functions."

Theorem 2.1 (Vignéras). *A unique hierarchy of functions exists* *that satisfy*

$$\begin{aligned}
&(1)\quad G_r(z+1) = G_{r-1}(z)G_r(z),\\
&(2)\quad G_r(1) = 1,\\
&(3)\quad \frac{d^{r+1}}{dz^{r+1}} \log G_r(z+1) \geq 0 \quad \textit{for } z \geq 0,\\
&(4)\quad G_0(z) = z.
\end{aligned}$$

Applying Dufresnoy and Pisot's results [**11**], she showed that the functions satisfying the above properties are uniquely determined. We note some particular cases. In the case when $r = 1$, $G_1(z)$ is characterized by the following relation:

$$\begin{aligned}
&G_1(z+1) = zG_1(z), \quad G_1(1) = 1,\\
&\frac{d^2}{dz^2} \log G_1(z+1) \geq 0 \quad \text{for } z \geq 0.
\end{aligned}$$

These are nothing but the defining relations for Euler's gamma function, and we can see that $G_1(z)$ is Euler's gamma function. Theorem 2.1 is a generalization of Bohr-Mollerup's theorem. Furthermore, $G_2(z)$ is known as Barnes's G-function [**4**]. It is applied to derive a representation for the Toeplitz determinants [**41**]. The hierarchy in Theorem 2.1 includes such functions. Vignéras's multiple gamma function can be regarded as a special case of Barnes's multiple gamma function. However, for simplicity, we will use the word "multiple gamma function" to refer to Vignéras's multiple gamma function in this article.

2.2 *The Weierstrass product representation.* Vignéras [**45**] showed that $G_r(z+1)$ has the following infinite product representation

$$\begin{aligned}
G_r(z+1) = \exp\bigg[&-zE_r(1) + \sum_{h=1}^{r-1} \frac{p_h(z)}{h!}(\psi_{r-1}^{(h)}(0) - E_r^{(h)}(1))\bigg]\\
\times \prod_{\mathbf{m}\in\mathbf{N}^{n-1}\times\mathbf{N}^*} &\bigg[\bigg(1+\frac{z}{s(\mathbf{m})}\bigg)^{(-1)^r} \exp\bigg\{\sum_{l=0}^{r-1} \frac{(-1)^{r-l}}{r-l}\bigg(\frac{z}{s(\mathbf{m})}\bigg)^{r-l}\bigg\}\bigg],
\end{aligned}$$

where $\mathbf{N}^*$ is defined as $\mathbf{N}/\{0\}$,

$$\begin{aligned} E_r(z) &:= \sum_{\mathbf{m}\in\mathbf{N}^{r-1}\times\mathbf{N}^*} \left[\left\{ \sum_{l=0}^{r-1} \frac{(-1)^{r-l}}{r-l} \left(\frac{z}{s(\mathbf{m})}\right)^{r-l} \right\} \right. \\ &\qquad\qquad \left. + (-1)^r \log\left(1+\frac{z}{s(\mathbf{m})}\right)\right], \\ \psi_{r-1}(z) &: = \log G_{r-1}(z+1), \\ \frac{e^{tz}-1}{e^z-1} &=: 1+\sum_{k=0}^{\infty} p_k(t)\frac{z^k}{k!} \end{aligned}$$

and $s(\mathbf{m}) := m_1 + m_2 + \cdots + m_n$ for $\mathbf{m} = (m_1, m_2, \ldots, m_n)$.

This representation for $G_r(z)$ is defined inductively by using the logarithmic derivative of $G_{r-1}(z)$. We give a more explicit representation for $G_r(z)$ [**43**], which can be regarded as the Weierstrass product representation for the multiple gamma functions.

Theorem 2.2. *For $r \in \mathbf{N}$, we have*

$$G_r(z+1) = \exp(F_r(z)) \prod_{k=1}^{\infty} \left\{ \left(1+\frac{z}{k}\right)^{-\binom{-k}{r-1}} \exp(\Phi_r(z,k)) \right\},$$

where

$$\begin{aligned} F_r(z) &: = \sum_{j=0}^{n-1} G_{n,j}(z) Q_j(z) + \sum_{n=0}^{r-2} \frac{\zeta'(-n)}{n!} \left[\frac{\partial^n}{\partial u^n} \binom{z-u}{r-1} \right]_{u=0}^{u=z} \\ &\qquad - \gamma \int_0^z \binom{z-u}{r-1} \, du, \\ \binom{z-u}{n-1} &=: \sum_{j=0}^{n-1} G_{n,j}(z) u^j, \\ \Phi_r(z,k) &: = \sum_{\mu=-1}^{r-2} \left\{ \sum_{n=\mu+1}^{r-1} \frac{{}_{r-1}S_n}{n-\mu} z^{n-\mu} \right\} \frac{(-1)^{\mu+1} k^{\mu}}{(r-1)!}, \\ Q_j(z) &:= P_j(z+1) - \sum_{n=0}^{j} \binom{j}{r} z^n P_{j-n}(1) \end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{n=1}^{j+1} \binom{j+1}{r} \frac{B_{j+1-n}(z)}{j+1} \sum_{l=1}^{n} \frac{(-z)^l}{l}, \\
P_j(x) &:= \sum_{n=0}^{j+1} \frac{B_n}{n!} \varphi_{j \cdot n} x^{j-n+1}, \\
\varphi_{j,n} &:= \left(\frac{d}{dt}\right)^n \left\{ \frac{t^{j+1}}{j+1} \log t - \frac{t^{j+1}}{(j+1)^2} \right\}\Big|_{t=1},
\end{aligned}
$$

$\zeta(s)$ is the Riemann zeta-function and $\zeta'(s)$ is the first derivative.

Let us give some examples of the Weierstrass product representation for the multiple gamma functions. In the case when $r = 1$, we have

$$
G_1(z+1) = \Gamma(z+1) = e^{-\gamma z} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k}\right)^{-1} e^{-z/k} \right\}.
$$

This is the Weierstrass product representation for the gamma function.

In the case when $r = 2$, we have

$$
\begin{aligned}
G_2(z+1) &= G(z+1) \\
&= e^{-z\zeta'(0) - \frac{z^2}{2}\gamma - \frac{z^2+z}{2}} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k}\right)^k \exp\left(-z + \frac{z^2}{2k}\right) \right\}.
\end{aligned}
$$

Since $\zeta'(0) = -(1/2)\log(2\pi)$, this is the Weierstrass product representation for the Barnes G-function [**4**].

2.3 *Asymptotic expansion.* By means of the Euler-MacLaurin summation formula

$$
\begin{aligned}
\sum_{r=M}^{N-1} f(r) = \int_M^N f(t)\,dt &+ \sum_{k=1}^{n} \frac{B_k}{k!} \{ f^{(k-1)}(N) - f^{(k-1)}(M) \} \\
&+ (-1)^{n-1} \int_M^N \frac{\overline{B}_n(t)}{n!} f^{(n)}(t)\,dt \quad \text{for } f \in C^n[M,N],
\end{aligned}
$$

we can derive an expansion formula of the multiple gamma function from the Weierstrass product representation. This formula gives an asymptotic expansion of $G_r(z+1)$ as $|z| \to \infty$ [**43**]. We call it *higher Stirling formula.*

Theorem 2.3. *Let $0 < \delta < \pi$. As $|z| \to \infty$ in the sector $\{z \in \mathbf{C} \| \arg z| < \pi - \delta\}$, then we have*

$$\begin{aligned}\log G_r(z+1) \sim & \left\{ \binom{z+1}{r} + \sum_{r=1}^{r} \frac{B_n}{n!} \left(-\frac{d}{dz} \right)^{n-1} \binom{z}{r-1} \right\} \log(z+1) \\ & - \sum_{n=1}^{r} \left\{ \left(-\frac{d}{dz} \right)^{n-1} \binom{z}{r-1} \right\} \frac{1}{n!n} \{(z+1)^n - 1\} \\ & - \sum_{j=0}^{r-1} G_{r,j}(z) \left\{ \zeta'(-j) + \frac{1}{(j+1)^2} \right\} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} F_{r,2n-1}(z)\end{aligned}$$

where

$$F_{n,r-1}(z) := \left(\frac{d}{dt} \right)^{r-1} \left\{ \binom{-t}{n-1} \log \left(\frac{z+t}{z+1} \right) \right\} \Bigg|_{t=1}.$$

We next give some examples of the higher Stirling formula. In the case when $r = 1$, we obtain

$$\begin{aligned}\log G_1(z+1) = \log \Gamma(z+1) \sim & \left(z + \frac{1}{2} \right) \log(z+1) \\ & - (z+1) - \zeta'(0) + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r-1)_2} \frac{1}{(z+1)^{2r-1}}\end{aligned}$$

where $(r)_k$ is the so-called *Pochanmmer's symbol*, defined as

$$(r)_k := r(r+1)(r+2) \cdots (r+k-1).$$

This is the Stirling formula since $\zeta'(0) = -(1/2) \log(2\pi)$.

In the case when $r = 2$, we obtain

$$\begin{aligned}\log G_2(z+1) \sim & \left(\frac{z^2}{2} - \frac{1}{12} \right) \log(z+1) \\ & - \frac{3}{4} z^2 - \frac{z}{2} + \frac{1}{4} - z\zeta'(0) + \zeta'(-1) \\ & - \frac{1}{12} \frac{1}{z+1} + \sum_{r=2}^{\infty} \frac{B_{2r}}{(2r-2)_3} \frac{1}{(z+1)^{2r-1}} (z - 2r + 1),\end{aligned}$$

which coincides with the formula in Barnes [**4**].

2.4 *Logarithmic derivative.* We define a function $\psi_r(z)$ by the logarithmic derivative of $G_r(z+1)$,

$$\begin{aligned}\psi_r(z) &:= \frac{d}{dz}\log G_r(z+1)\\ &= \frac{d}{dz}F_r(z) + \sum_{k=1}^{\infty}\left\{-\frac{1}{z+k}\binom{-k}{r-1} + \phi_r(z,k)\right\},\end{aligned}$$

where $\phi_r(z,k) := (d/dz)\Phi_r(z,k)$ and $F_r(z)$ is a polynomial introduced in (2.3). We can see the following relations between the logarithmic derivatives [**31**]:

Proposition 2.4. (i) *$\psi_r(z)$ satisfies a recurrence formula*

$$\psi_{r+1}(z) = \frac{z-r+1}{r}\psi_r(z) + p_r(z), \tag{1}$$

where $p_r(z)$ is a polynomial of degree less than or equal to r.

(ii) *$\psi_{r+1}(z)$ is transformed by the following formula*:

$$\psi_{r+1}(z) = \binom{z}{r}\psi_1(z) + P_r(z), \tag{2}$$

where $P_r(z)$ is a polynomial of degree less than or equal to r.

As a corollary to the proposition, we can easily show that the multiple gamma function satisfies no algebraic differential equation. We denote $\mathbf{C}(z)$ by the rational function field generated by z over the complex number field $\mathbf{C}$.

Theorem 2.5. *Vignéras's multiple gamma function does not satisfy any algebraic differential equation over $\mathbf{C}(z)$. In other words, for any $r, n \in \mathbf{Z}_{\geq 0}$ there is no $(n+1)$-variable polynomial*

$$f(t_0, t_1, t_2, \ldots, t_n) \in \mathbf{C}(z)[t_0, t_1, t_2, \ldots, t_n],$$

such that

$$f(G_r(z), G_r^{(1)}(z) \cdots, G_r^{(n)}(z)) = 0 \tag{3}$$

where $G_r^{(l)}(z)$ *is the* l*th derivative of* $G_r(z)$.

An outline of the proof is as follows: We remark that the logarithmic derivative of a solution for an algebraic differential equation also satisfies some algebraic differential equation, cf. Pastro [**34**]. Therefore, if some polynomial satisfies (3), then an algebraic differential equation exists for $\psi_r(z)$. However, from Proposition 2.4 (2), it follows that an algebraic differential equation exists for $\psi_1(z)$ over $\mathbf{C}(z)$. This contradicts Hölder's theorem for Euler's gamma function, cf. Komatsu [**20**].

In fact, we can obtain this theorem as a special case of Barnes's theorem [**8**] for the more general case. He proved that a class of functions, including Barnes's multiple gamma function, does not satisfy any algebraic differential equation. However, in the general case, the proof is complicated and we cannot see whether relations like (2) exist or not. For Vignéras's multiple gamma function, we can find the relation explicitly and prove this theorem directly.

3. Multiple q-gamma function.

3.1 *Definition and uniqueness.* In this section we suppose that $0 < q < 1$. A q-analogue of Euler's gamma function is

$$\Gamma(z+1;q) = (1-q)^{-z} \prod_{n=1}^{\infty} \left(\frac{1-q^{z+n}}{1-q^n} \right)^{-1}. \tag{4}$$

It is one of the most fundamental functions in the theory of q-special functions. Askey [**3**] showed that the q-gamma function $\Gamma(z;q)$ satisfies a q-analogue of the Bohr-Mollerup theorem

$$\Gamma(z+1;q) = [z]\Gamma(z;q), \quad \Gamma(1;q) = 1,$$
$$\frac{d^2}{dz^2} \log \Gamma(z+1;q) \geq 0 \quad \text{for } z \geq 0,$$

where

$$[z] := \frac{1-q^z}{1-q}.$$

The author [**29**] constructed a function $G_r(z;q)$ satisfying the q-analogue of the generalized Bohr-Mollerup theorem.

Theorem 3.1. *A unique hierarchy of functions exists which satisfy*

$$\begin{aligned}
&\text{(i)} \quad G_r(z+1;q) = G_{r-1}(z;q)G_r(z;q),\\
&\text{(ii)} \quad G_r(1;q) = 1,\\
&\text{(iii)} \quad \frac{d^{r+1}}{dz^{r+1}} \log G_{r+1}(z+1;q) \geq 0 \ \textit{for}\ z \geq 0,\\
&\text{(iv)} \quad G_0(z;q) = [z],
\end{aligned}$$

where

$$g_r(z,u) = \binom{z-u}{n-1} - \binom{-u}{n-1}.$$

$G_r(z;q)$ *is represented as the following infinite product representation:*

$$G_r(z+1;q) := (1-q)^{-\binom{z}{n}} \prod_{k=1}^{\infty} \left\{ \left(\frac{1-q^{z+k}}{1-q^k} \right)^{\binom{-k}{n-1}} (1-q^k) g_r(z,k) \right\} \tag{5}$$

for $r \geq 1$.

In the case when $r = 1$, this theorem corresponds to Askey's theorem. So the sequence $\{G_r(z;q)\}$ includes a q-gamma function. We call an element of the sequence a *multiple q-gamma function*. The representation (5) can be regarded as an analogue of the Weierstrass product representation for the multiple gamma function.

3.2 *Infinite product representation.* It is well known that Euler's gamma function has infinite product representations. The representation

$$\Gamma(z+1) = \lim_{N\to\infty} \frac{N!}{(z+1)(z+2)\cdots(z+N)}(N+1)^z$$

is known as the Gauss product representation and

$$\Gamma(z+1) = \prod_{n=1}^{\infty} \left\{ \left(1+\frac{1}{n}\right)^z \left(1+\frac{z}{n}\right)^{-1} \right\}$$

as the Euler product representation. We remark that there are q-analogues of these formulas. They are represented as

$$\Gamma(z+1;q) = \lim_{N\to\infty} \frac{[1][2]\cdots[N]}{[z+1][z+2]\cdots[z+N]}[N+1]^z \tag{6}$$

and as

$$\Gamma(z+1;q) = \prod_{n=1}^{\infty} \left\{ \left(\frac{[n+1]}{[n]}\right)^z \left(\frac{[z+n]}{[n]}\right)^{-1} \right\}. \tag{7}$$

We can generalize these formulas. As a counterpart to these representations, we can derive the following representations:

Proposition 3.2. *If* $\Re z > 0$*, then*

(i)

$$G_r(z+1;q) = \lim_{N\to\infty} \left\{ \frac{G_{r-1}(1;q)\cdots G_{r-1}(N;q)}{G_{r-1}(z+1;q)\cdots G_{r-1}(z+N;q)} \times \prod_{m=1}^{r} G_{r-m}(N+1;q)^{\binom{z}{m}} \right\}.$$

(ii)

$$G_r(z+1;q) = \prod_{n=1}^{\infty} \left\{ \frac{G_{r-1}(n;q)}{G_{r-1}(z+n;q)} \prod_{m=1}^{r} \left(\frac{G_{r-m}(n+1;q)}{G_{r-m}(n;q)}\right)^{\binom{z}{m}} \right\}.$$

In the case when $r=1$, we obtain the formula (6) and (7). It should be considered where Vignéras's multiple gamma function has such an infinite product representation or not. However, the author has not proved it yet.

3.3 *Classical limit.* In this section we consider the classical limit of the multiple q-gamma function. In the case for q-gamma function it was proved rigorously by Koornwinder [**21**]. By means of the Euler-MacLaurin summation formula, we can obtain an expansion formula of the multiple q-gamma functions. We call it Euler-MacLaurin expansion [**43**].

Proposition 3.3. *Suppose that $\Re z > -1$ and $m > n$, then*

$$\begin{aligned}
\log G_n(z+1;q) = &\left\{ \binom{z+1}{n} + \sum_{r=1}^{n} \frac{B_r}{r!} \left(-\frac{d}{dz} \right)^{r-1} \binom{z}{n-1} \right\} \\
&\times \log\left(\frac{1-q^{z+1}}{1-q}\right) \\
&+ \sum_{r=1}^{n} \left\{ \left(-\frac{d}{dz} \right)^{r-1} \binom{z}{n-1} \right\} \int_1^{z+1} \frac{\xi^r}{r!} \frac{q^\xi \log q}{1-q^\xi}\, d\xi \\
&+ \sum_{j=0}^{n-1} G_{n,j}(z) C_j(q) + \sum_{r=1}^{m} \frac{B_r}{r!} F_{n,r-1}(z;q) - R_{n,m}(z;q),
\end{aligned}$$

where

$$\begin{aligned}
F_{n,r-1}(z;q) :&= \left[\frac{d^{r-1}}{dt^{r-1}} \left\{ \binom{-t}{n-1} \log\left(\frac{1-q^{z+t}}{1-q^{z+1}} \right) \right\} \right]_{t=1}, \\
C_j(q) :&= -\sum_{r=1}^{n+1} \frac{B_r}{r!} f_{j+1,r-1}(1;q) \\
&\quad + \int_1^\infty \left[\frac{(-1)^n \bar{B}_{n+1}(t)}{(n+1)!} f_{j+1,n+1}(t;q) \right] dt, \\
f_{j+1,r-1}(t;q) :&= \left[\frac{d^{r-1}}{dt^{r-1}} \left\{ t^j \log\left(\frac{1-q^t}{1-q} \right) \right\} \right] \\
R_{n,m}(z;q) :&= \frac{(-1)^{m-1}}{m!} \\
&\quad \times \int_1^\infty \left[\overline{B}_m(t) \left\{ \frac{d^m}{dt^m} \left\{ \binom{-t}{n-1} \log\left(\frac{1-q^{z+t}}{1-q^{z+1}} \right) \right\} \right\} \right] dt.
\end{aligned}$$

This is a generalization of Moak's formula [**28**] for the q-gamma function, see also [**42**]. As remarked by Daalhuis [**10**], this formula is not an asymptotic expansion. However, we can see that each term of the above formula converges uniformly as $q \to 1$. Thus we obtain the following theorem:

Theorem 3.4. *As $q \to 1-0$, $G_r(z+1;q)$ converges to $G_r(z+1)$ uniformly on any compact set in the domain $\mathbf{C} \setminus \mathbf{Z}_{<0}$.*

4. Multiple elliptic gamma function.

4.1 *Definition and uniqueness.* First we fix some notations. For $0 \le j \le r$, τ_j are complex parameters satisfying $\Im \tau_j > 0$. We put $q_j := e^{2\pi\sqrt{-1}\tau_j}$, $x := e^{2\pi\sqrt{-1}z}$, $\underline{\tau} := (\tau_0, \tau_1, \dots, \tau_r)$ and $\underline{q} := (q_0, q_1, \dots, q_r)$. For the set $\underline{\tau}$, we define $\underline{\tau}^+(j)$, $\underline{\tau}^-(j)$ and $|\underline{\tau}|$ as

$$\begin{aligned}
\underline{\tau}^+(j) &:= (\tau_0, \tau_1, \dots, \tau_r, \tau_j), \\
\underline{\tau}^-(j) &:= (\tau_0, \tau_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_r), \\
\underline{\tau}[j] &:= (\tau_0, \tau_1, \dots, \tau_{j-1}, -\tau_j, \tau_{j+1}, \tau_r), \\
|\underline{\tau}| &:= \sum_{j=0}^{r} \tau_j.
\end{aligned}$$

Similarly, we introduce the following notation:

$$\begin{aligned}
\underline{q}^-(j) &:= (q_0, q_1, \dots, q_{j-1}, q_{j=1}, \dots, q_r), \\
\underline{q}[j] &:= (q_0, q_1, \dots, q_{j-1}, q_j^{-1}, q_{j+1}, \dots, q_r).
\end{aligned}$$

The principal value is the branch of the logarithm chosen here. We define "multiple $\underline{q}$-shifted factorial" as the following formula:

Definition 4.1.

$$(x;\underline{q})_\infty^{(r)} := \prod_{i_1, i_2, \dots, i_r = 0}^{\infty} (1 - x q_0^{i_0} q_1^{i_1} \cdots q_r^{i_r}).$$

This product was introduced and studied by Appell [**2**] and is also known as the generating function of the multipletite partition, cf. Andrews [**1**]. Furthermore, this product can be regarded as a generalization of Kirilov's q-polylogarithms [**19**]. This product can be represented as

$$(x;\underline{q})_\infty^{(r)} = \exp(-\mathrm{Li}_{r+2}(x;\underline{q}))$$

for $|x| < 1$, where

$$\mathrm{Li}_r(x;(q_1,\cdots,q_r)) := \sum_{j=1}^{\infty} \frac{x^j}{j\prod_{i=1}^{r}(1-q_i^j)}.$$

In the case $\tau_0 = \tau_1 = \cdots = \tau_r = \tau$, we can see that

$$(x;q)_\infty^{(r)} = \exp(-\mathrm{Li}_{r+2}(x;q))$$

where $\mathrm{Li}_r(x;q)$ is Kirillov's q-polylogarithm [**19**]. The function $(x;\underline{q})_\infty^{(r)}$ satisfies a functional relation

$$(q_j x;\underline{q})_\infty^{(r)} := \frac{(x;\underline{q})_\infty^{(r)}}{(x;\underline{q}^-(j))_\infty^{(r-1)}}.$$

By using this product, we introduce an "elliptic analogue" of the multiple gamma function. We call it *multiple elliptic gamma function.*

Definition 4.2.

$$G_r(z \mid \underline{\tau}) := (x^{-1}q_0q_1\cdots q_r;\underline{q})_\infty^{(r)}\{x;\underline{q})_\infty^{(r)}\}^{(-1)^r}.$$

This function is meromorphic. We note that $G_1(z \mid (\tau_0,\tau_1))$ is the elliptic gamma function introduced by Ruijsenaars [**35**] and by Felder and Varchenko [**12**].

From the definition of the multiple elliptic gamma function, we have the following functional relation of $G_r(z \mid \underline{\tau})$:

Proposition 4.3. (i) *$G_r(z \mid \underline{\tau})$ satisfies a relation*

$$\begin{aligned} G_r(z+1 \mid \underline{\tau}) &= G_r(z \mid \underline{\tau}), \\ G_r(z+\tau_j \mid \underline{\tau}) &= G_{r-1}(z \mid \underline{\tau}(j))G_r(z \mid \underline{\tau}) \quad \textit{for } j = 0,\ldots,r, \\ G_0(z \mid (\tau_0)) &= \theta_0(z,q_0), \end{aligned}$$

where

$$\theta_0(z;q) := \prod_{k=0}^{\infty}(1-xq^k)(1-x^{-1}q^{k+1}).$$

(ii) *At the point* $z = |\underline{\tau}|/2$, $G_r(z \mid \underline{\tau})$ *take the following value*:

$$G_r\left(\frac{|\underline{\tau}|}{2}\Big|\underline{\tau}\right) = \begin{cases} \{(q_0^{1/2}q_1^{1/2}\cdots q_r^{1/2};\underline{q})_\infty^{(r)}\}^2 & r\text{: even} \\ 1 & r\text{: odd.} \end{cases}$$

By reversing the argument, we can prove a kind of uniqueness of the function which satisfies the relation in Proposition 4.3.

Proposition 4.4. *If* $G_{r-1}(z|(\tau_0,\dots,\tau_{r-1}))$, $\Im\tau_j > 0$, *is given, we can determine the unique meromorphic function* $u(z)$ *which satisfies*

(i) $u(z)$ *is holomorphic upper half plane*,

(ii) $u(z+1) = u(z)$, $u(z+\tau_r) = G_{r-1}(z \mid \underline{\tau})u(z)$,

(iii) $u(|\underline{\tau}|/2) = \begin{cases} \{(q_0^{1/2}q_1^{1/2}\cdots q_r^{1/2};\underline{q})_\infty^{(r)}\}^2 & r\text{: even,} \\ 1 & r\text{: odd.} \end{cases}$

This can be proved through a similar argument to Felder-Varchenko's proof [**12**] of the uniqueness of the elliptic gamma function. On the other side we can see a slightly different uniqueness theorem.

Proposition 4.5. *If there are such* $\tau_j, \tau_k \in \underline{\tau} = (\tau_0, \tau_1, \cdots \tau_r)$ *that 1,* τ_j *and* τ_k *are linearly independent over* $\mathbf{Q}$, *then we can determine unique meromorphic function* $u(z)$ *satisfying*

$$\begin{aligned} u(z+1) &= u(z), \\ u(z+\tau_j) &= G_{r-1}(z \mid \underline{\tau}(j))u(z), \\ u(z+\tau_k) &= G_{r-1}(z \mid \underline{\tau}(k))u(z), \\ u\left(\frac{|\underline{\tau}|}{2}\right) &= \begin{cases} \{(q_0^{1/2}q_1^{1/2}\cdots q_r^{1/2};\underline{q})_\infty^{(r)}\}^2 & r\text{: even,} \\ 1 & r\text{: odd.} \end{cases} \end{aligned}$$

4.2 *Elementary properties.* From the definition of the multiple elliptic gamma function, we derive some formulas by straightforward calculation.

Proposition 4.6. (i) $G_r(z \mid \underline{\tau})\{G_r(|\underline{\tau}| - z|\underline{\tau})\}^{(-1)^{r-1}} = 1$,

(ii) $G_r\left(z\middle|\left(\frac{\tau_0}{N}, \frac{\tau_1}{N}, \cdots \frac{\tau_r}{N}\right)\right) = \prod_{n_1,n_2,\dots,n_r=0}^{N-1} G_r\left(z+\frac{n_0\tau_0+\cdots+n_r\tau_r}{N}\middle|\underline{\tau}\right)$.

Claim (i) is an analogue of the complementary formula for Euler's gamma function and Claim (ii) is an analogue of the Gauss-Legendre multiplication formula for the function.

Next we can represent $G_r\ (z \mid \underline{\tau})$ by using trigonometric series.

Proposition 4.7 (Summation formula). *For*

$$z \in \left\{ z \in \mathbf{C} \mid |\Im(2z - |\underline{\tau}|)| < \sum |\Im\tau_j| \right\},$$

then we obtain

$$G_r(z \mid \underline{\tau}) = \begin{cases} \exp\left[\left[(1/(2\sqrt{-1})^r\right] \sum_{l=1}^{\infty}(\sin(\pi l(2z - |\underline{\tau}|))/l \prod_{j=1}^{r} \sin \pi l \tau_j)\right] \\ \qquad r: \ \textit{odd}, \\ \exp\left[(1/2^r(\sqrt{-1})^{r+1}) \sum_{l=1}^{\infty}(\cos(\pi l(2z - |\underline{\tau}|))/l \prod_{j=1}^{r} \sin \pi l \tau_j)\right] \\ \qquad r: \ \textit{even}. \end{cases}$$

These are a generalization of the so-called "summation formula" in Felder and Varchenko [**12**]. We can also see a differential relation between the multiple elliptic gamma functions.

Proposition 4.8. *The multiple elliptic gamma function satisfies the following differential relation*:

$$\left(\frac{\partial}{\partial\tau_j} G_r(z \mid \underline{\tau})\right) G_{r+1}(z \mid \underline{\tau}^+(j)) + \frac{\partial}{\partial z}\{G_{r+1}(z \mid \underline{\tau}^+(j)) G_r(z \mid \underline{\tau})\} = 0.$$

4.3 *Trigonometric limit.* We can relate our multiple elliptic gamma functions to multiple q-gamma functions through a kind of the trigonometric limit.

Proposition 4.9. *In the case when* $\tau_1 = \tau_2 = \cdots = \tau_r = \tau$, *as* $\tau_0 \to \sqrt{1}\infty$,

$$G_r(\tau z \mid \underline{\tau}) \prod_{k=0}^{r} \left(\left(q; (q_0, \underbrace{q, q, \dots, q}_{k}) \right)_{\infty}^{(r)} \right)^{\binom{z-1}{r-k}} \longrightarrow G_r(z;q)$$

for z *in any compact set in the domain* $\mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$, *where* $G_r(z;q)$ *is the multiple* q*-gamma function introduced in Section* 3.

Acknowledgment. The author expresses his deep gratitude to the organizing committee of the workshop "Special Function 2000." He also thanks Professor M.E.H. Ismail, Professor K. Mimachi, Professor M. Noumi and Doctor S. Kakei for their valuable comments and to Professor K. Ueno for his encouragement. The author is partially supported by Grant-in-Aid for Scientific Research from the Ministry of Education (12640046) and Waseda University Grant for Special Research Project 2000A-172, 2001A-119.

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