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GENERALIZED UMEMURA POLYNOMIALS

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ABSTRACT. We introduce and study generalized Umemura polynomials $U_{n,m}^{(k)}(z,w;a,b)$ which are a natural generalization of the Umemura polynomials $U_n(z,w;a,b)$ related to Painlevé VI equation. We will show that if $a=b$ or $a=0$ or $b=0$, then polynomials $U_{n,m}^{(0)}(z,w;a,b)$ generate solutions to Painlevé VI. We will describe a connection between polynomials $U_{n,m}^{(0)}(z,w;a,0)$ and certain Umemura polynomials $U_k(z,w;\alpha,\beta)$.

1. Introduction. There is a vast body of literature devoted to the Painlevé VI equation $P_{\mathrm{VI}} := P_{\mathrm{VI}}(\alpha,\beta,\gamma,\delta)$:
(1.1)
$$\frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{q-t}\right)\left(\frac{dq}{dt}\right) + \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left(\alpha-\beta\frac{t}{q^2}+\gamma\frac{(t-1)}{(q-1)^2}+\delta\frac{t(t-1)}{(q-t)^2}\right)$$

where $t\in \mathbf{C}$, $q := q(t;\alpha,\beta,\gamma,\delta)$ is a function of t and $\alpha,\beta,\gamma,\delta$ are arbitrary complex parameters. It is well known and goes back to Painlevé that any solution $q(t)$ of the equation P_{VI} satisfies the so-called Painlevé property:

• the critical points 0, 1 and ∞ of the equation (1.1) are the only *fixed singularities* of $q(t)$.

• any *movable singularity* of $q(t)$, the position of which depends on integration constants, is a pole.

In this paper we introduce and initiate the study of certain special polynomials related to the Painlevé VI equation, namely, the generalized Umemura polynomials $U_{n,m}^{(k)}(z,w;a,b)$. These polynomials have many interesting combinatorial and algebraic properties and in the particular case $n=0=k$ coincide with Umemura's polynomials $U_m(z^2,w^2;a,b)$,

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see, e.g., [**2, 5**]. In the present paper we study recurrence relations between polynomials $U_{n,m}^{(k)}(z,w;a,b)$. Our main result is Theorem 4.1 which gives a generalization of the recurrence relation between Umemura's polynomials [**5**]. As a corollary, we obtain that polynomials $U_{n,m}^{(0)}(z,w;a,0)$ also generate solutions to the equation Painlevé VI. The main tool in our proofs is Lemma 4.2 from Section 4. For example, we prove a new recurrence relation between Umemura's polynomials, Theorem 4.9, and describe explicitly connections between polynomials $U_{n,m}(0,b)$ and Umemura's polynomials $U_m(b_1,b_2)$, see Lemma 4.7. Finally, in Section 5, we state a conjecture which describes the Plücker relations between certain Umemura's polynomials.

2. Painlevé VI. In this section we collect together some basic results about equation Painlevé VI. More detail and proofs may be found in a familiar series of papers by Okamoto [**3**]. We refer the reader to the proceedings of conference, "The Painlevé property. One century later," [**1**] where different aspects of the theory of Painlevé equations may be found.

2.1 *Hamiltonian form.* It is well known and goes back to a paper by Okamoto [**3**] that the sixth Painlevé equation (1.1) is equivalent to the following Hamiltonian system

$$\mathcal{H}_{\mathrm{VI}}(\mathbf{b};t,q,p):\begin{cases} dq/dt=(\partial H/\partial p)\\ (dp/dt)=-(\partial H/\partial q)\end{cases} \tag{2.1}$$

with the Hamiltonian

$$\begin{aligned}H:=H_{\mathrm{VI}}(\mathbf{b};t,q)=\frac{1}{t(t-1)}&[q(q-1)(q-t)p^2-\{(b_1+b_2)(q-1)(q-t)\\ &+(b_1-b_2)q(q-t)+(b_3+b_4)q(q-1)\}p\\ &+(b_1+b_3)(b_1+b_4)(q-t)],\end{aligned}$$

where $\mathbf{b}=(b_1,b_2,b_3,b_4)$ belongs to the parameters space $\mathbf{C}^4$; the parameters $(\alpha,\beta,\gamma,\delta)$ and (b_1,b_2,b_3,b_4) are connected by the following relations

$$\begin{aligned}&\alpha=\frac{1}{2}(b_3-b_4)^2,\beta=-\frac{1}{2}(b_1+b_2)^2,\\ &\gamma=\frac{1}{2}(b_1-b_2)^2,\delta=-\frac{1}{2}(b_3-b_4)(b_3+b_4-2).\end{aligned} \tag{2.2}$$

Proposition 2.1. [**3**]. *If* $(q(t), p(t))$ *is a solution to the Hamiltonian system* (2.1), *the function*

$$h(\mathbf{b}, t) = t(t-1)H_{\mathrm{VI}}(\mathbf{b}; t, q(t), p(t)) + e_2(b_1, b_3, b_4) - \frac{1}{2}e_2(b_1, b_2, b_3, b_4)$$

satisfies the equation $E_{\mathrm{VI}}(\mathbf{b})$:

$$\frac{dh}{dt}\left[t(t-1)\frac{d^2h}{dt^2}\right]^2 + \left[\frac{dh}{dt}\left\{2h - (2t-1)\frac{dh}{dt}\right\} + b_1b_2b_3b_4\right] = \prod_{k=1}^{4}\left(\frac{dh}{dt} + b_k^2\right). \tag{2.3}$$

Conversely, for a solution $h := h(\mathbf{b}, t)$ *to the equation* $E_{\mathrm{VI}}(\mathbf{b})$ *such that* $d^2h/dt^2 \neq 0$), *there exists a solution* $(q(t), p(t))$, *to the Hamiltonian system* (2.1). *Furthermore, the function* $q := q(t)$ *is a solution to the Painlevé equation* (1.1), *whose parameters* $(\alpha, \beta, \gamma, \delta)$ *are determined by the relations* (2.2).

We will call the equation $E_{\mathrm{VI}}(\mathbf{b})$, see equation (0.3), by the Painlevé-Okamoto equation.

2.2 *Bäcklund transformation.* Consider the following linear transformation of the parameters space $\mathbf{C}^4$:

$$\begin{aligned}
s_1 &:= (b_1, b_2, b_3, b_4) \longmapsto (b_2, b_1, b_3, b_4),\\
s_2 &:= (b_1, b_2, b_3, b_4) \longmapsto (b_1, b_3, b_2, b_4),\\
s_3 &:= (b_1, b_2, b_3, b_4) \longmapsto (b_1, b_2, b_4, b_3),\\
s_0 &:= (b_1, b_2, b_3, b_4) \longmapsto (b_1, b_2, -b_3, -b_4),\\
l_3 &:= (b_1, b_2, b_3, b_4) \longmapsto (b_1, b_2, b_3 + 1, b_4).
\end{aligned}$$

Denote by $W = \langle s_0, s_1, s_2, s_3, l_3\rangle$ the subgroup of $\mathbf{Aut}\,\mathbf{C}^4$ generated by these transformations. It is not difficult to see, that $W \cong W(D_4^{(1)})$, i.e., W is isomorphic to the affine Weyl group of type $D_4^{(1)}$.

Proposition 2.2. ([**3**]. *For each* $w \in W$, *a birational transformation*

$$L_w : \{\textit{solutions to } \mathcal{H}_{\mathrm{VI}}(\mathbf{b})\} \longmapsto \{\textit{solutions to } \mathcal{H}_{\mathrm{VI}}(w(\mathbf{b}))\}.$$

The birational transformations L_w, $w \in W(D_4^{(1)})$ are called by Bäcklund transformations associated with the equation Painlevé VI.

2.3 *τ-function.* Let $(q(t), p(t))$ be a solution to the Hamiltonian system (2.1), the τ-function $\tau(t)$ corresponding to the solution $(q(t), p(t))$ is defined by the following equation

$$\frac{d}{dt} \log \tau(t) = H_{\mathrm{VI}}(\mathbf{b}; t, q(t), p(t));$$

in other words,

$$\tau(t) = (\text{constant}) \exp\left(\int H_{\mathrm{VI}}(\mathbf{b}; t, q(t), p(t))\, dt \right)$$

2.4 *Umemura polynomials.* Suppose that $b_3 = -1/2$, $b_4 = 0$. Then it is well known and goes back to Umemura's paper [**5**] that the pair

$$(q_0, p_0) = \left(\frac{(b_1 + b_2)^2 - (b_1^2 - b_2^2)\sqrt{t(1-t)}}{(b_1 - b_2)^2 + 4b_1 b_2 t}, \frac{b_1 q_0 - \frac{1}{2}(b_1 + b_2)}{q_0(q_0 - 1)} \right)$$

defines a solution to the Hamiltonian system (2.1) with parameters $\mathbf{b} = (b_1, b_2, -1/2, 0)$. Note, see, e.g., [**5**]

$$\begin{aligned} H_0 &= H_{\mathrm{VI}}\left((b_1, b_2, -\frac{1}{2}, 0); t, q_0(t), p_0(t) \right) \\ &= \frac{1}{t(t-1)} \Big\{ b_1(b_1 - 1)(1 - 2t) + 2b_1^2 \sqrt{t(t-1)} \\ &\qquad + 2b_2(b_1 - t) + b_2(b_2 - 1)(1 - 2t) - 2b_2^2 \sqrt{t(t-1)} \Big\}, \end{aligned}$$

and

$$\tau_0(t) = \exp\left\{ \int H_0(t)\, dt \right\}.$$

To introduce Umemura's polynomials, let (q_m, p_m) be a solution to the Hamiltonian system $\mathcal{H}_{\mathrm{VI}}(b_1, b_2, -(1/2)+m, 0) = \mathcal{H}_{VI}(l_3^m(b_1, b_2, -(1/2), 0)$

obtained from the solution (q_0, p_0) by applying m times the Bäcklund transformation l_3. Consider the corresponding τ-function τ_m:

$$\frac{d}{dt}\log\tau_m = H_{\mathrm{VI}}\left(\left(b_1, b_2, -\frac{1}{2}+m, 0\right); t, q_m(t), p_m(t)\right).$$

It follows from Proposition 2.1, see e.g. [**3, 5**], that the τ-functions $\tau_n := \tau_n(t)$ satisfy the Toda equation

$$\frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2} = \frac{d}{dt}\left(t(t-1)\frac{d}{dt}(\log\tau_n)\right) + (b_1+b_2+n)(b_3+b_4+n). \tag{2.4}$$

Following Umemura [**5**], define a family of functions $T_n(t)$, $n = 0, 1, 2, \ldots$, by

$$\tau_n(t) = T_n(t)\exp\left(\int\left(H_0(t) - \frac{n(b_1t-(1/2)(b_1+b_2))}{t(t-1)}\right)dt\right).$$

Proposition 2.3 (Umemura [**5**]). *$T_n(t)$ is a polynomial in the variable $v := \sqrt{t/(t-1)} + \sqrt{(t-1)/t}$ with rational coefficients.*

For example, $T_0 = 1$, $T_1 = 1$, $T_2 = (1/2)(-4b_1^2+1)(2-v)/4 + (-4b_2^2+1)(2+v)/4$. It follows from the Toda equation (2.4) that polynomials $T_n := T_n(v)$ satisfy the following recurrence relation [**5**]:

$$\begin{aligned} T_{n-1}T_{n+1} = {} & \left\{\frac{1}{4}(-2b_1^2 - 2b_2^2 + (b_1^2-b_2^2)v) + \left(n-\frac{1}{2}\right)^2\right\}T_n^2 \\ & + \frac{1}{4}(v^2-4)^2\left\{T_n\frac{d^2T_n}{dv^2} - \left(\frac{dT_n}{dv}\right)^2\right\} \\ & + \frac{1}{4}(v^2-4)vT_n\frac{dT_n}{dv} \end{aligned} \tag{2.5}$$

with initial conditions $T_0 = T_1 = 1$.

Definition 2.4. Polynomials $U_n := U_n(z, w, b_1, b_2) := 2^{n(n-1)}T_n(v)$ where $z = (2-v)/4$, $w = (2+v)/4$, are called by Umemura polynomials.

The formula (2.6) below was stated as a conjecture by Okada, Noumi, Okamoto and Umemura [**2**] and has been proved recently by Taneda and Kirillov

(2.6)
$$2^{n(n-1)}T_n(v) := U_n(z,w,b_1,b_2) = \sum_{I\subset[n-1]} \mathbf{d}_n(I)c_I d_{[n-1]\setminus I} z^{|I|}w^{|I^c|},$$

where

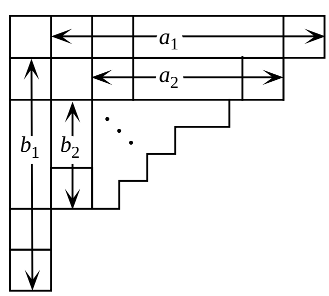
(i) $[n-1] = \{1,2,\dots,n-1\}$; for any subset $I = \{i_1 > i_2 > \cdots > i_p\} \subset [n-1]$, $\mathbf{d}_n(I) = \dim^{GL(n)}_{\lambda(I)}$ stands for the dimension of irreducible representation of the general linear group $\mathrm{GL}\,(n)$ corresponding to the highest weight $\lambda(I)$ with the Frobenius symbol $\lambda(I) = (i_1, i_2, \dots, i_p | i_1 - 1, i_2 - 1, \dots, i_p - 1)$;

(ii) $c = -4b_1^2$, $d = -4b_2^2$, $z = (2-v)/4$, $w = (2+v)/4$;

(iii) $\bar{c} = c + (2k-1)^2$, $\bar{d} = d + (2k-1)^2$, $c_k = \bar{c}_1\bar{c}_2\cdots\bar{c}_k$, $d_k = \bar{d}_1\bar{d}_2\cdots\bar{d}_k$;

(iv) $|I| = i_1 + i_2 + \cdots + i_p$.

Recall that Frobenius's symbol $(a_1, a_2, \dots, a_p | b_1, b_2, \dots, b_p)$ denotes the partition which corresponds to the following diagram



3. Generalized Umemura polynomials. Let n, m, k be fixed nonnegative integers, $k \le n$. Denote by $[n;m]$ the set of integers $\{1, 2, \dots, n, n+2, n+4, \dots, n+2m\}$. Let I be a subset of the set

$[n;m]$. Follow [**6**], define the numbers

$$\mathbf{d}_{n,m}(I)=\prod_{\substack{i\in I\\ j\in[n;m]\setminus I}}\left|\frac{i+j}{i-j}\right|,\quad \mathbf{c}(I)=\sum_{\substack{i\in I\\ i>n}}\frac{i-n}{2}. \tag{3.1}$$

It has been shown in [**6**] that, in fact, $\mathbf{d}_{n,m}(I)$ are integers for any subset $I\subset[n;m]$. Now we are going to introduce the generalized Umemura polynomials

$$\begin{aligned}U_{n,m}^{(k)}&:=U_{n,m}^{(k)}(z,w;a,b)\\ &=\sum_{[k]\subset I\subset[n;m]}\prod_{\substack{i\in I\setminus[k]\\ j\in[k]}}\left(\frac{i+j}{i-j}\right)\mathbf{d}_{n,m}(I)(-1)^{\mathbf{c}(I)}e_I^{(n,m,k)}(z,w),\end{aligned}$$

where

(i) $[k]$ stands for the set $\{1,2,\dots,k\}$;

(ii) $\bar{a}_k=a+(k-1)^2$, $\bar{b}_k=b+(k-1)^2$ and $a_{2k}=\bar{a}_2\bar{a}_4\cdots\bar{a}_{2k}$, $a_{2k+1}=\bar{a}_1\bar{a}_3\cdots\bar{a}_{2k+1}$; $b_{2k}=\bar{b}_2\bar{b}_4\cdots\bar{b}_{2k}, b_{2k+1}=\bar{b}_1\bar{b}_3\cdots\bar{b}_{2k+1}$;

(iii) for any subset $I\subset[n;m]$, we set $a_I=\prod_{i\in I}a_i$, $b_I=\prod_{i\in I}b_i$;

(iv) $e_I^{(n,m,k)}(z,w)=a_{I\setminus[k]}b_{[n,m]\setminus I}z^{|I\setminus[k]|}w^{|[n;m]\setminus I|}$.

Note that the polynomial $U_{0,m}^{(0)}$ coincides with Umemura's polynomial $T_m(z^2,w^2;a,b)$. The formula for generalized Umemura polynomials stated below follows from the Cauchy identity, and in some particular case was used by van Diejen and Kirillov [**6**] in their study of q-spherical functions.

Lemma 3.1. *The generalized Umemura polynomials* $U_{n,m}^{(k)}(a,b;z,w)$ *admit the following determinantal expression*

$$\begin{aligned}U_{n,m}^{(k)}(a,b;z,w)=\det\Bigg|&a_iw^i\prod_{s\in[k]}\left(\frac{i+s}{i-s}\right)\delta_{i,j}\\ &+(-1)^{c(i)}\frac{2i}{i+j}\prod_{\substack{s\in[n,m],\\ s\neq i}}\left|\frac{i+s}{i-s}\right|b_iz^i\Bigg|_{i,j\in[n;m]\setminus[k]},\end{aligned}$$

where $c(i) = i$ *if* $i \le n$ *and* $c(i) = (i-n)/2$ *if* $i > n$.

In the particular case $k = 0$, $n = 0$, this formula gives a determinantal representation for Umemura's polynomials and has many applications.

4. Main result. Let us introduce notation $U_{n,m} := U^{(0)}_{n,m}(z, w; a, b)$. The main result of our paper describes a recurrence relation between polynomials $U_{n,m}$.

Theorem 4.1.

$$\begin{aligned} U_{n,m-1}U_{n,m+1} &= (-\bar{a}_{n+2m+2}z^2 + \bar{b}_{n+2m+2}w^2)U^2_{n,m} \\ &\quad + 8z^2w^2 D^2_x U_{n,m} \circ U_{n,m} \\ &\quad - \frac{4}{(n+2m+1)^2} ab(a-b)z^2w^2(U^{(1)}_{n,m})^2, \end{aligned} \tag{4.1}$$

where for any two functions $f = f(x)$ and $g = g(x)$

$$D^2_x f \circ g = f''g - 2f'g' + fg''$$

denotes the second Hirota derivative and $\prime = (d/dx)$; here variables z, w and x are connected by the relations $z = (1/2)(e^x + e^{-x} - 2)^{1/2}$, $w = (1/2)(e^x + e^{-x} + 2)^{1/2}$.

Below we give a sketch of our proof of Theorem 4.1. Detailed exposition will appear elsewhere. The main step of the proof is to establish the following algebraic identity which appears to have an independent interest.

Lemma 4.2. *For any two subsets* I, J *of the set* $[n; m]$, *we have*

$$\prod_{\lambda \in I}\left(\frac{x+2+\lambda}{x+2-\lambda}\right)\prod_{\lambda \in J}\left(\frac{x-\lambda}{x+\lambda}\right) + \prod_{\lambda \in I}\left(\frac{x-\lambda}{x+\lambda}\right)\prod_{\lambda \in I}\left(\frac{x+2+\lambda}{x+2-\lambda}\right) = 2 + \sum_{\lambda \in I\cup J}\frac{b^{I,J}_{\lambda}}{(x+2-\lambda)(x+\lambda)},$$

where $b^{I,J}_{\lambda}$ *are some constants, depending on* λ, I *and* J, *which may be computed explicitly.*

One can prove this lemma by using the residue theorem.

Lemma 4.3. *For any two subsets I, J of the set $[n;m]$, we have*

$$\sum_{\lambda \in I \cup J} b_\lambda^{I,J} = 4(|I| - |J|)^2 - 4(|I| + |J|).$$

This lemma follows from Lemma 4.2.

Lemma 4.4. *For an element $\lambda \in I \cap J$, we have $b_\lambda^{I,J} = 0$ if and only if $\lambda - 2 \in I \cap J$. For an element $\lambda \in I \setminus (I \cap J)$, we have $b_\lambda^{I,J} = 0$ if and only if $\lambda - 2 \in J$.*

This lemma follows and Lemma 4.4 can be deduced by direct calculations. □

Remarks 1. If $n = 0$, then $U_{0,m} = U_{m+1}(z^2, w^2; a, b)$ coincides with the Umemura polynomial and $U_{0,m}^{(1)} = 0$. In this case the recurrence relation (4.1) has been used by Taneda in his proof of Okada-Noumi-Okamoto-Umemura's conjecture (2.6).

2. Note that $U_{0,m} = U_{2,m-1}^{(1)}/(2m+1)$, and more generally $U_{k,m}^{(k)} = U_{k+2,m-1}^{(k+1)}(2k+1)!!(2m-1)!!/(2k+2m+1)!!$, where $(2n+1)!! = 1 \cdot 3 \cdot 5 \cdots (2n+1)$.

3. "Unwanted term" in (4.1) which contains $(U_{n,m}^{(1)})^2$ vanishes if either $a = 0$ or $b = 0$ or $a = b$.

In the case $a = b$ and $k = 0$ the expression $e_I^{(n,m,k)}(z, w)$ doesn't depend on a subset $I \subset [n;m]$ and is equal to $a_{[n;m]} z^{|I|} w^{|[n;m] \setminus I|}$. Hence, in this case

(4.2)

$$\begin{aligned} U_{n;m}^{(0)}(z, w; a, a) &= a_{[n;m]} \sum_{I \subset [n;m]} \mathbf{d}_{n,m}(I)(-1)^{\mathbf{c}(I)} z^{|I|} w^{|[n;m] \setminus I|} \\ &= a_{[n;m]} (z+w)^{\binom{n+m+1}{2}} (z-w)^{\binom{m+1}{2}}. \end{aligned}$$

Recall that $\bar{a}_i = a + (i-1)^2$, $a_{2i} = \bar{a}_2\bar{a}_4\cdots\bar{a}_{2i}, a_{2i+1} = \bar{a}_1\bar{a}_3\cdots\bar{a}_{2i+1}$ and $a_{[n,m]} = \prod_{i\in[n;m]} a_i$. The last equality in (4.2) has been proved for the first time by van Diejen and Kirillov [**6**]. On the other hand, we can show that the polynomials

$$X_{n,m}(z,w;a) = a_{[n;m]}(z+w)^{\binom{n+m+1}{2}}(z-w)^{\binom{m+1}{2}}$$

also satisfy the recurrence relation (4.1) and coincide with polynomials $U^{(0)}_{n,m}(z,w;a,a)$ if $m=0$. From this observation we can deduce the equality $X_{n,m}(z,w;a) = U^{(0)}_{n,m}(z,w;a,a)$, which is equivalent to the main identity from [**6**]. Another case when "unwanted term" in (4.1) vanishes is the case when either $a=0$ or $b=0$. In this case we have

Corollary 4.5. *Assume that* $b=0$. *Then the polynomial* $U_{n,m}(z,w;a,0)$ *defines a solution to the equation Painlevé* VI.

Finally we compare polynomials $U_{n,m}(z,w;a,0)$ and $U_m(z,w;\alpha,\beta)$. For this goal let us consider functions

$$h_0 := h_0(t) = \{b_1^2(\sqrt{t}-\sqrt{t-1})^2 + b_2^2(\sqrt{t}+\sqrt{t+1})^2\}/4,$$

and

$$h_{n,m} := h_{n,m}(b_1,b_2) = t(t-1)\log(U_{n,m})' - h_0.$$

Proposition 4.6. (i) $h_{0,m}$ *satisfies the Painlevé-Okamoto equation* $E_{\mathrm{VI}}(b_1,b_2,m+1/2,0)$;

(ii) $h_{1,m} = -(2t-1)(m+1)^2/2$ *satisfies the equation* $E_{\mathrm{VI}}(0,m+1,b_3,b_4)$;

(iii) $h_{n,m}(0,b_2)$ *satisfies the equation* $E_{\mathrm{VI}}(0,b_2,(n/2),(n+2m+1)/2)$.

Proposition 4.6 follows from Lemma 4.7 and Lemma 4.8 below. Let us define $U_{n,m}(b_1,b_2) := U_{n,m}(z,w;-4b_1^2,-4b_2^2)$, then

Lemma 4.7.

$$U_{n,m}(0,b_2) = \begin{cases} b_{[n;m]+\mathrm{odd}} w^{(n/2)^2} U_{0,m+(n/2)}((n/2),b_2) & \textit{if } n \textit{ is even} \\ b_{[n;m]_{\mathrm{odd}}} w^{\frac{n+2m+1}{2})^2} U_{0,\frac{n-1}{2}}(m+\frac{n+1}{2},b_2) & \textit{if } n \textit{ is odd,} \end{cases} \tag{4.3}$$

where $[n;m]_{\mathrm{odd}} = \{i \in [n;m] \mid i \text{ is odd}\}$.

From Lemma 4.7 we can deduce the following

Lemma 4.8.

$$h_{n,m}(0,b_2) = \begin{cases} h_{0,m+\frac{n}{2}}(\frac{n}{2},b_2) & \textit{if } n \textit{ is even,} \\ h_{0,\frac{n-1}{2}}(m+\frac{n+1}{2},b_2) & \textit{if } n \textit{ is odd.} \end{cases}$$

It follows from Lemma 4.7, (4.3) and Theorem 4.1 that Umemura's polynomials $U_m(b_1,b_2)$ satisfy a new recurrence relation with respect to the first argument b_1.

Theorem 4.9.

$$U_m(b_1-1,b_2)U_m(b_1+1,b_2)(b_1^2-b_2^2) = (b_1^2-b_2^2)U_m^2(b_1,b_2) + 2z^2 D_x^2 U_m(b_1,b_2) \circ U_m(b_1,b_2).$$

Recall that D_x^2 *denotes the second Hirota derivative.*

5. Conjecture. We define $q_m := q_m(t)$ by

$$q_m - t = 4U_m^2\left\{\left(m+\frac{1}{2}\right)t(t-1)\frac{d}{dt}\log U_{m+1} - \left(m+\frac{3}{2}\right)t(t-1)\frac{d}{dt}\log U_m - \frac{1}{2}b_1b_2 + \frac{1}{4}\left(b_1^2\frac{z}{w} + b_2^2\frac{w}{z}\right)\right\} \Big/ \big(U_{m+1}U_{m-1} - (2m+1)^2U_m^2\big).$$

One can check that q_m is a solution to both equations $P_{\mathrm{VI}}(b_1, b_2, m+\frac{1}{2}, 0)$ and $P_{\mathrm{VI}}(b_1, b_2, 0, m+\frac{1}{2})$. It follows from Okamoto's theory [**3**] that the function

$$\begin{aligned}\bar{h}_{1,m} &= t(t-1)\frac{d}{dt}\log U_{m+1} - \frac{1}{4}\left(b_1^2\frac{z}{w} + b_2^2\frac{w}{z}\right) \\ &\quad + \left(m+\frac{1}{2}\right)q_m - \frac{1}{2}\left(m+\frac{1}{2}\right)\end{aligned} \tag{5.1}$$

is also a solution to $E_{\mathrm{VI}}(b_1, b_2, n+\frac{1}{2}, 1)$. Based on the latter expression for the function $\bar{h}_{1,m}$, and using Lemmas 5 and 6, we come to the following

Conjecture 5.1. *If $b_1 = 0$, then we have*

$$U_{m+1}U_{m-1} - (2m+1)^2 U_m^2 = \frac{1}{4b_2^2}U_{2,m-1}^2,$$

where $U_m := U_m(0, b_2)$ is a special case of Umemura's polynomial, and $U_{2,m-1} = U_{2,m-1}(0, b_2)$.

REFERENCES

1. R. Conte, editor, *The Painlevé property. One century later*, CRM Series Math. Physics, Springer-Verlag, New York, 1999.

2. M. Noumi, K. Okamoto, S. Okada and H. Umemura, *Special polynomials associated with the Painlevé equation* II, in *Integrable systems and algebraic geometry*, World Scientific, 1998.

3. K. Okamoto, *Studies on Painlevéquation* I, Ann. Mat. Pura Appl. **146** (1987), 337–381; II, Jap. J. Math. **13** (1987), 47–76; III, Math. Ann. **275** (1986, 221–255); IV, Funkcial. Ekvac. **30** (1987), 305–332.

4. M. Taneda, *Polynomials associated with an algebraic solution of the sixth Painlevé equation*, Jap. J. Math, to appear.

5. H. Umemura, *Special polynomials associated with the Painlevé equation* I, Proceedings of Workshop on the Painlevé transcendents, Montreal, 1996.

6. J.F. van Deijen and A.N. Kirillov, *Determinantal formulas for zonal spherical functions of hyperboloids*, Math. Ann. **319** (2001), 215–234.

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