ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 2, Summer 2002

AN INVERSE TO THE ASKEY-WILSON OPERATOR

MOURAD E.H. ISMAIL AND MIZAN RAHMAN

ABSTRACT. We study properties of the kernel of a right inverse of the Askey-Wilson divided difference operator on L^2 weighted with the weight function of the continuous q-Jacobi polynomials. This operator is embedded in a one-parameter family of integral operators, denoted by $\mathcal{D}_q^{-t}$ whose kernel is related to the Poisson kernel. It is shown that as $t \to 1^-$, the t-commutator $(\mathcal{D}_q\mathcal{D}_q^{-t} - t\mathcal{D}_q^{-t}\mathcal{D}_q)f$ tends to the constant term in the orthogonal expansion of f in continuous q-Jacobi polynomials.

1. Introduction. Given a function $f(x)$ with $x = \cos\theta$, then $f(x)$ can be viewed as a function of $e^{i\theta}$. Let

$$\breve{f}(e^{i\theta}) := f(x), \quad x = \cos\theta. \tag{1.1}$$

In this notation the Askey-Wilson divided difference operator $\mathcal{D}_q$ [**4**] is defined by

$$(\mathcal{D}_q f)(x) := \frac{\breve{f}(q^{1/2}e^{i\theta}) - \breve{f}(q^{-1/2}e^{i\theta})}{\breve{e}(q^{1/2}e^{i\theta}) - \breve{e}(q^{-1/2}e^{i\theta})}, \tag{1.2}$$

where $e(x) = x$. It follows easily from (1.2) that

$$(\mathcal{D}_q f)(x) = \frac{\breve{f}(q^{1/2}e^{i\theta}) - \breve{f}(q^{-1/2}e^{i\theta})}{i(q^{1/2} - q^{1/2})\sin\theta}. \tag{1.3}$$

The operator $\mathcal{D}_q$ was introduced in [**4**] and is a q-analogue of the differentiation operator d/dx. Note that $\mathcal{D}_q$ remains invariant if q is

---

AMS *Mathematics subject classification.* Primary 33D15, Secondary 47B34.
Keywords and phrases. Weight functions, inverse operator, integral operators, divided difference operators, continuous q-Jacobi polynomials, monotonicity of kernels, t-commutators.
Research of the first author partially supported by NSF grant DMS 99-70865.
Research of the second author partially supported by NSERC grant A6197.
Received by the editors on September 1, 2000, and in revised form on October 30, 2001.

Copyright ©2002 Rocky Mountain Mathematics Consortium

replaced by $1/q$. In this work we will avoid having q on the unit circle; thus, there is no loss of generality in assuming $|q| < 1$.

The integral operator $\int_a^x$ is a right inverse to d/dx. Recall that the Chebyshev polynomials of the first and second kinds, respectively, are

$$T_n(\cos\theta) = \cos n\theta, \quad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{1.4}$$

Brown and Ismail utilized the fact

$$\mathcal{D}_q T_n(x) = q^{-(n-1)/2}\frac{1-q^n}{1-q}U_{n-1}(x)$$

to define a right inverse to $\mathcal{D}_q$ on $L_2[-1,1,(1-x^2)^{1/2}]$ through the action of the inverse operator on the Chebyshev polynomials of the second kind, that is, they sought formal expansions of f and g, ($f = \mathcal{D}_q^{-1}g$) in the form

$$f(x) \sim \sum_{n=0}^{\infty} f_n T_n(x), \quad g(x) \sim \sum_{n=1}^{\infty} g_n U_{n-1}(x), \tag{1.5}$$

so that

$$f_n = q^{(n-1)/2}\frac{1-q}{1-q^n}g_n, \quad n > 0. \tag{1.6}$$

A straightforward calculation [**6**] gives the following expression

$$\mathcal{D}_q^{-1}g(\cos\theta) = \frac{1-q}{4\pi q^{1/2}}\int_{-\pi}^{\pi}\frac{\vartheta_4'((\theta-\phi)/2|q^{1/2})}{\vartheta_4((\theta-\phi)^2|q^{1/2})}g(\cos\phi)\sin\phi\, d\phi, \tag{1.7}$$

where ϑ_4 is the Jacobian theta function [**15**]

$$\begin{aligned}\vartheta_4(\theta|q) &= 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2n\theta \\ &= \prod_{n=0}^{\infty}(1-q^{2n+2})(1+2q^{2n+1}\cos 2\theta + q^{4n+2}).\end{aligned} \tag{1.8}$$

Brown, Evans and Ismail [**7**] defined a q-differentiable function f on the space $L^2[(1-x^2)^{-1/2}]$, $|x| \le 1$, as a function that has Fourier-Chebyshev expansions $f(x) \sim \sum_{n=0}^{\infty} f_n T_n(x)$ with the property that

$$\sum_{n=0}^{\infty} |q^{-n/2}(1-q^n) f_n|^2 < \infty, \tag{1.9}$$

and its q-derivative defined as

$$\mathcal{D}_q f(x) = \sum_{n=1}^{\infty} q^{(n-1)/2} \frac{1-q^n}{1-q} f_n U_{n-1}(x). \tag{1.10}$$

Clearly the polynomials form a dense subset of $L^2[(1-x^2)^{-1/2}]$, and their image under $\mathcal{D}_q$ is a dense subset of $L^2[(1-x^2)^{1/2}]$.

In a later paper [**12**], Ismail and Zhang extended these results to subsets of $L^2[w_\beta(x|q)]$ for $\mathcal{D}_q$ and of $L^2[w_{\beta q}(x|q)]$ for $\mathcal{D}_q^{-1}$, where

$$w_\beta(x|q) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(\beta e^{2i\theta}, \beta e^{-2i\theta}; q)_\infty} \frac{1}{\sqrt{1-x^2}}, x = \cos\theta, \tag{1.11}$$

the infinite products above being defined through the q-shifted factorials. We shall follow the notation in [**9**] and [**1**].

In [**12**], the kernel of $\mathcal{D}_q^{-1}$ was found to be

$$\begin{aligned} K_\beta(x,y|q) = {} & \frac{(1-q)(q, q^2\beta^2; q)_\infty}{4\pi(\beta, \beta q; q)_\infty} \\ & \times \sum_{n=1}^{\infty} \frac{1-\beta q^n}{(q^2\beta^2; q)_{n-1}} (q;q)_{n-1} q^{(n-1)/2} \\ & \times C_n(x;\beta|q) C_{n-1}(y;\beta q|q), \end{aligned} \tag{1.12}$$

where

$$\begin{aligned} C_n(\cos\theta;\beta|q) &= \sum_{k=0}^{n} \frac{(\beta;q)_k(\beta;q)_{n-k}}{(q;q)_k(q;q)_{n-k}} e^{i(n-2k)\theta} \\ &= \frac{(\beta^2;q)_n}{(q;q)_n} \beta^{-n/2} {}_4\phi_3\left(\left. \begin{matrix} q^{-n}, \beta^2 q^n, \beta^{1/2}e^{i\theta}, \beta^{1/2}e^{-i\theta} \\ \beta q^{1/2}, -\beta, -\beta q^{1/2} \end{matrix} \right| q, q \right), \end{aligned} \tag{1.13}$$

are the continuous q-ultraspherical polynomials, see [**2, 9**] and ${}_{r+1}\phi_r$ is a basic hypergeometric series whose properties are stated in detail in [**9**]. A further extension was made by Ismail, Rahman and Zhang [**10**] by taking the operator $\mathcal{D}_q$ on the space $L^2[w_{\alpha,\beta}(x|q)]$, and the inverse operator $\mathcal{D}_q^{-1}$ on $L^2[w_{\alpha+1,\beta+1}(x|q)]$, where

(1.14)
$$w_{\alpha,\beta}(x|q^2) = \frac{(e^{2i\theta}, e^{-2i\theta}; q^2)_\infty / \sin\theta}{(q^{\alpha+1/2}e^{i\theta}, q^{\alpha+1/2}e^{-i\theta}, -q^{\beta+1/2}e^{i\theta}, -q^{\beta+1/2}e^{-i\theta}; q)_\infty},$$

$0 \le \theta \le \pi$, is the weight function for the continuous q-Jacobi polynomials of Askey and Wilson [**9, 4**]:

(1.15)
$$P_n^{(\alpha,\beta)}(x|q) = \frac{(q^{\alpha+1}; q)_n}{(q;q)_n} {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, q^{(2\alpha+1)/4}e^{i\theta}, q^{(2\alpha+1)/4}e^{-i\theta} \\ q^{\alpha+1}, -q^{(\alpha+\beta+1)/2}, -q^{(\alpha+\beta+2)/2} \end{matrix}\,\middle|\, q, q\right).$$

It was found in [**10**] that the kernel of the inverse operator is

(1.16)
$$K_{\alpha,\beta}(x,y|q) = \sum_{n=0}^\infty \frac{(1-q)(1+q^{(\alpha+\beta+1)/2})(1+q^{(\alpha+\beta+2)/2})}{2(1-q^{\alpha+\beta+n+2})h_n^{(\alpha+1,\beta+1)}(q)} q^{n-(2\alpha+1)/2} \times P_{n+1}^{(\alpha,\beta)}(x|q) P_n^{(\alpha+1,\beta+1)}(y|q),$$

where

(1.17)
$$h_n^{(a,b)}(q) = \frac{2\pi(1-q^{a+b+1})(q^{(a+b+2)/2}; q^{1/2})_\infty}{(q, q^{a+1}, q^{b+1}; q)_\infty (-q^{(a+b+1)/2}; q^{1/2})_\infty} \times \frac{(q^{a+1}, q^{b+1}, -q^{(a+b+3)/2}; q)_n q^{n(2a+1)/2}}{(1-q^{2n+a+b+1})(q, q^{a+b+1}, -q^{(a+b+1)/2}; q)_n},$$

are the normalization constants in the orthogonality relation

(1.18)
$$\int_{-1}^1 P_n^{(a,b)}(x|q) P_m^{(a,b)}(x|q) w_{a,b}(x|q)\, dx = h_n^{(a,b)}(q)\delta_{m,n}.$$

No attempt was made in [**12**], respectively [**10**], to compute the sum on the righthand side of (1.12), respectively (1.16). Instead, the problem of diagonalizing the inverse operators was studied in detail, with eigenvalues and eigenvectors determined explicitly. However, a closed-form expression of the kernel was found quite useful in [**6**] in proving the boundedness of the operator as well as its other analytic properties. Also, the concept of an indefinite integral, or an anti-q-derivative, was not dealt with to any degree of seriousness in any of the previous papers. In particular, the analogue of the so-called "arbitrary constant" in q-calculus was not discussed.

Furthermore, no attempt was made in either [**12**] or [**10**] to give a proof of the essential property $\mathcal{D}_q\mathcal{D}_q^{-1} = I$. Our objective in this paper is to address some of those questions. First of all, we will give in Section 2 an exact evaluation of the kernel in (1.16):

(1.19)
$$\begin{aligned} K_{\alpha,\beta}(\cos\theta, \cos\phi|q) &= \frac{(1-q)(q,q;q)_\infty}{4\pi c q^{1/2}} \\ &\quad \times \frac{h(\cos\theta; -cq^{1/2} - q^{1/2}/c)h(\cos\phi; aq^{1/2}, aq, -cq^{1/2})}{h(\cos\phi; q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta}, -1/c)} \\ &\quad - L_{\alpha,\beta}(\cos\phi|q), \quad ac \neq 0, \end{aligned}$$

where $a = q^{(2\alpha+1)/2}$, $c = q^{(2\beta+1)/2}$, with

(1.20)
$$\begin{gathered} h(\cos\theta; a_1, \dots, a_n) = \prod_{j=1}^{n} h(\cos\theta; a_j), \\ h(\cos\theta; a) = \prod_{j=0}^{\infty}(1 - 2aq^j\cos\theta + a^2q^{2j}) = (ae^{i\theta}, ae^{-i\theta}; q)_\infty, \end{gathered}$$

and

(1.21)
$$\begin{aligned} L_{\alpha,\beta}(\cos\phi|q) &= \frac{(1-q)(q, a^2q^{3/2}, c^2q^{3/2}; q)_\infty}{4\pi c q^{1/2}(qa^2, acq^{1/2}; q^{1/2})_\infty} \\ &\quad \times \frac{h(\cos\phi; -qa^2c)}{h(\cos\phi; -1/c)}(q^{1/2}, -aq^{1/2}/c; q)_\infty \\ &\quad \times {}_8W_7\left(a^2; a^2q^{1/2}, -ac, -acq^{1/2}, -\frac{e^{i\phi}}{c}, -\frac{e^{-i\phi}}{c}; q, q\right). \end{aligned}$$

In (1.21) the W function is a very well-poised series [**1, 9**]

(1.22)
$$\begin{aligned} &{}_{2r+2}W_{2r+1}(a; b_1, b_2, \dots, b_{2r-1}; q, z) \\ &\quad := {}_{2r+2}\phi_{2r+1}\left(\begin{matrix} a, qa^{1/2}, -qa^{1/2}, b_1, \dots, b_{2r-1} \\ a^{1/2}, -a^{1/2}, aq/b_1, \dots, aq/b_{2r-1} \end{matrix}\middle| q, z\right). \end{aligned}$$

It follows from the above considerations that an indefinite q-integral, or an inverse to $\mathcal{D}_q$, may be defined as

(1.23)
$$\begin{aligned}(\mathcal{D}_q^{-1}f)(\cos\theta) &= F(\cos\theta)\\ &= \int_0^\pi K_{\alpha,\beta}(\cos\theta,\cos\phi|q)f(\cos\phi)w_{\alpha+1,\beta+1}(\cos\phi|q)\,d\phi\\ &= \frac{(1-q)(q,q;q)_\infty}{4\pi cq^{1/2}}h(\cos\theta;-cq^{1/2},-q^{1/2}/c)\\ &\quad\times\int_0^\pi \frac{(e^{2i\varphi},e^{-2i\varphi};q)_\infty f(\cos\theta)\,d\theta}{h(\cos\phi;q^{1/2}e^{i\theta},q^{1/2}e^{-i\theta})h(\cos\phi;-1/c,-cq)}\\ &\quad-\int_0^\pi L_{\alpha,\beta}(\cos\phi|q)f(\cos\phi)w_{\alpha,\beta}(\cos\phi|q)\,d\phi.\end{aligned}$$

The second integral on the righthand side of (1.23) may seem to be troublesome but, in fact, it is a θ-independent constant that is absorbed in the "constant of integration." In the q-calculus, this constant need not be an absolute constant, rather, a function whose q-derivative is zero. A second point that may cause some concern is the appearance of $1/c$ in an h-function in the integrand of the first term on the righthand side of (1.23). However, the apparent singularity that could arise if $c > 1$ is neutralized by $h(\cos\phi;-cq)$. All we really need to assume is that a positive integer r exists such that $cq < q^r < c$.

In Section 3 we shall discuss some other properties of the kernel in (1.19) as well as show that

$$\lim_{t\to 1^-}\mathcal{D}_q\mathcal{D}_q^{-t}g(x) = g(x), \tag{1.24}$$

where $0 < t < 1$ and the kernel of $\mathcal{D}_q^{-t}$ is $K^t_{\alpha,\beta}(x,y|q)$ which we will define to be almost the same as that in (1.16) except for a factor t^n inside the infinite series. There are many instances of kernels of this type in the literature and the reader may consult [**3**] for interesting examples. In Section 4 we shall prove that the derivative of the kernel in the first integral on the righthand side of (1.23), with respect to $x = \cos\theta$, is positive, thus establishing its monotonicity in $\cos\theta$. In Section 5 we give an integral representation of the t-commutator $\mathcal{D}_q\mathcal{D}_q^{-t} - t\mathcal{D}_q^{-t}\mathcal{D}_q$ and set up a general procedure for representing this commutator in terms of Poisson kernels. The limiting value as $t \to 1^-$

is an analogue of

$$\frac{d}{dx}\int_a^x f(y)\,dy - \int_a^x \frac{df(y)}{dy}\,dy = f(a). \tag{1.25}$$

The reason is that $f(a)$ is the constant term in the expansion of f in the basis $\{(x-a)^n : n = 0, 1, \dots\}$, the Taylor series, while the limit of $(\mathcal{D}_q\mathcal{D}_q^{-t} - t\mathcal{D}_q^{-t}\mathcal{D}_q f)(x)$ as $t \to 1^-$ is the constant term of the expansion of f in a series of continuous q-Jacobi polynomials.

2. Computation of the kernel of $\mathcal{D}_q^{-1}$. Setting $a = q^{(2\alpha+1)/2}$ and $c = q^{(2\beta+1)/2}$ in (1.16), and using (1.15) and (1.17), we find that

$$K_{\alpha,\beta}(x,y|q) = \frac{(1-q)(-ac, -acq^{1/2}, q, a^2q^{3/2}, c^2q^{3/2}; q)_\infty}{4\pi a(acq^{3/2}, acq; q)_\infty} G(x,y), \tag{2.1}$$

where

$$\begin{aligned} G(x,y) &= \sum_{n=0}^{\infty} \frac{(1+acq)(1-a^2c^2q^{2n+2})(a^2c^2q^2;q)_n(a^2q^{1/2};q)_{n+1}a^{-2n}}{(1-a^2c^2q^2)(1+acq^{n+1})(q;q)_{n+1}(c^2q^{3/2};q)_n(1-ac^2c^2q^{n+1})} \\ &\quad \times p_{n+1}(x; a, aq^{1/2}, -c, -cq^{1/2}|q)p_n(y; aq^{1/2}, aq, -cq^{1/2}, -cq|q), \end{aligned} \tag{2.2}$$

and

$$p_k(\cos\theta; a, b, c, d|q) = {}_4\phi_3\left(\begin{matrix} q^{-k}, abcdq^{k-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}\middle| q, q\right), \tag{2.3}$$

is an Askey-Wilson polynomial, see [**4**] and [**9**]. Ismail and Wilson [**11**] applied the Sears transformation [**9**] to obtain the representation

$$\begin{aligned} p_n(\cos\theta; a, b, c, d|q) = \frac{a^n(q, cd; q)_n}{(ac, ad; q)_n} \sum_{k=0}^{n} \frac{(ae^{i\theta}, be^{i\theta}; q)_k}{(ab, q; q)_k} \\ \times \frac{(ce^{-i\theta}, de^{-i\theta}; q)_{n-k}}{(q, cd; q)_{n-k}} e^{i(n-2k)\theta}. \end{aligned} \tag{2.4}$$

This shows that, for $\max\{|a|, |b|, |c|, |d|\} < 1$, there is a constant C which may depend on a, b, c, d and q but is independent of n such that

$$|p_n(x; a, b, c, d|q)| \le Cna^n, \quad x \in [-1, 1]. \tag{2.5}$$

By [**9**]

$$
\begin{aligned}
&p_n(\cos\theta; a,b,c,d|q)\\
&\quad = \frac{(bc;q)_n}{A(\theta)(ad;q)_n}\int_{qe^{i\theta}/d}^{qe^{-i\theta}/d} \frac{(due^{i\theta}, due^{-i\theta}, abcdu/q;q)_\infty}{(dau/q, dbu/q, dcu/q;q)_\infty}\\
&\qquad\qquad \times \frac{(q/u;q)_n}{(abcdu/q;q)_n}(adu/q)^n\, d_q u
\end{aligned}
$$

where

(2.7)

$$
\begin{aligned}
A(\theta; a,b,c,d) = -\frac{iq(1-q)}{2d}&(q, ab, ac, bc;q)_\infty\\
&\times h(\cos\theta; d) w(\cos\theta; a,b,c,d|q),
\end{aligned}
$$

$$
w(\cos\theta; a,b,c,d|q) := \frac{(e^{2i\theta}, e^{-2i\theta};q)_\infty}{h(\cos\theta; a,b,c,d|q)\sqrt{1-x^2}}, \tag{2.8}
$$

and the q-integral is defined by

$$
\int_a^b f(x)\, d_q x = \int_0^b f(x)\, d_q x - \int_0^a f(x)\, d_q x, \tag{2.9}
$$

$$
\int_0^a f(x)\, d_q x = a(1-q)\sum_{n=0}^\infty f(aq^n)q^n, \tag{2.10}
$$

provided the infinite series on the righthand side of (2.10) converges. For the two Askey-Wilson polynomials in (2.2), we need to make judicious choice of the parameters that simplify the calculations. To that end, we take $(a, aq^{1/2}, -cq^{1/2}, -c)$ for (a,b,c,d) in $p_{n+1}(x; a, aq^{1/2}, -c, -cq^{1/2}|q)$ and $(aq^{1/2}, aq, -cq^{1/2}, -cq)$ for the same quartet in $p_n(y; aq^{1/2}, aq, -cq^{1/2}, -cq|q)$. This gives

$$
\begin{aligned}
&p_{n+1}(x; a, aq^{1/2}, -c, -cq^{1/2}|q)\\
&\qquad = \frac{(1+acq^{n+1})}{(1+ac)B(\theta)}\\
&\qquad\quad \times \int_{-qe^{i\theta}/c}^{-qe^{-i\theta}/c} \frac{(-cve^{i\theta}, -cve^{-i\theta}, a^2c^2v;q)_\infty}{(-acv/q, -acvq^{-1/2}, c^2vq^{-1/2};q)_\infty}\\
&\qquad\quad \times \frac{(q/v;q)_{n+1}}{(a^2c^2v;q)_n}(-acv/q)^{n+1}\, d_q v
\end{aligned} \tag{2.11}
$$

and

$$(2.12)\quad p_n(y; aq^{1/2}, aq, -cq^{1/2}, -cq|q)$$
$$= \frac{1}{A(\phi)} \int_{-e^{i\phi}/c}^{-e^{-i\phi}/c} \frac{(-cque^{i\phi}, -cque^{-i\phi}, a^2c^2q^2u; q)_\infty}{(-acq^{1/2}u, -acqu, c^2q^{1/2}u; q)_\infty} \times \frac{(q/u;q)_n}{(a^2c^2q^2u;q)_n}(-acuq^{1/2})^{n+1}\, d_qu$$

where $A(\phi) = A(\phi; aq^{1/2}, aq, -cq^{1/2}, -cq)$ and $B(\theta) = A(\theta; a, aq^{1/2}, -cq, -cq^{1/2})$, that is,

(2.13)
$$A(\phi) = \frac{i(1-q)}{2c}(q, -acq, -acq^{3/2}, a^2q^{3/2}; q)_\infty \times h(\cos\phi; -cq)w(\cos\phi; aq^{1/2}, aq, -cq^{1/2}, -cq|q),$$

(2.14)
$$B(\theta) = \frac{iq(1-q)}{2c}(q, -acq^{1/2}, -acq, a^2q^{1/2}; q)_\infty \times h(\cos\theta; -c)w(\cos\theta; a, aq^{1/2}, -c, -cq^{1/2}|q).$$

The bound in (2.5) establishes the uniform convergence of the infinite series in (2.2). Thus we find, after some simplifications, that

$$(2.15)\quad A(\phi)B(\theta)G(\cos\theta, \cos\phi)$$
$$= \int_{-e^{i\phi}/c}^{-e^{-i\phi}/c} \frac{(-cque^{i\phi}, -cque^{-i\phi}, a^2c^2q^2u; q)_\infty}{(-acq^{1/2}u, -acqu, c^2q^{1/2}u; q)_\infty} \times \int_{-qe^{i\theta}/c}^{-qe^{-i\theta}/c} \frac{(-cve^{i\theta}, -cve^{-i\theta}, a^2c^2v; q)_\infty}{(-acv/q, -acq^{-1/2}v, c^2q^{-1/2}v; q)_\infty} J(u,v)\, d_qu\, d_qv,$$

where

(2.16)
$$J(u,v) = \frac{a(1-c^2q^{1/2})}{cq^{1/2}(1+ac)(1-qac)(1-qa^2c^2)} \times \left\{ \frac{(qa^2c^2, c^2q^{1/2}u, c^2q^{-1/2}v, a^2c^2uv, qu; q)_\infty}{(c^2q^{1/2}, a^2c^2q^2u, a^2c^2v, c^2q^{-1/2}uv, u; q)_\infty} - \frac{1-qa^2c^2u}{1-u} \right\},$$

which is obtained by the use of the ${}_6\phi_5$-summation formula [**9**]. We wish to remark here that this simplification would not be so easily possible if we did not choose the parameters as we did in (2.11) and (2.12).

So we get

$$G(x,y)=G_1(x,y)-G_2(x,y), \tag{2.17}$$

where

$$\begin{aligned} G_1(x,y)=\frac{aq^{-1/2}}{A(\phi)B(\theta)}&\frac{(a^2c^2q^2;q)_\infty/(1+ac)}{c(1-qac)(c^2q^{3/2};q)_\infty} \\ &\times\int_{-e^{i\phi}/c}^{-e^{-i\phi}/c}\frac{(-cque^{i\phi},-cque^{-i\phi},qu;q)_\infty}{(-acq^{1/2}u,-acqu,u;q)_\infty}\,d_qu \\ &\times\int_{-e^{i\theta}/c}^{-e^{-i\theta}/c}\frac{(-cve^{i\theta},-cve^{-i\theta},a^2c^2uv;q)_\infty\,d_qv}{(-acv/q,-acvq^{-1/2},c^2uv/q;q)_\infty}, \end{aligned}\tag{2.18}$$

and

(2.19)

$$\begin{aligned} G_2(x,y)=\frac{aq^{-1/2}}{A(\phi)B(\theta)}&\frac{(1-c^2q^{1/2})/(1+ac)}{c(1-qac)(1-qa^2c^2)} \\ &\times\int_{-e^{i\phi}/c}^{-e^{-i\phi}/c}\frac{(-cque^{i\phi},-cque^{-i\phi},qa^2c^2u,qu;q)_\infty}{(-acq^{1/2}u,-acqu,c^2q^{1/2}u,u;q)_\infty}\,d_qu \\ &\times\int_{-e^{i\theta}/c}^{-e^{-i\theta}/c}\frac{(-cve^{i\theta},-cve^{-i\theta},a^2c^2v;q)_\infty\,d_qv}{(-acv/q),-acvq^{-1/2},c^2q^{-1/2}v;q_\infty}. \end{aligned}$$

Using [**9**] twice we find that

(2.20)

$$\begin{aligned} G_1(x,y)=&\frac{aq^{-1/2}(q,a^2c^2q^2;q)_\infty h(x;-cq^{1/2},-q^{1/2}/c)}{c(1-qac)(-ac;q^{1/2})_\infty(a^2q^{3/2},c^2q^{3/2},-acq,-acq^{3/2};q)_\infty} \\ &\times\frac{h(y;aq^{1/2},aq,-cq^{1/2})}{h(y;q^{1/2}e^{i\theta},q^{1/2}e^{-i\theta},-1/c)}. \end{aligned}$$

For $G_2(x,y)$ we see that the u and v integrals are decoupled and that the v-integral is, clearly, $B(\theta)$, by [**9**] and (2.14). Also, by [**9**] the u-integral is a simple ${}_8W_7$ series that leads to

(2.21)

$$\begin{aligned} G_2(x,y)=&\frac{aq^{-1/2}(1-c^2q^{1/2})(q^{1/2},-aq^{1/2}/c;q^{1/2})_\infty h(y;-a^2cq)}{c(1-qac)(1-acq^{1/2})(-ac,qa^2;q^{1/2})_\infty h(y;-1/c)} \\ &\times{}_8W_7(a^2;a^2q^{1/2},-ac,-acq^{1/2},-e^{i\phi}/c,-e^{-i\phi}/c;q,q). \end{aligned}$$

Substituting (2.20) and (2.21) in (2.17) and (2.1) then simplifying the coefficients, we finally obtain (1.19) and (1.21).

In order to analyze the limit in (1.20) we introduce the one-parameter family of kernels

$$\begin{aligned}K^t_{\alpha,\beta}(x,y|q) = \sum_{n=0}^{\infty} &\frac{(1-q)(1+q^{(\alpha+\beta+1)/2})(1+q^{(\alpha+\beta+2)/2})}{2(1-q^{\alpha+\beta+n+2})h_n^{(\alpha+1,\beta+1)}(q)} \\ &\times q^{n-(2\alpha+1)/2} P_{n+1}^{(\alpha,\beta)}(x|q) P_n^{(\alpha+1,\beta+1)}(y|q) t^n,\end{aligned} \tag{2.22}$$

and the corresponding family of integral operators

$$\begin{aligned}(\mathcal{D}_q^{-t} f)(\cos\theta) = \int_0^{\pi} &K^t_{\alpha,\beta}(\cos\theta, \cos\phi|q) f(\cos\phi) \\ &\times w_{\alpha+1,\beta+1}(\cos\phi|q)\sin\phi\, d\phi,\end{aligned} \tag{2.23}$$

where $h_n^{(a,b)}(q)$ is as in (1.17).

3. Some limiting properties of the kernel. Using $c = q^{(2\beta+1)/4}$, the q-gamma function

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad 0 < q < 1, \quad x \neq 0, -1, -2, \dots, \tag{3.1}$$

see [**1**] and the notation $(a;q)_\alpha = (a;q)_\infty/(aq^\alpha;q)_\infty$, for real α, one can show that the first term on the extreme righthand side of (1.23) can be written in the following form

(3.2)

$$\begin{aligned}\frac{\Gamma_q^2(1/2)}{2\pi} &q^{-(2\beta+3)/4}(-q^{(2\beta+3)/4}e^{i\theta}, -q^{(2\beta+3)/4}e^{-i\theta}; q)_{-(2\beta+1)/4} \\ &\times (-q^{(1-2\beta)/4}e^{i\theta}, -q^{-(2\beta+1)/4}e^{-i\theta}; q)_{(2\beta+1)/4} \\ &\times \int_0^{\pi} \frac{(e^{2i\theta}, e^{-2i\theta}; q^2)_{1/2}}{(-q^{1/2}e^{i\phi}, -q^{1/2}e^{-i\phi}; q)_{(2\beta+3)/4}} \\ &\qquad \times \frac{M(\cos\theta, \cos\phi|q) g(\cos\phi)\, d\phi}{(-q^{1/2}e^{i\phi}, -q^{1/2}e^{-i\phi}; q)_{-(2\beta+3)/4}},\end{aligned}$$

where
(3.3)
$$M(\cos\theta,\cos\phi)=\frac{(q^{1/2},q^{1/2}e^{i\phi},q^{1/2}e^{-i\phi},-q^{1/2}e^{i\theta},-q^{1/2}e^{-i\theta};q)_\infty^2}{2(q^{1/2}e^{i(\theta+\phi)},q^{1/2}e^{i(\theta-\phi)},q^{1/2}e^{i(\phi-\theta)},q^{1/2}e^{-i(\theta+\phi)};q)_\infty}.$$
It was shown in [**14**] that
$$\lim_{q\to1^-}M(\cos\theta,\cos\phi)=H(\cos\theta-\cos\phi), \tag{3.4}$$
where $H(t)$ is the Heaviside unit function. Since
$$\begin{gathered}\lim_{q\to1^-}(a;q)_\alpha=(1-a)^\alpha,\qquad \lim_{q\to1^-}\Gamma_q(1/2)=\sqrt{\pi},\\ \lim_{q\to1^-}(e^{2i\phi},e^{-2i\phi};q)_{1/2}=2\sin\phi,\end{gathered} \tag{3.5}$$
we have that the limit of the expression in (3.2) as $q\to1^-$ is
$$\int_{-1}^{1}H(x-y)g(y)\,dy=\int_{-1}^{x}g(y)\,dy, \tag{3.6}$$
which is of course the indefinite integral of elementary calculus.

It is also of interest to compare the Chebyshev limits, that is, $a\to1^-$, $c\to1^-$, of (1.19) with the expression found in [**6**]. First of all, note that [**9**] gives
$$\begin{aligned}&(-a^2cqe^{i\phi},-a^2cqe^{-i\phi};q)_\infty\,{}_8W_7(a^2;a^2q^{1/2},-ac,\\&\qquad -acq^{1/2},-e^{i\phi}/c,-e^{-i\phi}/c;q,q)\\&=\frac{(qa^2,q^{1/2},q^{1/2}/c^2,aqe^{i\phi},aqe^{-i\phi},aq^{1/2}e^{i\phi},aq^{1/2}e^{-i\phi},qa^2c^2;q)_\infty}{(-aq/c,-aq^{1/2}/c,-acq^{1/2},-acq,-q^{1/2}e^{i\phi}/c,-q^{1/2}e^{-i\phi}/c;q)_\infty}\\&\quad+\frac{(qa^2,-ac,-e^{i\phi}/c,-e^{-i\phi}/c,-a^2cq^{3/2}e^{i\phi},-a^2cq^{3/2}e^{-i\phi},q^{3/2},-aq^{3/2}/c;q)_\infty}{q^{-1/2}(q^{1/2},-aq^{1/2}/c,-acq,a^2q^2,-q^{1/2}e^{i\phi}/c,-q^{1/2}e^{-i\phi}/c;q)_\infty}\\&\quad\times{}_8W_7(qa^2;a^2q^{1/2},-acq^{1/2},-acq,-q^{1/2}e^{i\phi}/c,-q^{1/2}e^{-i\phi}/c;q,q).\end{aligned} \tag{3.7}$$
Setting $a=1$, $c=1$ on the righthand side of (3.7), simplifying and using Exercise 15 in [**15**, page 489], we find that
$$\begin{aligned}&\lim_{(a,c)\to(1,1)}{}_8W_7(a^2;a^2q^{1/2},-ac,-acq^{1/2},-e^{i\phi}/c,-e^{-i\phi}/c;q,q)\\&\qquad=\left[\frac{(q^{1/2};q)_\infty}{(-q^{1/2};q)_\infty}\right]^2\frac{h(\cos\phi;q^{1/2},q)}{h(\cos;-q^{1/2},-q)}\\&\qquad\quad-\frac{1+\cos\phi}{\sin\phi}q^{1/2}\frac{\vartheta_3'(\frac{\phi}{2}|q^{1/2})}{\vartheta_3(\frac{\phi}{2}|q^{1/2})},\end{aligned} \tag{3.8}$$

where $\vartheta_3(x|q) = \vartheta_4(x+\pi/2|q)$, see [**15**] and (1.8). However, the lefthand side of (3.8) is clearly

$$
\begin{aligned}
(3.9) \qquad & 1+2\sum_{n=1}^{\infty}\frac{(-e^{\phi},-e^{-i\phi};q)_n}{(-qe^{\phi},-qe^{-i\phi};q)_n}q^n \\
&= 1-\frac{1+\cos\phi}{\sin\phi}\left[\tan(\frac{\phi}{2})+\frac{\vartheta_2'(\frac{\phi}{2}|q^{1/2})}{\vartheta_2(\frac{\phi}{2}|q^{1/2})}\right] \\
&= \quad -\cot(\frac{\phi}{2})\frac{\vartheta_2'(\frac{\phi}{2}|q^{1/2})}{\vartheta_2(\frac{\phi}{2}|q^{1/2})}.
\end{aligned}
$$

In going from the second line to the third we used Exercise 15, [**15**, page 489]. This leads to the following identity

$$
\begin{aligned}
(3.10) \quad & q^{1/2}\frac{\vartheta_3'(\frac{\phi}{2}|q^{1/2})}{\vartheta_3(\frac{\phi}{2}|q^{1/2})}-\frac{\vartheta_2'(\frac{\phi}{2}|q^{1/2})}{\vartheta_2(\frac{\phi}{2}|q^{1/2})} \\
&= \left[\frac{(q^{1/2};q)_\infty}{(-q^{1/2};q)_\infty}\right]^2\tan(\frac{\phi}{2})\frac{h(\cos\phi;q^{1/2},q)}{h(\cos\phi;-q^{1/2},-q)},
\end{aligned}
$$

which seems to be new. Also in view of equations (2.5) and (2.8) of [**6**], this results in another seemingly new identity, namely,

$$
\begin{aligned}
(3.11) \quad & \frac{\vartheta_4'((\theta+\phi)/2|q^{1/2})}{\vartheta_4((\theta+\phi)/2|q^{1/2})}-\frac{\vartheta_4'((\theta-\phi)/2|q^{1/2})}{\vartheta_4((\theta-\phi)/2|q^{1/2})} \\
&= \sin\phi\frac{(q;q)_\infty^2h(\cos\theta;-q^{1/2},-q^{1/2})h(\cos\phi;q^{1/2},q,-q^{1/2})}{h(\cos\phi;q^{1/2}e^{i\theta},q^{1/2}e^{-i\theta},-1)} \\
&\qquad -\frac{1}{(1-q)^{1/2}}\frac{\vartheta_2'((\frac{\phi}{2}|q^{1/2}))}{2\vartheta_2((\frac{\phi}{2}|q^{1/2}))}.
\end{aligned}
$$

Before considering the limit (1.24) we would like to point out that it is not possible to obtain the q-Hermite limit $(a,c\to 0)$ from (1.19) for the obvious reason that the representation (2.3) for $p_n(x;a,aq^{1/2},-c,-cq^{1/2}|q)$ is not valid in this limit. Brown and Ismail [**6**] found the kernel for the q-Hermite case by a separate calculation. The only hope of finding a general formula that will contain all the special and limiting cases is to densely define the $\mathcal{D}_q$ operator on the larger space $L^2[-1,1]$ weighted by $w(x;a,b,c,d|q)$ where a,b,c,d are only restricted by their absolute values being less than 1.

To prove (1.24) we first observe that

(3.12)
$$\mathcal{D}_q K^t_{\alpha,\beta}(x,y|q) = P_t(x,y;aq^{1/2},aq,-cq^{1/2},-cq|q), \quad 0<t<1,$$

where P_t is the Poisson kernel as defined in [**13**] and

(3.13)
$$K^t_{\alpha,\beta}(x,y|q) = \sum_{n=0}^{\infty} \frac{(1-q)(1+q^{(\alpha+\beta+1)/2})(1+q^{(\alpha+\beta+2)/2})t^n}{2(1-q^{\alpha+\beta+n+2})h_n^{(\alpha+1,\beta+1)}(q)} q^{n-\alpha-1/2} \times P_{n+1}^{(\alpha,\beta)}(x|q)P_n^{(\alpha+1,\beta+1)}(y|q),$$

with $a=q^{(2\alpha+1)/4}$, $c=q^{(2\beta+1)/4}$. Thus we have

(3.14)
$$\mathcal{D}_q D_q^{-t} g(\cos\theta) = \int_0^\pi P_t(\cos\theta,\cos\phi;aq^{1/2},aq,-cq^{1/2},-cq|q)w_{\alpha+1,\beta+1}(\cos\phi|q)\,d\phi.$$

However, by [**13**],

(3.15)
$$P_t(\cos\theta,\cos\phi;aq^{1/2},aq,-cq^{1/2},-cq|q)$$

$$= \frac{(q,a^2q^{3/2},c^2q^{3/2},-acq,-acq^{3/2};q)_\infty}{2\pi(acq^{3/2},acq^2;q)_\infty}$$
$$\times\Bigg\{ \frac{(1-t^2)(-acq^{3/2}t;q)_\infty}{(-tq^{-3/2}/ac;q)_\infty} \sum_{n=0}^{\infty} \frac{(acq^{3/2},acq^2,-cqe^{i\theta},-cqe^{-i\theta};q)_n}{(q,c^2q^{3/2},-acq^{3/2},-cq^{3/2}/a;q)_n}$$
$$\times \frac{(-ace^{i\phi},-ace^{-i\phi};q)_n}{(-actq^{3/2},-acq^{3/2}/t;q)_n} q^n$$
$$\times {}_{10}W_9(-aq^{-n-\frac{1}{2}}/c;q^{-n-1/2}/c^2,-q^{-n-1}/(ac),q^{-n},\ aq^{1/2}e^{i\theta}, aq^{1/2}e^{-i\theta},aq^{1/2}e^{i\phi},aq^{1/2}e^{-i\phi};q,q)$$
$$+\frac{(t,-at/c,a^2c^2q^3;q)_\infty}{(a^2q^{3/2},tc^2q^{3/2},-acq,-acq^{3/2},-acq^2,-a/c,-acq^{3/2}/t;q)_\infty}$$
$$\times \frac{h(\cos\theta;aq,-ctq^{1/2})h(\cos\phi;aq^{1/2},-ctq)}{h(\cos\phi;te^{i\theta},te^{-i\theta})}$$

$$\times \sum_{n=0}^{\infty} \frac{(-t, tq^{1/2}, -tq^{1/2}, tc^2q^{3/2}; q)_n}{(q, qt^2, -at/c, -tq^{-1/2}/ac; q)_n} \left| \frac{(te^{i(\theta+\phi)}, te^{i(\theta-\phi)}; q)_n}{(-ctq^{1/2}e^{i\theta}, -ctqe^{i\phi}; q)_n} \right|^2 q^n$$
$$\times {}_{10}W_9(c^2tq^{n+1/2}; tq^n, -ctq^n/a, -actq^{n+1/2}, -cqe^{i\theta}, -cqe^{-i\theta},$$
$$-cq^{1/2}e^{i\phi}, -cq^{1/2}e^{-i\phi}; q, q)$$
$$+ \frac{(t, -ct/a, a^2c^2q^3; q)_\infty}{(c^2q^{3/2}, ta^2q^{3/2}, -acq, -acq^{3/2}, -acq^2, -c/a, -acq^{3/2}/t; q)_\infty}$$
$$\times \frac{h(\cos\theta; -cq, atq^{1/2})h(\cos\phi; -cq^{1/2}, atq)}{h(\cos\phi; te^{i\theta}, te^{-i\theta})}$$
$$\times \sum_{n=0}^{\infty} \frac{(-t, tq^{1/2}, -tq^{1/2}, ta^2q^{3/2}; q)_n}{(q, qt^2, -ct/a, -tq^{-1/2}/ac; q)_n} \left| \frac{(te^{i(\theta+\phi)}, te^{i(\theta-\phi)}; q)_n}{(atq^{1/2}e^{i\theta}, atqe^{i\phi}; q)_n} \right|^2 q^n$$
$$\times {}_{10}W_9(a^2tq^{n+1/2}; tq^n, -atq^n/c, -actq^{n+1/2}, aqe^{i\theta}, aqe^{-i\theta},$$
$$\left. aq^{1/2}e^{i\phi}, aq^{1/2}e^{-i\phi}; q, q) \right\}.$$

In the above expressions it is assumed that all parameters and variables are real. If $|\theta \pm \phi| \neq 0$ or 2π, then

$$\lim_{t\to 1^-} P_t(\cos\theta; \cos\phi; aq^{1/2}, aq, -cq^{1/2}, -cq|q) = 0.$$

On the other hand when $|\theta \pm \phi| = 0$ or 2π, then the first term on the righthand side of (3.15) vanishes as do all the terms but the $n = 0$ term in both series on the righthand side, in the limit $t \to 1^-$. So we have, after some simplification,

(3.16)
$$\lim_{t\to 1^-} \mathcal{D}_q \mathcal{D}_q^{-t} g(\cos\theta) = \lim_{t\to 1^-} \frac{1-t^2}{2\pi \sin\theta} \int_{-\pi}^{\pi} \frac{g(\cos\phi)\sin\phi\, d\phi}{1 - 2t\cos(\theta-\phi) + t^2},$$

which is exactly the same expression that Brown and Ismail [**6**] had in proving the limiting result $\lim_{p\to q^+} \mathcal{D}_p \mathcal{D}_q^{-1} g(x) = g(x)$, $t = q/p$, at the points of continuity of g.

4. Properties of q-indefinite integral. We shall follow the notation for theta functions in Chapter 21 of Whittaker and Watson

[**15**], namely,

$$\begin{aligned}\vartheta_1(z,q) &= \sum_{-\infty}^{\infty}(-1)^n q^{(n+1/2)^2}\sin(2n+1)z \\ &= 2q^{1/4}\sin z(q^2, q^2e^{2iz}, q^2e^{-2iz}; q^2)_\infty,\end{aligned}$$

and

$$\vartheta_2(z,q) = \vartheta_1(z+\pi/2, q), \quad \vartheta_3(z,q) = \vartheta_4(z+\pi/2, q). \tag{4.2}$$

The main result in this section is Theorem 4.1 below.

Theorem 4.1. *Let $f(y) \geq 0$ and not be identically zero for all $y \in [-1,1]$. Then $F(x)$ defined in* (1.23) *is an increasing function of x.*

Proof. Let $c = e^{-\gamma}$, with $\gamma > 0$. To prove the monotonicity of $K_{\alpha,\beta}(x,y|q)$ in x, we need only to consider the θ-dependent part of the expression on the righthand side of (1.19), namely, $\mathcal{F}(\theta,\phi)$,

$$\mathcal{F}(\theta,\phi) = \frac{h(\cos\theta; -cq^{1/2}, -q^{1/2}/c)}{h(\cos\phi; q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta})}. \tag{4.3}$$

It follows that

$$\mathcal{F}(\theta,\phi) = \frac{\vartheta_4((\theta+\psi)/2, \sqrt{q})\vartheta_4((\theta-\psi)/2, \sqrt{q})}{\vartheta_4((\theta+\phi)/2, \sqrt{q})\vartheta_4((\theta-\phi)/2, \sqrt{q})}, \tag{4.4}$$

where

$$\psi := \gamma + i\pi. \tag{4.5}$$

Problem 18 of Chapter 21 in [**15**, page 490] asserts that

$$\begin{aligned}\frac{\vartheta_4'(y,q)}{\vartheta_4(y,q)} + \frac{\vartheta_4'(z,q)}{\vartheta_4(z,q)} = \frac{\vartheta_4'(y+z,q)}{\vartheta_4(y+z,q)} &+ \vartheta_2(0,q)\vartheta_3(0,q) \\ &\times \frac{\vartheta_1(y,q)\vartheta_1(z,q)\vartheta_1(y+z,q)}{\vartheta_4(y,q)\vartheta_4(z,q)\vartheta_4(y+z,q)}.\end{aligned} \tag{4.6}$$

Therefore it follows that

$$\begin{aligned}\frac{\partial_\theta \mathcal{F}(\theta,\phi)}{\mathcal{F}(\theta,\phi)} &= \vartheta_2(0,\sqrt{q})\vartheta_3(0,\sqrt{q})\frac{\vartheta_1(\theta,\sqrt{q})}{\vartheta_4(\theta,\sqrt{q})}\\ &\quad\times\bigg[\frac{\vartheta_1((\theta+\psi)/2,\sqrt{q})\vartheta_1((\theta-\psi)/2,\sqrt{q})}{\vartheta_4((\theta+\psi)/2,\sqrt{q})\vartheta_4((\theta-\psi)/2,\sqrt{q})}\\ &\qquad-\frac{\vartheta_1((\theta+\phi)/2,\sqrt{q})\vartheta_1((\theta-\phi)/2,\sqrt{q})}{\vartheta_4((\theta+\phi)/2,\sqrt{q})\vartheta_4((\theta-\phi)/2,\sqrt{q})}\bigg].\end{aligned}\tag{4.7}$$

Problem 1 of Chapter 21 in Whittaker and Watson [**15**, page 487] shows that the numerator inside the square brackets in (4.7) satisfies

$$\begin{aligned}&\vartheta_1\big(\frac{\theta+\psi}{2},\sqrt{q}\big)\vartheta_1\big(\frac{\theta-\psi}{2},\sqrt{q}\big)\vartheta_4\big(\frac{\theta+\phi}{2},\sqrt{q}\big)\vartheta_4\big(\frac{\theta-\phi}{2},\sqrt{q}\big)\\ &\quad-\vartheta_4\big(\frac{\theta+\psi}{2},\sqrt{q}\big)\vartheta_4\big(\frac{\theta-\psi}{2},\sqrt{q}\big)\vartheta_1\big(\frac{\theta+\phi}{2},\sqrt{q}\big)\vartheta_1\big(\frac{\theta-\phi}{2},\sqrt{q}\big)\\ &\quad=\big[\vartheta_1^2\big(\frac{\theta}{2},\sqrt{q}\big)\vartheta_4^2\big(\frac{\psi}{2},\sqrt{q}\big)-\vartheta_4^2\big(\frac{\theta}{2},\sqrt{q}\big)\vartheta_1^2\big(\frac{\psi}{2},\sqrt{q}\big)\big]\\ &\qquad\times\big[\vartheta_4^2\big(\frac{\theta}{2},\sqrt{q}\big)\vartheta_4^2\big(\frac{\phi}{2},\sqrt{q}\big)-\vartheta_1^2\big(\frac{\theta}{2},\sqrt{q}\big)\vartheta_1^2\big(\frac{\phi}{2},\sqrt{q}\big)\big]\vartheta_4^{-4}(0,\sqrt{q})\\ &\qquad-\big[\vartheta_4^2\big(\frac{\theta}{2},\sqrt{q}\big)\vartheta_4^2\big(\frac{\psi}{2},\sqrt{q}\big)-\vartheta_1^2\big(\frac{\theta}{2},\sqrt{q}\big)\vartheta_1^2\big(\frac{\psi}{2},\sqrt{q}\big)\big]\\ &\qquad\times\big[\vartheta_1^2\big(\frac{\theta}{2},\sqrt{q}\big)\vartheta_4^2\big(\frac{\phi}{2},\sqrt{q}\big)-\vartheta_4^2\big(\frac{\theta}{2},\sqrt{q}\big)\vartheta_4^2\big(\frac{\phi}{2},\sqrt{q}\big)\big]\vartheta_4^{-4}(0,\sqrt{q}).\end{aligned}$$

Upon the application of Exercises 1 on page 487 and 4 on page 488 in [**15**] the above expression simplifies to

$$\begin{aligned}&\vartheta_4^{-4}(0,\sqrt{q})\big[\vartheta_1^4\big(\frac{\theta}{2},\sqrt{q}\big)-\vartheta_4^4\big(\frac{\theta}{2},\sqrt{q}\big)\big]\\ &\qquad\times\big[\vartheta_4^2\big(\frac{\phi}{2},\sqrt{q}\big)\vartheta_1^2\big(\frac{\psi}{2},\sqrt{q}\big)-\vartheta_1^2\big(\frac{\phi}{2},\sqrt{q}\big)\vartheta_4^2\big(\frac{\psi}{2},\sqrt{q}\big)\big]\\ &\qquad\qquad=\vartheta_4(0,\sqrt{q})\vartheta_4(\theta,\sqrt{q})\vartheta_1\big(\frac{\phi+\psi}{2},\sqrt{q}\big)\vartheta_1\big(\frac{\phi-\psi}{2},\sqrt{q}\big).\end{aligned}$$

Now we use formulas (1.8), (4.1) and (4.2) to find

$$\begin{aligned}\vartheta_1\big(\frac{\phi\pm\psi}{2}),\sqrt{q}\big)&=\vartheta_1\big(\frac{\phi\pm i\gamma}{2}\pm\pi/2,\sqrt{q}\big)=\pm\vartheta_2\frac{\phi\pm i\gamma}{2},\sqrt{q}\big),\\ \vartheta_4\big(\frac{\phi\pm\psi}{2}),\sqrt{q}\big)&=\vartheta_4\big(\frac{\phi\pm i\gamma}{2}\pm\pi/2,\sqrt{q}\big)=\vartheta_3\big(\frac{\phi\pm i\gamma}{2},\sqrt{q}\big).\end{aligned}$$

These relationships imply $(\partial\mathcal{F}(\theta,\phi)/\partial\theta) < 0$, hence $\mathcal{F}(\theta,\phi)$ increases with $\cos\theta$ for $\theta \in [0,\pi]$. This completes the proof of Theorem 4.1.

The case $\alpha = \beta = -1/2$ of Theorem 3.1 was proved in [**7**].

5. Commutation relations. The main results of this section are formulas (5.16) and (5.17) below. We need some material from [**7**], which we now state. We shall use the inner product

$$\langle f, g\rangle := \int_{-1}^{1} f(x)\overline{g(x)}\frac{dx}{\sqrt{1-x^2}}. \tag{5.1}$$

Observe that the definition (5.1) requires $\breve{f}(z)$ to be defined for $|q^{\pm 1/2}z| = 1$ as well as for $|z| = 1$. In particular, $\mathcal{D}_q$ is well defined on $H_{1/2}$, where

$$H_\nu := \{f : f((z+1/z)/2) \text{ is analytic for } q^\nu \le |z| \le q^{-\nu}\}. \tag{5.2}$$

Theorem 5.1 (Integration by parts [**5**]). *The Askey-Wilson operator $\mathcal{D}_q$ satisfies*

$$\begin{aligned}\langle \mathcal{D}_q f, g\rangle = \frac{\pi\sqrt{q}}{1-q}&[f((q^{1/2}+q^{-1/2})/2))\overline{g(1)}\\ &- f(-(q^{1/2}+q^{-1/2})/2))\overline{g(-1)}]\\ &- \langle f, \sqrt{1-x^2}\mathcal{D}_q(g(x)(1-x^2)^{-1/2})\rangle,\end{aligned} \tag{5.3}$$

for $f, g \in H_{1/2}$.

The adjoint of $\mathcal{D}_q$ is $\mathcal{D}_q^*$,

$$(\mathcal{D}_q^* g)(x) = -\sqrt{1-x^2}\mathcal{D}_q\left(\frac{g(x)}{\sqrt{1-x^2}}\right) \tag{5.4}$$

which follows from applying the integration by parts formula (5.3), the orthogonality (1.18), [**5**].

We shall also need

$$\mathcal{D}_q P_n^{(\alpha,\beta)}(x|q) = \frac{2q^{-n+\frac{2\alpha+5}{4}}(1-q^{\alpha+\beta+n+1})}{(1+q^{\frac{\alpha+\beta+1}{2}})(1+q^{\frac{\alpha+\beta+2}{2}})(1-q)} P_{n-1}^{(\alpha+1,\beta+1)}(x|q), \tag{5.5}$$

[**9**] and its adjoint relation

$$\mathcal{D}_q(w_{\alpha+1,\beta+1}(x|q)P_{n-1}^{(\alpha+1,\beta+1)}(x|q)) = \frac{2(1-q^n)}{(q-1)}(1+q^{(\alpha+\beta+1)/2})(1+q^{(\alpha+\beta+2)/2}) \times q^{-(2\alpha+1)/4}w_{\alpha,\beta}(x|q)P_n^{(\alpha,\beta)}(x|q). \tag{5.6}$$

Instead of carrying out this calculation for the continuous q-Jacobi polynomials, we shall outline a formal general procedure whose steps can be easily justified in the case of continuous q-Jacobi polynomials.

Let $\{p_n(x;\mathbf{a})\}$ be a multi-parameter family of polynomials satisfying the orthogonality relation

$$\int_E p_m(x;\mathbf{a})p_n(x;\mathbf{a})w(x;\mathbf{a})\,dx = h_n(\mathbf{a})\delta_{m,n}, \tag{5.7}$$

where $\mathbf{a}$ stands for the multi-parameter vector $(a_1,\dots,a_r)$. Assume further that we have a lowering operator T so that

$$Tp_n(x;\mathbf{a}) = u_n(\mathbf{a})p_{n-1}(x;\mathbf{a}+\mathbf{1}), \tag{5.8}$$

where $\mathbf{a}+1 = (1+a_1,\dots,1+a_r)$. Let T^* be the adjoint of T with respect to the inner product

$$\langle f,g\rangle = \int_E f(x)\overline{g(x)}\frac{dx}{v(x)}. \tag{5.9}$$

It is important that v does not depend on any of the a-parameters and to assume that $v\ge 0$ on E together with the finiteness of $\int_E dx/v(x)$. Now consider the inner product space of functions with norms $\sqrt{\langle f,f\rangle}$. Thus (5.7) and (5.8) give

$$\begin{aligned} h_{n-1}(\mathbf{a}+1)\delta_{m,n} &= \langle p_{m-1}(.;\mathbf{a}+1), p_{n-1}(.;\mathbf{a}+1)v(.)w(.;\mathbf{a}+1)\rangle \\ &= \frac{1}{u_m(\mathbf{a})}\langle Tp_m(.;\mathbf{a}), p_{n-1}(.;\mathbf{a}+1)v(.)w(.;\mathbf{a}+1)\rangle \\ &= \frac{1}{u_m(\mathbf{a})}\langle p_m(.;\mathbf{a}), T^*p_{n-1}(.;\mathbf{a}+1)v(.)w(.;\mathbf{a}+1)\rangle. \end{aligned} \tag{5.10}$$

If the functions $\{\sqrt{w(x;\mathbf{a})}p_n(x;\mathbf{a})\}$ are complete in $\mathcal{H}$, then T^* is a raising operator in the sense

(5.11)
$$T^*(v(x)w(x;\mathbf{a}+1)p_{n-1}(x;\mathbf{a}+1)) = \frac{h_{n-1}(\mathbf{a}+1)}{h_n(\mathbf{a})}u_n(\mathbf{a})v(x)w(x;\mathbf{a})p_n(x;\mathbf{a}).$$

Sometimes (5.11) holds without the completeness assumption. In the case of continuous q-Jacobi polynomials $v(x) = \sqrt{1-x^2}$ and $T = \mathcal{D}_q$. Hence u_n is as in (5.5). According to (5.1)

$$(T^*f)(x) = -\sqrt{1-x^2}\mathcal{D}_q(f(x)/\sqrt{1-x^2}).$$

This leads to (5.6).

In general the analogues of (2.22) and (2.23) are

$$K^t(x,y;\mathbf{a}) = \sum_{n=0}^{\infty} \frac{p_{n+1}(x;\mathbf{a})}{h_n(\mathbf{a}+1)}\frac{p_n(y;\mathbf{a}+1)}{u_{n+1}(\mathbf{a})}t^n, \tag{5.12}$$

and

$$(T^{-t}f)(x) = \int_E K^t(x,y;\mathbf{a})w(y;\mathbf{a}+1)f(y)\,dy, \tag{5.13}$$

respectively. The general Poisson kernel is

$$P_t(x,y;\mathbf{a}) = \sum_{n=0}^{\infty} \frac{p_n(x;\mathbf{a})p_n(y;\mathbf{a})}{h_n(\mathbf{a})}t^n. \tag{5.14}$$

Recall that the t-commutator $[A,B]_t$ is $AB - tBA$. Therefore,

(5.15)
$$\begin{aligned}
([T,T^{-t}]_t f)(x) &= (TT^{-t}f)(x) - t(T^{-t}Tf)(x) \\
&= \int_E \sum_{n=0}^{\infty} \frac{p_n(x;\mathbf{a}+1)}{h_n(\mathbf{a}+1)}p_n(y;\mathbf{a}+1)t^n f(y)w(y;\mathbf{a}+1)\,dy \\
&\quad - \int_E \sum_{n=0}^{\infty} \frac{p_{n+1}(x;\mathbf{a})}{h_n(\mathbf{a}+1)}\frac{p_n(y;\mathbf{a}+1)}{u_{n+1}(\mathbf{a})}t^{n+1}(Tf)(y)w(y;\mathbf{a}+1)\,dy.
\end{aligned}$$

The second term in the last equation is

$$
\begin{aligned}
&-\langle Tf, \sum_{n=0}^{\infty} \frac{p_{n+1}(x;\mathbf{a})}{h_n(\mathbf{a}+1)u_{n+1}(\mathbf{a})} t^{n+1} p_n(.;\mathbf{a}+1)v(.)w(.;\mathbf{a}+1)\rangle \\
&\quad = -\langle f, \sum_{n=0}^{\infty} \frac{p_{n+1}(x;\mathbf{a})}{h_n(\mathbf{a}+1)u_{n+1}(\mathbf{a})} t^{n+1} T^* p_n(\cdot;\mathbf{a}+1)v(.)w(.;\mathbf{a}+1)\rangle \\
&\quad = -\int_E \sum_{n=0}^{\infty} \frac{p_{n+1}(x;\mathbf{a})p_{n+1}(y;\mathbf{a})}{h_{n+1}(\mathbf{a})} t^{n+1} w(y;\mathbf{a})f(y)\,dy.
\end{aligned}
$$

Thus (5.15) becomes

(5.16)

$$
\begin{aligned}
&([T, T^{-t}]_t f)(x) \\
&\quad = \int_E \Bigg[\frac{P_0^2(x;\mathbf{a})}{h_0(\mathbf{a})} w(y;a) + P_t(x,y;\mathbf{a}+1)w(y;\mathbf{a}+1) \\
&\qquad\qquad\qquad - P_t(x,y;\mathbf{a})w(y;\mathbf{a}) \Bigg] f(y)\,dy.
\end{aligned}
$$

As $t \to 1^-$, one would expect $\int_E P_t(x,y;\mathbf{a})w(y;\mathbf{a})\,dy$ to converge to $f(x)$ for all admissible $\mathbf{a}$. Thus with $P_0(x;\mathbf{a}) = 1$ for all $\mathbf{a}$ we expect (5.16) to yield

$$
\lim_{t\to 1^-} ([T, T^{-t}]_t f)(x) = \frac{1}{h_0(\mathbf{a})} \int_E w(y;\mathbf{a})f(y)\,dy. \tag{5.17}
$$

The analysis in Section 3 proves (5.17) for the continuous q-Jacobi polynomials.

REFERENCES

1. G.E. Andrews, R.A. Askey and R. Roy, *Special functions*, Cambridge University Press, Cambridge, 1999.

2. R. Askey and M.E.H. Ismail, *A generalization of ultraspherical polynomials*, in *Studies in pure mathematics*, P. Erdös, editor, Birkhauser, Basel, 1983.

3. R. Askey, M. Rahman and S.K. Suslov, *On a general q-Fourier transformation with nonsymmetric kernels*, J. Comp. Appl. Math. **68** (1996), 25–55.

4. R. Askey and J. Wilson, *Some basic hypergeometric polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. **319** (1985)

5. B.M. Brown, W.D. Evans and M.E.H. Ismail, *The Askey-Wilson polynomials and q-Sturm-Liouville problems*, Math. Proc. Camb. Philos. Soc. **119** (1996), 1–16.

6. B.M. Brown and M.E.H. Ismail, *A right inverse of the Askey-Wilson operator*, Proc. Amer. Math. Soc. **123** (1995), 2071–2079.

7. B.M. Brown, W.D. Evans and M.E.H. Ismail, *Titchmarsh Weyl theory for the Askey-Wilson operator*, in preparation.

8. G. Gasper and M. Rahman, *Positivity of the Poisson kernel for the continuous q-Jacobi polynomials and some quadratic transformation formulas for basic hypergeometric series*, SIAM J. Math. Anal. **17** (1986), 970–999.

9. ———, *Basic hypergeometric series*, Cambridge University Press, Cambridge, 1990.

10. M.E.H. Ismail, M. Rahman and R. Zhang, *Diagonalization of certain integral operators* II, J. Comp. Appl. Math. **68** (1996), 163–196.

11. M.E.H. Ismail and J. Wilson, *Asymptotic and generating relations for the q-Jacobi and* ${}_4\phi_3$ *polynomials*, J. Approximation Theory **36** (1982), 43–54.

12. M.E.H. Ismail and R. Zhang, *Diagonalization of certain integral operators*, Adv. Math. **109** (1994), 1–33.

13. B. Nassrallah and M. Rahman, *Projection formulas, a reproducing kernel and a generating function for the q-Wilson polynomials*, SIAM J. Math. Anal. **16** (1985), 186–197.

14. M. Rahman, *Askey-Wilson functions of the first and second kinds, series and integral representations for* $C_n^2(x;\beta|q) + D_n^2(x;\beta|q)$, J. Math. Anal. Appl. **164** (1992), 263–284.

15. E.T. Whittaker and G.N. Watson, *A course of modern analysis*, fourth edition, Cambridge University Press, Cambridge, 1927.

Department of Mathematics, University of South Florida, Tampa, FL 33620-5700
Email address: ismail@math.usf.edu

School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada KIS 5B6
Email address: mrahman@math.carleton.ca