

ELLIPTIC BETA INTEGRALS AND MODULAR HYPERGEOMETRIC SUMS: AN OVERVIEW

J.F. VAN DIEJEN AND V.P. SPIRIDONOV

ABSTRACT. Recent results on elliptic generalizations of various beta integrals are reviewed. Firstly, a single variable Askey-Wilson type integral describing an elliptic extension of the Nassrallah-Rahman integral is presented. Then a multiple Selberg-type integral defining an elliptic extension of the Macdonald-Morris constant term identities for nonreduced root systems is described. The Frenkel-Turaev sum and its multivariable generalization, conjectured recently by Warnaar, follow from these integrals through residue calculus. A new elliptic Selberg-type integral, from which the previous one can be derived via a technique due to Gustafson, is defined. Residue calculus applied to this integral yields an elliptic generalization of the Denis-Gustafson sum—a modular extension of the Milne-type multiple basic hypergeometric sums.

1. Introduction. Elliptic generalizations of the very well-poised basic hypergeometric series were introduced by Frenkel and Turaev [12] in relation to elliptic solutions of the Yang-Baxter equation associated with the SOS-type solvable models of statistical mechanics [5]. These series were derived also in [30] through a different technique, as solutions of some particular spectral problems associated with new families of discrete biorthogonal rational functions generalizing the Wilson's functions [33, 34]. Various nice properties of the elliptic hypergeometric series were discovered in [12]. Firstly, under a balancing condition, they become invariant with respect to modular transformations. Secondly, there exist natural generalizations of the Bailey's transformation formula for a terminating $_{10}\Phi_9$ series and of the Jackson's sum for a terminating $_{8}\Phi_7$ series. Some new identities for terminating elliptic hypergeometric series were derived and a multiple extension of the Frenkel-Turaev sum was conjectured by Warnaar in [32].

Work supported in part by the Fondo Nacional de Desarrollo Científico y Tecnológico (FONDECYT) Grants #1989832 and #7980041, the Programa Formas Cuadráticas of the Universidad de Talca, the Cátedra Presidencial in Number Theory, and by the Russian Foundation for Basic Research (RFBR) grant #1980832.

Proceedings of the NSF Conference, Special Functions–2000 (Tempe, USA, May 29–June 9, 2000).

Received by the editors on October 31, 2000.

The present note is a brief status report on recent results concerning elliptic beta integrals and their relation to the modular hypergeometric series summation formulae of Frenkel-Turaev, Warnaar and to an elliptic analogue of the Milne-type sums [6, 17, 19, 20], to be defined below. The main new results were discussed in the authors' talks at the NSF conference accompanying the NATO ASI "Special functions-2000" and their detailed treatment can be found in [27, 28, 8-10].

We start from a description of notations. Let p and q be two complex variables with $|p|, |q| < 1$. The q -shifted factorials are defined as [13]

$$\begin{aligned}(a; q)_\infty &\equiv \prod_{k=0}^{\infty} (1 - aq^k), \\ (a; p)_s &= \frac{(a; p)_\infty}{(ap^s; p)_\infty}, \\ (a_1, \dots, a_k; p)_s &= \prod_{m=1}^k (a_m; p)_s.\end{aligned}$$

Define the doubly infinite product

$$(1) \quad (a; p, q)_\infty \equiv \prod_{j,k=0}^{\infty} (1 - ap^j q^k).$$

For $p = 0$, one has $(a; 0, q)_\infty = (a; q)_\infty$. The elliptic gamma function [25] is defined as a ratio of such double products

$$(2) \quad \Gamma(z; p, q) = \frac{(pqz^{-1}; p, q)_\infty}{(z; p, q)_\infty}.$$

It satisfies the first-order q - and p -difference equations

$$(3) \quad \Gamma(qz; p, q) = \theta(z; p)\Gamma(z; p, q), \quad \Gamma(pz; p, q) = \theta(z; q)\Gamma(z; p, q),$$

where the θ -function is defined as

$$(4) \quad \theta(z; p) = (z, pz^{-1}; p)_\infty.$$

This function is related to the Jacobi θ_1 -function in a simple way

$$\begin{aligned} \theta_1(x | \tau) &= 2 \sum_{m=0}^{\infty} (-1)^m p^{(2m+1)^2/8} \sin \pi(2m+1)x \\ &= p^{1/8} i e^{-\pi i x} (p; p)_{\infty} \theta(e^{2\pi i x}; p), \end{aligned}$$

where $p = e^{2\pi i \tau}$. The key properties of the $\theta(z; p)$ function used extensively in the calculations are described by the following functional relations

$$(6) \quad \theta(pz; p) = \theta(z^{-1}; p) = -z^{-1} \theta(z; p).$$

Elliptic Pochhammer symbols are defined naturally as quotients of elliptic gamma functions:

$$(7) \quad \theta(z; p; q)_m = \frac{\Gamma(zq^m; p, q)}{\Gamma(z; p, q)} = \prod_{j=0}^{m-1} \theta(zq^j; p), \quad m \in \mathbf{N}.$$

It is easy to see that, for $p = 0$, one has

$$\Gamma(z; 0, q) = \frac{1}{(z; q)_{\infty}}, \quad \theta(z; 0, q)_m = (z; q)_m = \prod_{j=0}^{m-1} (1 - zq^j).$$

Copying the q -shifted factorial notations, we introduce the following shorthand conventions

$$\begin{aligned} \Gamma(a_1, \dots, a_l; p, q) &= \prod_{r=1}^l \Gamma(a_r; p, q), \\ \theta(a_1, \dots, a_l; p; q)_m &= \prod_{r=1}^l \theta(a_r; p; q)_m, \\ \theta(a_1, \dots, a_l; p) &= \prod_{r=1}^l \theta(a_r; p). \end{aligned}$$

Some further properties of the elliptic gamma function are described in [25] and [11].

2. A one-variable elliptic beta integral. The Euler's beta integral $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$, $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$, and its various extensions play a fundamental role in the theory of special functions [1]. The most important one-variable q -beta integrals are the Askey-Wilson integral, determining the absolutely continuous part of the measure for Askey-Wilson polynomials [4], and the Nassrallah-Rahman integral [21], determining the measure of a family of continuous ${}_{10}\Phi_9$ biorthogonal rational functions [22]. For a comprehensive survey of the one-variable q -beta integrals, see [23].

Recently, one of us introduced a new Askey-Wilson type integral that amounts to a generalization of the Nassrallah-Rahman integral to the elliptic level [27, 28].

Theorem 1. *Let T be the positively oriented unit circle, and let us take two complex bases p and q satisfying the inequalities $|p|, |q| < 1$ and five complex parameters t_m , $m = 0, \dots, 4$, satisfying the constraints $|t_m| < 1$, $|pq| < |A|$, where $A = \prod_{m=0}^4 t_m$. Then the following identity holds true:*

$$(9) \quad \frac{1}{2\pi i} \int_T \frac{\prod_{m=0}^4 \Gamma(z t_m, z^{-1} t_m; p, q)}{\Gamma(z^2, z^{-2}, zA, z^{-1}A; p, q)} \frac{dz}{z} = \frac{2 \prod_{0 \leq m < s \leq 4} \Gamma(t_m t_s; p, q)}{(q; q)_\infty (p; p)_\infty \Gamma_{m=0}^4 \Gamma(At_m^{-1}; p, q)}.$$

The same formula remains true if one deforms the unit circle to a contour C which encircles the poles at $z = \{t_r p^l q^m\}_{l, m \in \mathbf{N}}$, $r = 0, \dots, 4$, $\{p^{l+1} q^{m+1} A^{-1}\}_{l, m \in \mathbf{N}}$ and separates them from the partner poles with inversed coordinates.

For $p = 0$, the equality (9) is reduced to the Nassrallah-Rahman q -beta integral [21, 22]:

$$(10) \quad \frac{1}{2\pi i} \int_T \frac{(z^2, z^{-2}, zA, z^{-1}A; q)_\infty}{\prod_{m=0}^4 (z t_m, z^{-1} t_m; q)_\infty} \frac{dz}{z} = \frac{2 \prod_{m=0}^4 (t_m^{-1} A; q)_\infty}{(q; q)_\infty \prod_{0 \leq m < s \leq 4} (t_m t_s; q)_\infty},$$

where it is assumed that $|t_m| < 1$. When one of the parameters, say t_4 , goes to zero, (10) is reduced to the celebrated Askey-Wilson integral

$$(11) \quad \frac{1}{2\pi i} \int_T \frac{(z^2, z^{-2}; q)_\infty}{\prod_{m=0}^3 (z t_m, z^{-1} t_m; q)_\infty} \frac{dz}{z} = \frac{2(t_0 t_1 t_2 t_3; q)_\infty}{(q; q)_\infty \prod_{0 \leq m < s \leq 3} (t_m t_s; q)_\infty}.$$

The method used by Askey for proving the Askey-Wilson and Nassrallah-Rahman integrals [2, 3] was partially lifted to the elliptic level in [27]. This resulted in the proof of the integral (9) for the following choice of parameters: $|p| < |t_0| < 1$ and $t_1 = q^{k_1}$, $t_2 = q^{k_2+1/2}$, $t_3 = -q^{k_3}$, $t_4 = -q^{k_4+1/2}$, where $k_i \in \mathbf{N}$ such that the condition $|pq| < |A|$ is not violated. Additional elements of the scheme, leading to the complete proof of the equality (9), have been discovered in [28]. It can be shown [29] that, similar to the situation with (10), (11), the elliptic beta integral (9) determines a measure for an elliptic generalization of the Rahman’s set of biorthogonal rational functions [22], which may be considered as a continuous analogue of the set of functions discovered in [30]. Actually, there is even a more general set of meromorphic functions whose biorthogonality is determined by the integral (9) [29].

3. An elliptic Selberg integral generalizing Macdonald-Morris conjectures. A natural multivariable generalization of the Euler’s beta integral is given by the Selberg integral [1]. Its various q -extensions play a vital role in the Macdonald’s theory of multivariable orthogonal polynomials and symmetric functions [18]. The most general known multiple q -beta integrals were derived by Gustafson [14, 15]. In this and the next sections we describe elliptic generalizations of the corresponding integrals which are reduced to (9) in the one-variable cases.

The following conjecture was put forward in [8] and its proof under some vanishing hypothesis (to be described below) is given in [9].

Conjecture. *Let $p, q, t, t_r, r = 0, \dots, 4$, be complex variables such that $|p|, |q|, |t|$ and $|t_r|$ are smaller than 1 and $|pq| < |B|$, where $B = t^{2n-2} \prod_{s=0}^4 t_s$. Then the following Selberg-type integration formula holds true*

$$\begin{aligned}
 (12) \quad & \int_{T^n} \Delta_n(\mathbf{z}; p, q) \frac{dz_1}{z_1} \dots \frac{dz_n}{z_n} \\
 &= \frac{2^n n!}{(q; q)_\infty^n (p; p)_\infty^n} \prod_{j=1}^n \frac{\Gamma(t^j; p, q)}{\Gamma(t; p, q)} \frac{\prod_{0 \leq r < s \leq 4} \Gamma(t^{j-1} t_r t_s; p, q)}{\prod_{r=0}^4 \Gamma(t^{n+j-2} t_r^{-1} \prod_{s=0}^4 t_s; p, q)},
 \end{aligned}$$

where

$$(13) \quad \Delta_n(\mathbf{z}; p, q) = \frac{1}{(2\pi i)^n} \prod_{1 \leq j < k \leq n} \frac{\Gamma(tz_j z_k, tz_j z_k^{-1}, tz_j^{-1} z_k, tz_j^{-1} z_k^{-1}; p, q)}{\Gamma(z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; p, q)} \\ \times \prod_{j=1}^n \frac{\prod_{r=0}^4 \Gamma(t_r z_j, t_r z_j^{-1}; p, q)}{\Gamma(z_j^2, z_j^{-2}, Bz_j, Bz_j^{-1}; p, q)}$$

and T denotes the unit circle with positive orientation.

For $n = 1$ this integration formula coincides with (9). For $p = 0$ the equality (12) is reduced to the multivariate Nassrallah-Rahman integral derived by Gustafson in [15]

$$(14) \quad \frac{1}{(2\pi i)^n} \int_{T^n} \prod_{1 \leq j < k \leq n} \frac{(z_j z_k, z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; q)_\infty}{(tz_j z_k, tz_j z_k^{-1}, tz_j^{-1} z_k, tz_j^{-1} z_k^{-1}; q)_\infty} \\ \times \prod_{1 \leq j \leq n} \frac{(z_j^2, z_j^{-2}, z_j B, z_j^{-1} B; q)_\infty}{\prod_{r=0}^4 (t_r z_j, t_r z_j^{-1}; q)_\infty} \frac{dz_1}{z_1} \dots \frac{dz_n}{z_n} \\ = 2^n n! \prod_{j=1}^n \frac{(t; q)_\infty \prod_{r=0}^4 (t^{n+j-2} t_r^{-1} \prod_{s=0}^4 t_s; q)_\infty}{(q, t^j; q)_\infty \prod_{0 \leq r < s \leq 4} (t_r t_s t^{j-1}; q)_\infty},$$

with $|q|, |t|$ and $|t_r| < 1$ for $r = 0, \dots, 4$.

The above integration formulae are nothing else than the statements that the constant terms of the Laurent expansions in \mathbf{z} of the functions standing under the integral signs coincide with the expressions given on the corresponding righthand sides. For $p = 0$ these constant terms are equal for special values of the parameters t_r to the constant terms associated with the classical root systems that were originally conjectured by Macdonald and Morris [16, 18].

Now we would like to describe briefly how one can derive the multiple Frenkel-Turaev summation formula conjectured by Warnaar in [32]. The corresponding sum appears to be new even after the reductions to the q and purely hypergeometric series levels.

For the parameter region for which the identity (12) is valid, the integrated function has poles in z_j lying inside the unit circle T at

$\mathbf{z} = \{t_r p^l q^m\}_{l,m \in \mathbf{N}}, r = 0, \dots, 4, \{z_k^{\pm 1} t p^l q^m\}_{l,m \in \mathbf{N}}, \{z_k^{\pm 1} p^{l+1} q^{m+1}\}_{l,m \in \mathbf{N}}$ and $\{p^{l+1} q^{m+1} B^{-1}\}_{l,m \in \mathbf{N}}$. Due to the $z_j \rightarrow z_j^{-1}$ reflection-invariant of the integrand, the poles located outside T are related to those inside by simple inversion.

Let us now dilate the parameter t_0 from the regime $|t_0| < 1$ to $|t_0| > 1$. As a result, a finite number of poles moves from the interior of T to the exterior and the same number of poles makes an opposite move. More precisely, let $0 < p, q < 1$ and $N \in \mathbf{N}$ be some integer such that $q^{-N} < |t_0| < q^{-N-1}$. Then, for $p < 1/|t_0|$ the poles in z_j at $t_0 q^m$, $m = 0, \dots, N$, relocate to the exterior of T and the poles related to these by inversion move to the interior of T . A theorem providing a residue formula taking into account such pole movements across the integration contour has been formulated in [8].

Theorem 2. *Let $\Delta_n(\mathbf{z}; p, q)$ be given by (13) with $0 < q, t < 1$ and t_0, \dots, t_4 generic such that $\#\{\arg(t_r), \arg(t_r^{-1}) \mid r = 0, \dots, 4\} = 10$ and $t_r^{-1} \prod_{s=0}^r t_s \notin [1 + \infty[$ for $r = 0, \dots, 4$. Let also $|t_0| > 1$ and $|t_r| < 1$ for $r = 1, \dots, 4$, and $0 < p < \min(|t_0|^{-1}, q^{-1}|B|)$. Then*

$$(15) \quad \int_{C^n} \Delta_n(\mathbf{z}; p, q) \frac{dz_1}{z_1} \dots \frac{dz_n}{z_n} = \sum_{m=0}^n 2^m m! \binom{n}{m} \sum_{\substack{0 \leq \lambda_1 \leq \dots \leq \lambda_m \\ |\tau_m q^{\lambda_m}| > 1}} \int_{T^{n-m}} \mu_m(\lambda, \mathbf{z}; p, q) \frac{dz_1}{z_1} \dots \frac{dz_{n-m}}{z_{n-m}},$$

where $\tau_j = t_0 t^{j-1}$, $j = 1, \dots, n$,

$$\mu_m(\lambda, \mathbf{z}; p, q) = \kappa_m \nu_m(\lambda; p, q) \delta_{m, n-m}(\lambda, \mathbf{z}) \Delta_{n-m}(\mathbf{z}; p, q),$$

with

$$\kappa_m = \prod_{1 \leq j < k \leq m} \frac{\Gamma(t \tau_k \tau_j^{-1}, t \tau_k^{-1} \tau_j^{-1}; p, q)}{\Gamma(\tau_k \tau_j^{-1}, \tau_k^{-1} \tau_j^{-1}; p, q)} \times \prod_{j=1}^m \frac{\prod_{r=1}^4 \Gamma(t_r \tau_j, t_r \tau_j^{-1}; p, q)}{(q; q)_\infty (p; p)_\infty \Gamma(\tau_j^{-2}, \tau_j^{-1} B, \tau_j B; p, q)},$$

$$\begin{aligned} \nu_m(\lambda; p; q) &= q^{\sum_{j=1}^m \lambda_j} t^{2 \sum_{j=1}^m (n-j)\lambda_j} \\ &\times \prod_{1 \leq j < k \leq m} \left(\frac{\theta(\tau_k \tau_j q^{\lambda_k + \lambda_j}, \tau_k \tau_j^{-1} q^{\lambda_k - \lambda_j}; p)}{\theta(\tau_k \tau_j, \tau_k \tau_j^{-1}; p)} \right) \\ &\times \frac{\theta(t \tau_k \tau_j; p; q)_{\lambda_k + \lambda_j}}{\theta(q t^{-1} \tau_k \tau_j; p; q)_{\lambda_k + \lambda_j}} \frac{\theta(t \tau_k \tau_j^{-1}; p; q)_{\lambda_k - \lambda_j}}{\theta(q t^{-1} \tau_k \tau_j^{-1}; p; q)_{\lambda_k - \lambda_j}} \\ &\times \prod_{j=1}^m \left(\frac{\theta(\tau_j^2 q^{2\lambda_j}; p)}{\theta(\tau_j^2; p)} \prod_{r=0}^t \frac{\theta(t_r \tau_j; p; q)_{\lambda_j}}{\theta(q t_r^{-1} \tau_j; p; q)_{\lambda_j}} \right), \end{aligned}$$

and

$$\delta_{m,n-m}(\lambda, \mathbf{z}) = \prod_{1 \leq j \leq m, 1 \leq k \leq n-m} \frac{\Gamma(t \tau_j q^{\lambda_j} z_k, t \tau_j q^{\lambda_j} z_k^{-1}, t \tau_j^{-1} q^{-\lambda_j} z_k, t \tau_j^{-1} q^{-\lambda_j} z_k^{-1}; p, q)}{\Gamma(\tau_j q^{\lambda_j} z_k, \tau_j q^{\lambda_j} z_k^{-1}, \tau_j^{-1} q^{-\lambda_j} z_k, \tau_j^{-1} q^{-\lambda_j} z_k^{-1}; p, q)}.$$

Here T is the positively oriented unit circle and the contour $C \subset \mathbf{C}$ is a smooth positively oriented Jordan curve around zero such that (i) every half-line parting from zero intersects C just once, (ii) $C^{-1} := \{z \in \mathbf{C} \mid z^{-1} \in C\} = C$ and (iii) C separates the poles in z_j at $\{t_r p^l q^m, p^{l+1} q^{m+1} B^{-1}\}_{l,m \in \mathbf{N}, r = 0, \dots, 4}$, (all in the interior of C) from those related to it by inversion (all in the exterior of C). Furthermore, t_5 , in $\nu_m(\lambda; p; q)$, is determined by q, t and t_0, \dots, t_4 via the balancing condition $q^{-1} t^{2n-2} \prod_{r=0}^t t_r = 1$.

The proof of the theorem is given in [8]; it uses a residue formula for a multivariate Askey-Wilson q -beta integral derived by Stokman in [31].

By analyticity, the formula (12) becomes valid for a wider region of parameters (for t_r dilated radially to the exterior of T provided we replace the contour of integration T by C which satisfies the conditions (i)–(iii) of the last theorem. Notice that these conditions avoid crossings over poles and that C also separates the interior poles in z_j at $\{z_k^{\pm 1} t p^l q^m\}_{l,m \in \mathbf{N}}$ from the ones with inversed coordinates.

By taking a special limit in the described residue formula, one can get the following multi-dimensional modular hypergeometric Frenkel-Turaev sum:

Corollary 3. *Take $N \in \mathbf{N}$. For parameters subject to*

$$(16) \quad \begin{aligned} q^{-1}t^{2n-2} \prod_{r=0}^5 t_r &= 1 \quad (\text{balancing condition}), \\ q^N t^{n-1} t_0 t_4 &= 1 \quad (\text{truncation condition}), \end{aligned}$$

one has the following meromorphic identity in the parameters

$$(17) \quad \sum_{0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq N} \nu_n(\lambda; p; q) = \mathcal{N}_n(p; q),$$

where $\nu_n(\lambda; p; q)$ is as stated in Theorem 2 and

$$(18) \quad \mathcal{N}_n(p; q) = \prod_{j=1}^n \frac{\theta(qt^{n+j-2}t_0^2; p; q)_N \prod_{1 \leq r < s \leq 3} \theta(qt^{1-j}t_r^{-1}t_s^{-1}; p; q)_N}{\theta(qt^{2-n-j} \prod_{r=0}^3 t_r^{-1}; p; q)_N \prod_{r=1}^3 \theta(qt^{j-1}t_0t_r^{-1}; p; q)_N},$$

provided none of the denominators of $\nu_n(\lambda; p; q)$ and $\mathcal{N}_n(p; q)$ vanishes.

Indeed, the lefthand side expression in (15) can be replaced by the combination of elliptic gamma functions standing on the righthand side of (12). Suppose that $t^{1-n}q^{-N} < |t_0| < t^{1-n}q^{-N-1}$ for some $N \in \mathbf{N}$. Then division of the residue formula (15) by $2^n n! \kappa_n$ and the limit $t_4 \rightarrow t_0^{-1}t^{1-n}q^{-N}$ yield the stated summation formula for a restricted parameter domain. However, the resulting formula may be analytically extended to a meromorphic identity in q, t and t_r , $r = 0, \dots, 5$ restricted only by the balancing and truncation conditions in (16). Further details of the derivation can be found in [8].

For $n = 1$ the sum (17) coincides with the elliptic generalization of the terminating ${}_8\Phi_7$ Jackson sum derived by Frenkel and Turaev [12]. The arbitrary n case of the sum (17) was conjectured by Warnaar in [32]. Its direct proof has been obtained recently in [24]. For completeness, let us give explicit forms of summation formulae appearing in two sequential $p \rightarrow 0$ and $q \rightarrow 1$ limits taken in (17).

Corollary 4. *Let parameters t, t_m be subject to the balancing and truncation conditions in (16). Then the following multiple generalization of the Jackson’s sum for terminating very well-poised balanced ${}_8\Phi_7$*

basic hypergeometric series holds true

$$(19) \quad \sum_{0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq N} \nu_n(\lambda; q) = \mathcal{N}_n(q),$$

where

$$(20) \quad \begin{aligned} \nu_n(\lambda; q) = & q^{\sum_{j=1}^n \lambda_j} t^{2 \sum_{j=1}^n (n-j)\lambda_j} \\ & \times \prod_{1 \leq j < k \leq n} \left(\frac{1 - \tau_k \tau_j q^{\lambda_k + \lambda_j}}{1 - \tau_k \tau_j} \frac{1 - \tau_k \tau_j^{-1} q^{\lambda_k - \lambda_j}}{1 - \tau_k \tau_j^{-1}} \right. \\ & \quad \times \frac{(t \tau_k \tau_j; q)_{\lambda_k + \lambda_j}}{(qt^{-1} \tau_k \tau_j; q)_{\lambda_k + \lambda_j}} \frac{(t \tau_k \tau_j^{-1}; q)_{\lambda_k - \lambda_j}}{(qt^{-1} \tau_k \tau_j^{-1}; q)_{\lambda_k - \lambda_j}} \Big) \\ & \times \prod_{1 \leq j \leq n} \left(\frac{1 - \tau_j^2 q^{2\lambda_j}}{1 - \tau_j^2} \prod_{r=0}^t \frac{(t_r \tau_j; q)_{\lambda_j}}{(qt^{r-1} \tau_j; q)_{\lambda_j}} \right), \end{aligned}$$

and

$$(21) \quad \mathcal{N}_n(q) = \prod_{1 \leq j \leq n} \frac{(qt^{n+j-2} t_0^2; q)_N \prod_{1 \leq r < s \leq 3} (qt^{1-j} t_r^{-1} t_s^{-1}; q)_N}{(qt^{2-n-j} \prod_{r=0}^3 t_r^{-1}; q)_N \prod_{r=1}^3 (qt^{j-1} t_0 t_r^{-1}; q)_N}.$$

If one uses relations (16) for elimination of t_4, t_5 and afterwards takes the limit $t_3 \rightarrow \infty$, then this summation formula reduces to the terminating multiple ${}_6\Phi_5$ sum derived in [7]. Other multi-dimensional generalizations of the terminating ${}_8\Phi_7$ sum that are different from the one described above can be found in the works of Milne, Denis-Gustafson and Schlosser, see, e.g., [6, 19, 26].

The plain ${}_7F_6$ hypergeometric series level simplification of the summation formula (19) is reached if one sets $t = q^g, t_r = q^{gr}, r = 0, \dots, 5$, and takes the limit $q \rightarrow 1$. The result generalizes the terminating multiple ${}_5F_4$ sum of [7].

Corollary 5. *Let the parameters $g, g_r, r = 0, \dots, 5$, be subject to the constraints*

$$(22) \quad \begin{aligned} (2n - 2)g + \sum_{r=0}^5 g_r - 1 &= 0 \quad (\text{balancing condition}) \\ (n - 1)g + g_0 + g_4 + N &= 0 \quad (\text{truncation condition}). \end{aligned}$$

Then the following multiple generalization of the Dougall's very well-posed 2-balanced terminating ${}_7F_6$ sum holds true:

$$(23) \quad \sum_{0 \leq \lambda_1 \lambda_2 \leq \dots \leq \lambda_n \leq N} \nu_n(\lambda) = \mathcal{N}_n,$$

where

$$\begin{aligned} \nu_n(\lambda) = & \prod_{1 \leq j < k \leq n} \left(\left(1 + \frac{\lambda_k + \lambda_j}{\rho_k + \rho_j} \right) \left(1 + \frac{\lambda_k - \lambda_j}{\rho_k - \rho_j} \right) \right. \\ & \times \frac{(g + \rho_k + \rho_j)_{\lambda_k + \lambda_j} (g + \rho_k - \rho_j)_{\lambda_k - \lambda_j}}{(1 - g + \rho_k + \rho_j)_{\lambda_k + \lambda_j} (1 - g + \rho_k - \rho_j)_{\lambda_k - \lambda_j}} \\ & \left. \times \prod_{j=1}^n \left(1 + \frac{\lambda_j}{\rho_j} \right) \prod_{r=0}^5 \frac{(g_r + \rho_j)_{\lambda_j}}{(1 - g_r + \rho_j)_{\lambda_j}} \right), \end{aligned}$$

with $\rho_j \equiv (j - 1)g + g_0, j = 1, \dots, n$, and

$$(25) \quad \mathcal{N}_n = \prod_{j=1}^n \frac{(1 + (n + j - 2)g + 2g_0)_N \prod_{1 \leq r < s \leq 3} (1 - (j - 1)g - g_r - g_s)_N}{(1 - (n + j - 2)g - \sum_{r=0}^3 g_r)_N \prod_{r=1}^3 (1 + (j - 1)g + g_0 - g_r)_N}.$$

Here $(a)_n$ is the standard Pochhammer symbol, $(a)_n = a(a + 1) \cdots (a + n - 1)$.

4. A multiparameter elliptic Selberg integral and a related summation formula. We would like to describe now another Selberg-type extension of the elliptic beta integral (9) proposed in [9]. This integral generalizes the multi-parameter q -beta integral derived by Gustafson in [14] and one can deduce from it the previous elliptic Selberg integration formula (12). The following vanishing hypothesis is needed for a proof of this new elliptic Selberg integral.

Hypothesis. Let $0 < p, q < 1$, and let t_0, \dots, t_{2n+1} be complex parameters such that $0 < |t_r| < 1$ for $r = 0, \dots, 2n$, $t_{2n+1} = \prod_{r=0}^{2n} t_r^{-1}$ and with generic argument values in the sense that $\#\{\arg(t_r), \arg(t_r^{-1})\}$

$r = 0, \dots, 2n + 1\} = 4n + 4$. Then

$$(26) \quad \int_{C^n} \prod_{1 \leq j < k \leq n} \Gamma^{-1}(z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; p, q) \\ \times \prod_{j=1}^n \frac{\prod_{r=0}^{2n+1} \Gamma(t_r z_j, t_r z_j^{-1}; p, q)}{\Gamma(z_j^2, z_j^{-2}; p, q)} \frac{dz_1}{z_1} \dots \frac{dz_n}{z_n} = 0,$$

where the contour $C \subset \mathbf{C}$ is a positively oriented Jordan curve around zero such that (i) the interior is star shaped around the origin: every half-line parting from zero intersects C just once, (ii) $C^{-1} := \{z \in \mathbf{C} \mid z^{-1} \in C\} = C$ and (iii) the points $t_r, r = 0, \dots, 2n + 1$, all lie in the interior of C .

Although there are some heuristic indications on the validity of equality (26), they did not get yet a firm mathematical ground. For $n = 1$ the conjecture (26) is confirmed by the elliptic beta integral (9). For arbitrary n and $p = 0$, it follows from the $Sp(n)$ Selberg-type Nassrallah-Rahman integral of Gustafson [14]. As a consequence of the vanishing hypothesis the following statement was inferred in [9].

Corollary 6. *Let the complex parameters $t_r, r = 0, \dots, 2n + 2$, satisfy the constraints $|t_m| < 1$ and $|pq| < |A|$ where $A = \prod_{r=0}^{2n+2} t_r$. Then the following integration formula holds true*

$$(27) \quad \frac{1}{(2\pi i)^n} \int_{T^n} \prod_{1 \leq j < k \leq n} \Gamma^{-1}(z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; q) \\ \times \prod_{j=1}^n \frac{\prod_{r=0}^{2n+2} \Gamma(t_r z_j, t_r z_j^{-1}; p, q)}{\Gamma(z_j^2, z_j^{-2}, Az_j, Az_j^{-1}; p, q)} \frac{dz_1}{z_1} \dots \frac{dz_n}{z_n} \\ = \frac{2^n n!}{(q; q)_\infty^n (p; p)_\infty^n} \frac{\prod_{0 \leq r < s \leq 2n+2} \Gamma(t_r t_s; p, q)}{\prod_{r=0}^{2n+2} \Gamma(At_r^{-1}; p, q)}.$$

Theorem 7. *The integral (27) implies the integral (12).*

Proof. Let us denote the integral on the lefthand side of (12) by $I_n(t, t_r)$. Using (27) one can evaluate the interior integral of the

following expression

$$\begin{aligned}
 (28) \quad & \frac{1}{(2\pi i)^{2n-1}} \int_{T^n} \int_{T^{n-1}} \prod_{1 \leq j < k \leq n} \Gamma^{-1}(z_j z_k, z_j z_k^{-1}, z_j^{-1} z_k, z_j^{-1} z_k^{-1}; p, q) \\
 & \times \prod_{j=1}^n \frac{\prod_{r=0}^4 \Gamma(t_r z_j, t_r z_j^{-1}; p, q)}{\Gamma(z_j^2, z_j^{-2}, z_j t^{n-1} \prod_{0 \leq s \leq 4} t_s, z_j^{-1} t^{n-1} \prod_{0 \leq s \leq 4} t_s; p, q)} \\
 & \times \prod_{\substack{1 \leq j \leq n \\ 1 \leq k \leq n-1}} \Gamma(t^{1/2} z_j w_k, t^{1/2} z_j w_k^{-1}, t^{1/2} z_j^{-1} w_k, t^{1/2} z_j^{-1} w_k^{-1}; p, q) \\
 & \times \prod_{1 \leq j < k \leq n-1} \Gamma^{-1}(w_j w_k, w_j w_k^{-1}, w_j^{-1} w_k, w_j^{-1} w_k^{-1}; p, q) \\
 & \times \prod_{j=1}^{n-1} \frac{\Gamma(w_j t^{n-3/2} \prod_{0 \leq s \leq 4} t_s, w_j^{-1} t^{n-3/2} \prod_{0 \leq s \leq 4} t_s; p, q)}{\Gamma(w_j^2, w_j^{-2}, w_j t^{2n-3/2} \prod_{0 \leq s \leq 4} t_s, w_j^{-1} t^{2n-3/2} \prod_{0 \leq s \leq 4} t_s; p, q)} \\
 & \times \frac{dw_1}{w_1} \dots \frac{dw_{n-1}}{w_{n-1}} \frac{dz_1}{z_1} \dots \frac{dz_n}{z_n},
 \end{aligned}$$

and arrive to the lefthand side of the integral (12) up to some factor. Evaluating the exterior integral in (28) one comes to $I_{n-1}(t, t^{1/2} t_r)$ up to a factor. Equating these two evaluations entails the recursion

$$(29) \quad \frac{I_n(t, t_r)}{I_{n-1}(t, t^{1/2} t_r)} = \frac{2n}{(q; q)_\infty (p; p)_\infty} \frac{\Gamma(t^n; p, q)}{\Gamma(t; p, q)} \frac{\prod_{0 \leq r < s \leq 4} \Gamma(t_4 t_s; p, q)}{\prod_{r=0}^4 \Gamma(t^{n-1} t_r^{-1} \prod_{s=0}^4 t_s; p, q)},$$

iteration of which, starting from the known value for $n = 1$ from Theorem 1, yields (12). \square

Thus the proof of the integral (12) is reduced to the proof of the equality (26). Comparing with (27) one can see that the hypothesis (26) is equivalent to the condition that the lefthand side of the integral (27) vanishes after an appropriate deformation of the contour T to C and an enforcement of the parameters to live on the hypersurface determined by the constraint $A = t_{2n+2}$.

The general guideline of the proof of the integration formula (27) given in [9] corresponds to an elliptic generalization of the Gustafson's method [14] which, in turn, may be considered as a multivariate

generalization of the method of Askey used in [3] for proving the Nassrallah-Rahman integral.

First, we prove that both sides of the integration formula (27) satisfy a set of $(n+2)$ -term difference equations with some elliptic coefficients (for $n=1$ these are the equations derived in [27]). After this, the integral (27) is proved by induction in n . Assuming that the equality (27) has been established for some $n=N$, it can be proved at $n=N+1$ for a particular finite discrete set of parameters t_{2n+1}, t_{2n+2} (it is only at one of the intermediate steps in this procedure that the vanishing hypothesis (26) is needed). After the application of a limiting procedure and some analyticity arguments, similar to the ones used in [28], the validity of the identity (27) with $n=N+1$ integrations is established for arbitrary values of the parameters. For a complete presentation of all the details, we refer to [9].

Let us describe now a new modular hypergeometric series summation formula derived in [10] which appears from the residue calculus for the integral (27) and defines an elliptic analogue of the Milne-type multiple sums [6, 17, 19, 20].

In order to get this sum we first dilate n parameters $t_j, j=1, \dots, n$, from the region $|t_j| < 1$ to $|t_j| > 1$. In the same way as in the previous case, a finite number of poles leaves the unit disk and the same number of different poles enters it. Denote as $N_j, j=1, \dots, n$, positive integers satisfying the relations $|t_j q^{N_j+1}| < 1 < |t_j q^{N_j}|$ and take $|p| < \min(|t_j|^{-1}, |q^{-1}A|)$. Then a residue formula for the integral (27), similar to (15), can be derived. Taking a special limit in the parameters one gets a finite sum of residues instead of the continuous integral.

Theorem 8. *Let t_0, \dots, t_{2n+3} be complex parameters subject to the constraints as just described with the parameter t_{2n+3} determined from the balancing condition $q^{-1} \prod_{r=0}^{2n+3} t_r = 1$. For such a choice of parameters, the integral (27) remains valid provided T is replaced by the integration contour C separating the interior poles at $\{t_r p^l q^m, p^{l+1} q^{m+1} A^{-1}\}_{l,m \in \mathbf{N}}, r=0, \dots, 2n+2$, from the exterior ones obtained by the inversion of coordinates. Then the limits $t_{n+j} \rightarrow t_j^{-1} q^{-N_j}, j=1, \dots, n$, in the formula (27) with the deformed integration contour C degenerate it into the following elliptic*

hypergeometric series summation formula

$$\begin{aligned}
 (30) \quad & \sum_{\substack{0 \leq \lambda_j \leq N_j \\ j=1, \dots, n}} q^{\sum_{j=1}^n j \lambda_j} \prod_{1 \leq j < k \leq n} \frac{\theta(t_j t_k q^{\lambda_j + \lambda_k}, t_j t_k^{-1} q^{\lambda_j - \lambda_k}; p)}{\theta(t_j t_k, t_j t_k^{-1}; p)} \\
 & \times \prod_{1 \leq j \leq n} \left(\frac{\theta(t_j^2 q^{2\lambda_j}; p)}{\theta(t_j^2; p)} \prod_{0 \leq r \leq 2n+3} \frac{\theta(t_j t_r; p; q)_{\lambda_j}}{\theta(q t_j t_r^{-1}; p; q)_{\lambda_j}} \right) \\
 & = \theta(q/ab, q/ac, q/bc; p; q)_{N_1 + \dots + N_n} \\
 & \times \prod_{1 \leq j < k \leq n} \frac{\theta(q t_j t_k; p; q)_{N_j} \theta(q t_j t_k; p; q)_{N_k}}{\theta(q t_j t_k; p; q)_{N_j + N_k}} \\
 & \times \prod_{1 \leq j \leq n} \frac{\theta(q t_j^2; p; q)_{N_j}}{\theta(q t_j/a, q t_j/b, q t_j/c, q^{1+N_1+\dots+N_n-N_j}/t_j abc; p; q)_{N_j}},
 \end{aligned}$$

where $a = t_{2n+1}$, $b = t_{2n+2}$, $c = t_0$.

For $p = 0$ (basic hypergeometric degeneration) (30) reduces to the Denis-Gustafson sum, (see Theorem 4.1 in [6], a similar result was obtained by Milne and Lilly in [20]). Further degeneration $q = 1$ (plain hypergeometric level) has been described in Theorem 4.5 of [6].

In order to analyze modular properties of the elliptic functions identity (30), it is necessary to introduce modular parameters $\sigma, \tau : q \equiv e^{2\pi i \sigma}$, $p \equiv e^{2\pi i \tau}$ and to redefine the parameters as $t_j = q^{g_j}$. Using the well-known modular transformation properties of the θ_1 -function it is now fairly easy to check that (30) is invariant with respect to the natural action of $SL_2(\mathbf{Z})$:

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \quad \sigma \longrightarrow \frac{\sigma}{c\tau + d},$$

where $a, b, c, d \in \mathbf{Z}$ such that $ad - bc = 1$ and the parameters g_j are assumed to be invariant under these transformations. Modular invariance of the multiple Frenkel-Turaev sum (17) is verified in the same way. Repeating the arguments used in [8] in the analysis of (17), one can conclude that, just from the validity of the original Denis-Gustafson sum and modular invariance of both sides of the equality (30), the elliptic summation formula (30) holds true at least up to the order σ^{10} for small σ .

REFERENCES

1. G.E. Andrews, R. Askey and R. Roy, *Special functions*, Cambridge University Press, Cambridge, 1999.
2. R. Askey, *An elementary evaluation of a beta type integral*, Indiana J. Pure Appl. Math. **14** (1983), 892–895.
3. ———, *Beta integrals in Ramanujan's papers, his unpublished work and further examples*, in *Ramanujan revisited*, G.E. Andrews, R.A. Askey, B.C. Berndt, K.G. Ramanathan and R.A. Rankin, eds., Academic Press, Boston, (1988), 561–590.
4. R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. **54** (1985), no. 319.
5. E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, *Exactly solvable SOS models II: Proof of the star-triangle relation and combinatorial identities*, in *Conformal field theory and lattice models*, M. Jimbo, T. Miwa and A. Tsuchiya, eds., Advanced Studies in Pure Math. **16** (1988), 17–122.
6. R.Y. Denis and R.A. Gustafson, *An $SU(n)$ q -beta integral transformation and multiple hypergeometric series identities*, SIAM J. Math. Anal. **23** (1992), 552–561.
7. J.F. van Diejen, *On certain multiple Bailey, Rogers, and Dougall type summation formulas*, Publ. Res. Inst. Math. Sci. **33** (1997), 483–508.
8. J.F. van Diejen and V.P. Spiridonov, *An elliptic Macdonald-Morris conjecture and multiple modular hypergeometric sums*, Math. Res. Letters **7** (2000), 729–746.
9. ———, *Elliptic Selberg integrals*, Internat. Math. Res. Notices, no. 20, (2001), 1083–1110.
10. ———, *Modular hypergeometric residue sums of elliptic Selberg integrals*, Lett. Math. Phys., **58** (2001), 223–238.
11. G. Felder and A. Varchenko, *The elliptic gamma function and $SL(3, \mathbf{Z}) \times \mathbf{Z}^3$* , Adv. Math. **156** (2000), 44–76.
12. I.B. Frenkel and V.G. Turaev, *Elliptic solutions of the Yang-Baxter equation and modular hypergeometric functions*, in *The Arnold-Gelfand mathematical seminars*, V.I. Arnold, I.M. Gelfand, V.S. Retakh and M. Smirnov, eds., Birkhäuser, Boston, (1997), 171–204.
13. G. Gasper and M. Rahman, *Basic hypergeometric series*, Cambridge University Press, Cambridge, 1990.
14. R.A. Gustafson, *Some q -beta and Mellin-Barnes integrals with many parameters associated to the classical groups*, SIAM J. Math. Anal. **23** (1992), 525–551.
15. ———, *Some q -beta integrals on $SU(n)$ and $Sp(n)$ that generalize the Askey-Wilson and Nassrallah-Rahman integrals*, SIAM J. Math. Anal. **25** (1994), 441–449.
16. K.W.J. Kadell, *A proof of the q -Macdonald-Morris conjecture for BC_n* , Mem. Amer. Math. Soc. **108** (1994), no. 516.
17. G.M. Lilly and S.C. Milne, *The C_l Bailey transform and Bailey lemma*, Constr. Approx. **9** (1993), 473–500.
18. I.G. Macdonald, *Constant term identities, orthogonal polynomials, and affine Hecke algebras*, Doc. Math. DMV Extra Volume ICM **1** (1998), 303–317.

- 19.** S.C. Milne, *The multi-dimensional ${}_1\Psi_1$ sum and Macdonald identities for $A_l^{(1)}$* , in *Theta functions Bowdoin 1987*, L. Ehrenpreis and R.C. Gunning, eds., American Math. Soc., Providence, RI, 1989; *A q -analog of a Whipple's transformation for hypergeometric series in $U(n)$* , Adv. Math. **108** (1994), 1–76.
- 20.** S.C. Milne and G.M. Lilly, *Consequences of the A_l and C_l Bailey transform and Bailey lemma*, Discrete Math. **139** (1995), 319–346.
- 21.** B. Nassrallah and M. Rahman, *Projection formulas, a reproducing kernel and a generating function for q -Wilson polynomials*, SIAM J. Math. Anal. **16** (1985), 186–197.
- 22.** M. Rahman, *An integral representation of a ${}_{10}\phi_9$ and continuous bi-orthogonal ${}_{10}\phi_9$ rational functions*, Canad. J. Math. **38** (1986), 605–618.
- 23.** M. Rahman and S.K. Suslov, *The Pearson equation and the beta integrals*, SIAM J. Math. Anal. **25** (1994), 646–693.
- 24.** H. Rosengren, *A proof of a multivariable elliptic summation formula conjectured by Warnaar*, q -series with applications to combinatorics, number theory, and physics (Urbana, IL, 2000), 193–202, Contemp. Math., 291, Amer. Math. Soc., Providence, RI, 2001.
- 25.** S.N.M. Ruijsenaars, *First order analytic difference equations and integrable quantum systems*, J. Math. Phys. **38** (1997), 1069–1146.
- 26.** M. Schlosser, *Summation theorems for multidimensional basic hypergeometric series by determinant equations*, Discrete Math. **210** (2000), 151–169.
- 27.** V.P. Spiridonov, *An elliptic beta integral*, Proc. Fifth International Conference on Differential Equations and Applications, S. Elaydi, J. Lopez Fenner, G. Ladas and M. Pinto, eds., Taylor and Francis, London, 2001, 273–282.
- 28.** ———, *On the elliptic beta function*, Russian Math. Surveys **56** (2001), 185–186.
- 29.** ———, *Elliptic beta integrals and special functions of hypergeometric type*, Proc. NATO ARW Integrable structures of exactly solvable two-dimensional models of quantum field theory, G. von Gehlen and S. Pakuliak, eds., Kluwer Academic Publishers, 2001, 305–313.
- 30.** V.P. Spiridonov and A.S. Zhedanov, *Spectral transformation chains and some new biorthogonal rational functions*, Comm. Math. Phys. **210** (2000), 49–83.
- 31.** J.V. Stokman, *On BC type basis hypergeometric orthogonal polynomials*, Trans. Amer. Math. Soc. **352** (2000), 1527–1579.
- 32.** S.O. Warnaar, *Summation and transformation formulas for elliptic hypergeometric series*, preprint.
- 33.** J.A. Wilson, *Hypergeometric series, recurrence relations and some new orthogonal functions*, Ph.D. thesis, University of Wisconsin, Madison, WI, 1978.
- 34.** ———, *Orthogonal functions from Gram determinants*, SIAM J. Math. Anal. **22** (1991), 1147–1155.

INSTITUTO DE MATEMÁTICA Y FÍSICA, UNIVERSIDAD DE TALCA, CASILLA 747,
TALCA, CHILE

Email address: diejen@inst-mat.otalca.cl

BOGOLIUBOV LABORATORY OF THEORETICAL PHYSICS, JOINT INSTITUTE FOR
NUCLEAR RESEARCH, DUBNA, MOSCOW REGION 141980, RUSSIA
Email address: svp@thsun1.jinr.ru