ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 2, Summer 2002

PARTIAL DIFFERENTIAL EQUATIONS SATISFIED BY POLYNOMIALS WHICH HAVE A PRODUCT FORMULA

WILLIAM C. CONNETT AND ALAN L. SCHWARTZ

ABSTRACT. The classical families of orthogonal polynomials arise as eigenfunctions of Sturm-Liouville problems. In 1929, Bochner addressed the converse question: Which linear second order differential operators can have an infinite family of polynomials, $\mathcal{P}$, as their eigenfunctions? The classification that he gave showed that there were, up to a linear change of variables, a unique differential operator associated with each such family, and the argument did not even require that the members of $\mathcal{P}$ were orthogonal, only that there were "enough" polynomials in the family (indeed, in some cases, the polynomials are not orthogonal). In this paper it will be shown that certain families of bivariate polynomials satisfy not just one but a pair $L^{(1)}$ and $L^{(2)}$ of partial differential operators with bounds on the order of the operators determined by properties of $\mathcal{P}$. We also give a simple condition that the polynomials be completely determined by the pair of operators (up to multiplicative constants). This article includes detailed discussions of five examples of such polynomial families. We shall also discuss $\Delta(\mathcal{P})$, the algebra of operators which have $\mathcal{P}$ as eigenfunctions, and we give sufficient conditions that every member of $\Delta(\mathcal{P})$ is given uniquely as a polynomial in $L^{(1)}$ and $L^{(2)}$. This will be the case in all five examples.

1. Introduction. The classical families of orthogonal polynomials arise as eigenfunctions of Sturm-Liouville problems. In 1929, Bochner addressed the converse question [**2**]: Which linear second order differential operators can have an infinite family of polynomials, $\mathcal{P}$ as their eigenfunctions? The classification that he gave showed that there were, up to a linear change of variables, a unique differential operator associated with each such family, and the argument did not even require that the members of $\mathcal{P}$ were orthogonal, only that there were "enough"

1991 AMS *Mathematics subject classification.* Primary 35P99, 47L80, 33C50, Secondary 43A62, 43A10.
Both authors are supported by the National Science Foundation grant DMS 9706965.
Received by the editors on November 20, 2000, and in revised form on September 19, 2001.

polynomials in the family (indeed, in some cases, the polynomials are not orthogonal).

The condition that there be "enough" polynomials is given here in terms of bivariate polynomials.

Definitions. We order the pairs $\{(n,k) : 0 \le k \le n\}$ by $(m,j) < (n,k)$ if either $m < n$ or $m = n$ and $j < k$. We say that the bivariate polynomial p has *bivariate degree* (n,k) if

$$p(x,y) = \sum_{(m,j)\le(n,k)} a_{m,j} x^{m-j} y^j$$

with $a_{n,k\neq 0}$; in this case we also say that p has *degree* n. Let Π be the set of bivariate polynomials, and let Π_n be the set of polynomials in Π with degree not exceeding n.

If $\mathcal{P}$ is a collection of bivariate polynomials, let $\mathcal{P}_n = \mathcal{P} \cap \Pi_n$. We say $\mathcal{P}$ is *algebraically complete* if $\mathcal{P}$ is linearly independent and the elements of $\mathcal{P}_n$ span Π_n for each n. An algebraically complete family $\mathcal{P}$ is a *family of orthogonal polynomials* if there is a positive Borel measure σ on $\mathbf{R}^2$ such that $\int p\bar{q}\, d\sigma = 0$ whenever $p, q \in \mathcal{P}$ with $p \neq q$.

Analogous definitions may be formulated for polynomials in any number of variables. But in this article, we will limit attention to bivariate polynomials.

Remark. Algebraic completeness is trivial for univariate polynomials– there must be one polynomial of each degree. For bivariate polynomials algebraic completeness requires that $\mathcal{P}$ contain $n+1$ linearly independent polynomials of degree n. For polynomials in three variables, algebraic completeness requires $(n+1)(n+2)/2$ linearly independent polynomials of degree n.

Bochner's classification for algebraically complete families of univariate polynomials was used to prove a converse to Gasper's theorem about the positivity of the product formula for Jacobi polynomials [**3, 5**]. Algebraic completeness for bivariate families was also useful in the classification of canonical Hermitian and canonical non-Hermitian hypergroups given in [**4**].

The problem we address is: *given an algebraically complete family of bivariate polynomials $\mathcal{P}$, find a system of linear partial differential* *operators which have the polynomials in $\mathcal{P}$ as a complete set of eigenfunctions.* Except in the following lemma, all linear partial differential operators will have finite order. Actually we will show how to find a pair of linear partial differential operators which have exactly $\mathcal{P}$ as its set of joint polynomial eigenfunctions. A sufficient condition for the existence of such linear partial differential operators is that $\mathcal{P}$ have a product formula of the right sort.

The following lemma shows that the problem becomes interesting only when we bound the order of the linear partial differential operators. Theorem 1 finds a bound for the order of the linear partial differential operators. Theorems 2 and 3 address the issue of showing that $\mathcal{P}$ is a complete set of eigenfunctions for the linear partial differential operators. We write $D_x = \partial/\partial x$ and $f^{(j,k)} = D_x^j D_y^k f$.

Koornwinder proves a version of the following for a specific family of polynomials [**10**, II, Lemma 6.1], but the argument given here is valid in a more general context, see also [**12**, Lemma 0.1] and [**13**, Lemma 3.1 and Theorem 3.1].

Lemma 1.1. *Let $\mathcal{P}$ be an algebraically complete family of bivariate polynomials, and let complex numbers $\{\lambda_p\}_{p\in\mathcal{P}}$ be given. Then there is a unique differential operator possibly of infinite order*

$$M = \sum_{0\le j\le m<\infty} \alpha_{m,j}(x,y) D_x^{m-j} D_y^j \tag{1.1}$$

with $\alpha_{m,j} \in \Pi_m$ such that

$$Mp = \lambda_p p, \quad p \in \mathcal{P} \tag{1.2}$$

for every $(x,y) \in \mathbf{R}^2$.

Moreover, suppose M is given by (1.1), *and assume that* (1.2) *holds for (x,y) belong to some open subset of $\mathbf{R}^2$. Then $\alpha_{m,j}$ are completely determined by $\mathcal{P}$ and $\{\lambda_p\}_{p\in\mathcal{P}}$ and* (1.2) *is valid on all of $\mathbf{R}^2$.*

Proof. Let $\mathcal{P}$ and $\{\lambda_p\}_{p\in\mathcal{P}}$ be as in the hypotheses, and suppose $\alpha_{m,j} \in \Pi_m$ for $0 \le j \le m < k$. $\mathcal{P}$ contains exactly $k+1$ polynomials $p_0, p_1, \dots, p_k$ of degree k. Let $\lambda_j = \lambda_{p_j}$. We can write

$$p_l = \sum_{j=0}^{k} c_{l,j} x^{k-j} y^j + \text{l.o.t.}, \quad l = 0, 1, \dots, k.$$

Since $\mathcal{P}$ is algebraically complete,

$$C_k = \begin{bmatrix} c_{0,0} & \cdots & c_{0,k} \\ c_{1,0} & \cdots & c_{1,k} \\ \vdots & & \vdots \\ c_{k,0} & \cdots & c_{k,k} \end{bmatrix}$$

is nonsingular. Now the relation

$$M \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_k \end{bmatrix} = \begin{bmatrix} \lambda_0 p_0 \\ \lambda_1 p_1 \\ \vdots \\ \lambda_k p_k \end{bmatrix}$$

yields

$$C_k \begin{bmatrix} k!0!\alpha_{k,0} \\ (k-1)!1!\alpha_{k,1} \\ \vdots \\ 0!k!\alpha_{k,k} \end{bmatrix} = -\sum_{0\le j\le m<k} \alpha_{m,j} D_x^{m-j} D_y^j \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_k \end{bmatrix} + \begin{bmatrix} \lambda_0 p_0 \\ \lambda_1 p_1 \\ \vdots \\ \lambda_k p_k \end{bmatrix}.$$

Thus, since C_k is nonsingular, $\alpha_{k,j}$ are determined and belong to Π_k for $j = 0, 1, \dots, k$.

The converse is a corollary of the above argument. □

Definitions. Assume H is a subset of $\mathbf{R}^2$, let $M(H)$ denote the regular complex-valued Borel measures on H, let $M_+(H)$ denote the positive measures in $M(H)$ and $M_1(H)$ the measures in $M_+(H)$ with unit total variation; these are called *probability measures*. If $\mu \in M(H)$, let $\operatorname{supp}\mu$ be the support of μ. For $z \in H$, δ_z is the unit mass concentrated on z. Let $C(H)$ be the Banach space of functions which are continuous on H with the norm $\|f\|_H = \sup_{(x,y)\in H} |f(x,y)|$, and let $C_c(H)$ be the functions in $C(H)$ which have compact support in H.

We say an algebraically complete family of polynomials $\mathcal{P}$ has a *product formula on H* if, for each $z, w \in H$, $\sigma_{z,w} \in M(H)$ exists such that

$$p(z)p(w) = \int_H p\, d\sigma_{z,w}, \quad p \in \mathcal{P}. \tag{1.3}$$

We say (1.3) is a *strong product formula on H with identity element e* if

$$\sigma_{z,w} \in M_+(H), \quad z, w \in H$$

and there is an element $e \in H$ which satisfies the following two conditions

$$\sigma_{z,e} = \delta_z, \quad z \in H, \tag{1.4}$$

$$\lim_{w \to e} \operatorname{diam}(\operatorname{supp}(\sigma_{z,w})) = 0, \quad z, w \in H \tag{1.5}$$

$\sigma_{z,w}$ are called the *product measures*.

Some elementary observations are contained in the following

Lemma 1.2. *Suppose $\mathcal{P}$ is an algebraically complete family of polynomials which has a strong product formula on H with identity element e and product measures $\sigma_{z,w}$.*

(i) *$p(e) = 1$ for every $p \in \mathcal{P}$.*

(ii) *$\sigma_{z,w} \in M_1(H)$ for each $z, w \in H$.*

(iii) *If H is a bounded set, then $\|p\|_H = 1$ for every $p \in \mathcal{P}$.*

(iv) *e is a boundary point of H.*

Proof. (i) For $w = e$, (1.3) and (1.4) yield $p(z)p(e) = p(z)$.

(ii) Apply the product formula to the constant polynomial in $\mathcal{P}$ which because of (1) must be $p_0 \equiv 1$.

(iii) From (1.3) and (ii) we have $\|p\|_H^2 \le \|p\|_H$, so $\|p\|_H \le 1$. Equality follows from (i).

(iv) This follows from (iii) applied to $\mathcal{P}_1$.

(v) H is bounded by (iii). □

1.1 *Strong product formulas and hypergroup product formulas.* Suppose $\mathcal{P}$ is an algebraically complete family of polynomials with strong product formula on H with identity e. This gives rise to a convolution $*$ on $M(H)$ as follows. Let $\mu, \nu \in M(H)$ and define $\mu * \nu$ by its action on $C(H)$ by setting

$$\int f\, d(\mu * \nu) = \iint \left[\int f\, d\sigma_{z,w} \right] d\mu(z)\, d\nu(w), \quad f \in C_c(H).$$

The resulting convolution is easily shown to be commutative and associative, so $(M(H), *)$ is a commutative Banach algebra of measures with identity δ_e and in which a product of probability measures is also a probability measure. If $(M(H), *)$ satisfies a number of other conditions, it becomes a hypergroup (or in current usage, DJS-hypergroup); for more background on hypergroups, see [**4, 16, 8**] and the references cited there. We say that $\mathcal{P}$ has a *hypergroup product formula* on H if $(M(H), *)$ is a hypergroup.

Now suppose (1.3) is a hypergroup product formula. Sometimes the set of characters of $(M(H), *)$ coincides with $\mathcal{P}$, and $\mathcal{P}$ becomes an orthogonal family. In this case we say that $(M(H), *)$ is a 2-*variable continuous polynomial hypergroup*. The work reported here was originally motivated by our goal of classifying the members of this category, see [**4, 5, 17**] and the references cited there. Theorem 1 provides a necessary condition that $\mathcal{P}$ be the characters of a hypergroup.

1.2 *Differential equations.* In order to find the differential operators, it is necessary to understand how the identity element e is situated in H.

Definition. A *path* from e to H is a continuously differentiable function $f : [0, 1] \to H$ with $f(0) = e$ and a unit speed parameterization, $|f'(s)| = 1$. H has a *cusp of order* r *at* e if r is the smallest positive integer such that two r-times differentiable paths f and g exist such that $f^{(k)}(0) = g^{(k)}(0)$, $0 \le k \le r-1$ and $f^{(r)}(0)$ and $g^{(r)}(0)$ are linearly independent vectors. For instance, if $H = \{(x, y) : 0 \le y \le x^r \le 1\}$, then H has a cusp of order r at $e = (0, 0)$ as can be seen by choosing f and g to be paths following the boundaries of H. The situation of e in H used in [**4**] is that e is a two-dimensional accumulation point of H; this is slightly weaker than H having a cusp of order 1 at e.

We can now state the result which guarantees the existence of linear partial differential operators with $\mathcal{P}$ as eigenfunctions. This generalizes our earlier result in [**4**]. By Lemma 1.1 any algebraically complete family of polynomials can be made to be eigenfunction with specified eigenvalues of a possibly infinite order differential equation. One important feature of this theorem is that it puts bounds on the orders of the linear partial differential operators.

Theorem 1. *Suppose that $\mathcal{P}$ is an algebraically complete family of polynomials which has a strong product formula on H with identity element e and that H has a cusp of order r at e. Then the members of $\mathcal{P}$ are eigenfunctions for a pair $L^{(1)}$ and $L^{(2)}$ of commuting linear partial differential operators*:

$$L^{(1)}p = \lambda_p^{(1)}p \quad \text{and} \quad L^{(2)}p = \lambda_p^{(2)}p, \quad p \in \mathcal{P}. \tag{1.6}$$

$L^{(1)}$ has order not exceeding 2, and $L^{(2)}$ has order not exceeding $2r$. Moreover, if $f = (f_1, f_2)$ and $g = (g_1, g_2)$ are paths from e to H as in the definition of cusp, we can take

$$\lambda_p^{(1)} = f_1'(0)p^{(1,0)}(e) + f_2'(0)p^{(0,1)}(e) \tag{1.7}$$

$$\lambda_p^{(2)} = [f_1^{(r)}(0) - g_1^{(r)}(0)]p^{(1,0)}(e) + [f_2^{(r)}(0) - g_2^{(r)}(0)]p^{(0,1)}(e). \tag{1.8}$$

If $r = 1$ we can take

$$\lambda_p^{(1)} = p^{(1,0)}(e) \quad \text{and} \quad \lambda_p^{(2)} = p^{(0,1)}(e). \tag{1.9}$$

The proof will be given in Section 2.

Definitions. Let $\mathcal{P}$ be an algebraically complete family of polynomials, and let $D(\mathcal{P})$ be the algebra of linear partial differential operators with polynomial coefficients which have all members of $\mathcal{P}$ as eigenfunctions. Suppose that $L^{(1)}, L^{(2)} \in D(\mathcal{P})$. We say that the pair $(L^{(1)}, L^{(2)})$ determines $\mathcal{P}$ if every polynomial joint eigenfunction of $L^{(1)}$ and $L^{(2)}$ is a constant multiple of some polynomial in $\mathcal{P}$. In this case we also say that the pair $(L^{(1)}, L^{(2)})$ is *deterministic*.

Remark. The definition could be extended to any set of linear partial differential operators; indeed, if $L^{(1)}$ and $L^{(2)}$ are real, $\mathcal{P}$ consists of real polynomials and the pair $(L^{(1)}, L^{(2)})$ determines $\mathcal{P}$, then $L^{(1)} + iL^{(2)}$ determines $\mathcal{P}$.

Theorem 2. *Suppose $\mathcal{P}$ is an algebraically complete family and suppose $L^{(1)}$ and $L^{(2)}$ belong to $D(\mathcal{P})$ with eigenvalues $\{\lambda_p^{(1)}\}_{p\in\mathcal{P}}$ and $\{\lambda_p^{(2)}\}_{p\in\mathcal{P}}$. Then $(L^{(1)}, L^{(2)})$ determines $\mathcal{P}$ if and only if*

$$p \longmapsto (\lambda_p^{(1)}, \lambda_p^{(2)}) \text{ is injective on } \mathcal{P}. \tag{1.10}$$

The proof will be given in Section 2. The theorem is analogous to [**9**, Lemma 2.4]; a discussion of the related idea of admissibility will be given in Sections 1.3 and 5.

The question arises whether Theorem 2 can be strengthened to say that $\mathcal{P}$ is the totality of joint eigenfunctions of $L^{(1)}$ and $L^{(2)}$. The answer depends on the definition or "eigenfunction"; we will use the following and find that the question can often be answered in the affirmative if $\mathcal{P}$ is a family of orthogonal polynomials.

Definition. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$; let Φ be a complete orthonormal system for $\mathcal{H}$; for $f \in \mathcal{H}$ and $\phi \in \Phi$, let $\hat{f}(\phi) = \langle f, \phi \rangle$. Suppose L is a (possibly unbounded) linear operator on $\mathcal{H}$ defined at least on the space of finite linear combinations of elements of Φ. Assume that for each $\phi \in \Phi$ there is a complex number λ_ϕ satisfying $L\phi = \lambda_\phi \phi$. Let M be any extension of L and suppose f belongs to the domain of M. Then, for each $\phi \in \Phi$, $\langle Mf, \phi \rangle = \langle f, M^* \phi \rangle = \langle f, \overline{\lambda_\phi} \phi \rangle = \lambda_\phi \hat{f}(\phi)$. It follows that

$$\sum_{\phi \in \Phi} |\lambda_\phi \hat{f}(\phi)|^2 < \infty, \tag{1.11}$$

and

$$Mf = \sum_{\phi \in \Phi} \lambda_\phi \hat{f}(\phi) \phi. \tag{1.12}$$

Thus the maximal extension M of L has domain $D = \{f : (1.11) \text{ holds}\}$ and M is given by (1.12). We say that $f \in D$ is an *eigenfunction* of L if there is a complex number λ such that $Mf = \lambda f$.

Theorem 3. *Suppose $\mathcal{P}$ is a family of polynomials orthogonal with respect to $\sigma \in M_1(\mathbf{R}^2)$; let $H = \operatorname{supp} \sigma$ be compact. Suppose $L^{(1)}$ and $L^{(2)}$ are linear partial differential operators which determine $\mathcal{P}$. Let f be a joint eigenfunction of $L^{(1)}$ and $L^{(2)}$. Then f is a constant multiple of some polynomial in $\mathcal{P}$.*

The proof is contained in the next section.

1.3 *Admissibility in the sense of Krall and Sheffer.* Krall and Sheffer introduce the notions of admissibility for ordinary [**12**] and partial linear differential operators [**13**] in order to study the relation between such operators and polynomial eigenfunctions. We review those two notions here.

Let $L = \sum_{k=1}^{\infty} M_k(x)D_x^k$ where each $M_k \in \Pi_k$ with leading coefficient m_k. Let $\lambda_n = nm_1 + n(n-1)m_2 + \cdots + n!m_n$. L is admissible if $n \to \lambda_n$ is injective. (We use *admissible*-1 to distinguish this from the term as used for linear partial differential operators below.) L has a unique monic polynomial eigenfunction of every degree with λ_n being the eigenvalue corresponding to the polynomial eigenfunction of degree n if and only if L is admissible-1 [**12**, Lemma 0.1] and [**9**, Lemma 2.4].

In [**13**, Section 3], a linear partial differential operator is designated admissible (we use *admissible*-2) if there is a sequence of parameters $\{\lambda_n\}_{n=0}^{\infty}$ such that $Lp = \lambda_n p$ has no nonzero solutions in Π_{n-1}, and it has exactly $n+1$ linearly independent solutions of degree n. They show that if L is admissible-2, then $\lambda_0 = 0$ and $n \to \lambda_n$ is injective.

Our notion of a deterministic pair of linear partial differential operators is a natural extension of admissible-1 to the bivariate case and Theorem 2 is analogue to [**12**, Lemma 0.1] and [**9**, Lemma 2.4]. Theorem 2 shows that when (1.10) holds the individual polynomials are determined (up to multiplicative constants).

The admissible-2 condition only leads to the determination of the space V_n spanned by the nth degree polynomial eigenfunctions. An added disadvantage of admissible-2 is that there are orthogonal polynomial families (which are designated "classical" by Koornwinder [**11**]) which do not satisfy this criterion. We will return to this issue and related ones in Section 5.

The rest of the paper is organized as follows. Section 3 is devoted to five examples. In Section 4 we will give sufficient conditions on $(L^{(1)}, L^{(2)})$ that every linear partial differential operator in $D(\mathcal{P})$ can be represented uniquely as a bivariate polynomial in $L^{(1)}$ and $L^{(2)}$. These sufficient conditions hold in four of the five examples; a separate argument will be presented for the fifth example. Section 5 is devoted to Krall and Sheffer's notion of admissibility, and Section 6 lists a few open questions.

2. Proof of Theorems 1, 2 and 3.

2.1 *Proof of Theorem* 1. Assume that $\mathcal{P}$ and H are as in the hypothesis of Theorem 1, and that $\mathcal{P}$ has a strong product formula with identity e and product measures $\sigma_{z,w}$. Before we turn to a proof, we introduce moments. We use the following notations: $z = (x, y)$, $w = (u, v)$ and $\zeta = (s, t)$. The *moments* are defined by

$$M_{j,k}(z, w) = \frac{1}{j!k!} \int_H (s - x)^j (t - y)^k \, d\sigma_{z,w}(\zeta), \quad j, k \in \mathbf{N}_0, z, w \in H.$$

Lemma 2.1. *If $n \in \mathbf{N}_0$ and $j + k > 2n$, then for each fixed z,*

$$M_{j,k}(z, w) = O(|w - e|^{n+1}), \quad w \to e, \quad w \in H.$$

Proof. Note that there are polynomials $c_p(z)$ such that

$$\frac{1}{j!k!}(s - x)^j (t - y)^k = \sum_{p \in \mathcal{P}_{j+k}} c_p(z) p(\zeta),$$

so

$$M_{j,k}(z, w) = \sum_{p \in \mathcal{P}_{j+k}} c_p(z) p(z) p(w)$$

is a polynomial. Let z be fixed. If $j + k > 0$, $M_{j,k}(z, e) = 0$, so if $m_{j,k}(w) = M_{j,k}(z, w)$,

$$m_{j,k}(w) = O(|w - e|), \quad j, k \in \mathbf{N},$$

since $m_{j,k}(e) = 0$ and $m_{j,k}(w)$ is a polynomial. This establishes the lemma for $n = 0$.

Now proceed by induction on n. Assume that whenever $j + k > 2n$

$$m_{j,k}(w) = O(|w - e|^{n+1}).$$

Assume $j, k \in \mathbf{N}_0$ and $j + k > 2(n + 1)$; we will show

$$m_{j,k}(w) = O(|w - e|^{n+2}).$$

Let μ be the greatest even integer such that $\mu \le j$, and let $\nu = 2n+2-\mu$. Then either $j > \mu$ or $k > \nu$, and since μ and ν are both even $(s-x)^\mu (t-y)^\nu \ge 0$, so

$$|m_{j,k}(w)| \le \sup_{\zeta \in \operatorname{supp} \sigma_{z,w}} |s-x|^{j-\mu}|t-y|^{k-\nu} m_{\mu\nu}(w),$$

and $\mu + \nu = 2n+2 > 2n$ so $m_{\mu,\nu}(w) = O(|w-e|^{n+1})$. Now $j-\mu > 0$ or $k - \nu > 0$ so

$$m_{j,k}(w) = o(|w-e|^{n+1}) = O(|w-e|^{n+2})$$

since $m_{j,k}$ is polynomial. □

We now turn to the proof of the theorem. Assume the hypotheses of the theorem hold, and let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ be paths from e to H as in the cusp definition. We compute $p(z)[p(f(s)) - p(g(s))]$ two different ways. First we expand the second factor in a Taylor series to obtain, see (1.8),

$$p(z)[p(f(s)) - p(g(s))] = \frac{s^r}{r!}\lambda_p^{(2)} p(z) + o(s^r). \tag{2.1}$$

We also have by expanding $p(\zeta)$ in a Taylor series around $(s,t) = (x,y)$

$$\begin{aligned} p(z)p(w) &= \int_H p(\zeta)\, d\sigma_{z,w}(\zeta) \\ &= \int_H \sum_{j+k \le \deg(p)} \frac{1}{j!k!}(s-x)^j (t-y)^k p^{(j,k)}(z)\, d\sigma_{z,w}(\zeta) \\ &= \sum_{j+k\le \deg(p)} M_{j,k}(z,w) p^{(j,k)}(z), \end{aligned}$$

so

$$\begin{aligned} &p(z)[p(f(s)) - p(g(s))] \\ &\quad = \sum_{j+k\le\deg(p)} [M_{j,k}(z,f(s)) - M_{j,k}(z,g(s))] p^{(j,k)}(z) \\ &\quad = \sum_{j+k\le 2r} [M_{j,k}(z,f(s)) - M_{j,k}(z,g(s))] p^{(j,k)}(z) + o(s^r) \end{aligned} \tag{2.2}$$

because, according to the lemma, if $j+k>2r$,

(2.3)
$$\begin{aligned} M_{j,k}(z,f(s))-M_{j,k}(z,g(s)) &= O(|f(s)-e|^{r+1})+O(|g(s)-e|^{r+1}) \\ &= o(s^r). \end{aligned}$$

Now equate the coefficients of $s^r/r!$ in (2.1) and (2.2) to obtain

$$\sum_{j+k\le 2r} A_{j,k}(z)p^{(j,k)}(z) = \lambda_p^{(2)}p(z)$$

where

$$\begin{aligned} A_{j,k}(z) &= D_s^r[M_{j,k}(z,f(s)) - M_{j,k}(z,g(s))|_{s=0} \\ &= [f_1^{(r)}(0) - g_1^{(r)}(0)]M_{j,k}^{(1,0)}(z,e) + [f_2^{(r)}(0) - g_2^{(r)}(0)]M_{j,k}^{(0,1)}(z,e) \end{aligned}$$

and $\lambda_p^{(2)}$ is given by (1.8).

We also obtain a second linear partial differential operator of order 2 or less by performing a similar computation on $p(z)p(f(s))$ and equating the coefficients of s in both expressions. The result is

$$\sum_{j+k\le 2} [f_1'(0)M_{j,k}^{(1,0)}(z,e) + f_2'(0)M_{j,k}^{(0,1)}(z,e)]p^{(j,k)}(z) = \lambda_p^{(1)}p(z)$$

where $\lambda_p^{(1)}$ is given by (1.7).

$L^{(1)}L^{(2)}$ and $L^{(2)}L^{(1)}$ have polynomial coefficients and they have $\mathcal{P}$ as eigenfunctions with the same eigenvalues, so they are equal. This establishes the theorem. □

Remark. Similar results can be obtained for more general classes of functions than polynomials, but this would involve more technical assumptions. See [**3**] for a treatment of the univariate case.

2.2 *Proof of Theorem* 2. Suppose that $\mathcal{P}$ is an algebraically complete family with $L^{(1)}, L^{(2)} \in D(\mathcal{P})$ such that

$$L^{(j)}p = \lambda_p^{(j)}p, \quad j=1,2 \text{ and } p\in\mathcal{P}.$$

Assume (1.10) holds. Suppose that f is a polynomial and that for $j = 1,2$, $L^{(j)}f = \lambda^{(j)}f$ for some $\lambda^{(j)} \in \mathbf{C}$. Since $\mathcal{P}$ is algebraically complete, there are constants c_p such that $f = \sum_{p\in\mathcal{P}} c_p p$ with $E = \{p : c_p \neq 0\}$ being a finite set. We have for $j = 1,2$,

$$\lambda^{(j)} \sum_{p\in E} c_p p = \lambda^{(j)} f = L^{(j)} f = L^{(j)} \Big[\sum_{p\in E} c_p p \Big] = \sum_{p\in E} c_p \lambda_p^{(j)} p.$$

Thus, for all $p \in E$, $\lambda^{(j)} = \lambda_p^{(j)}$; hence, by (1.10), E consists of a single element q and $f = c_q q$.

For the converse, assume there are $p, q \in \mathcal{P}$ with $p \neq q$, but $\lambda_p^{(j)} = \lambda_q^{(j)}$ for $j = 1,2$. Thus $p + q$ is a joint polynomial eigenfunction of $L^{(1)}$ and $L^{(2)}$ which is not a constant multiple of any member of $\mathcal{P}$.

2.3 *Proof of Theorem* 3. Suppose that $\mathcal{P}, H, \sigma, L^{(1)}, L^{(2)}$ are as in the statement of the theorem and that $\lambda_p^{(1)}$ and $\lambda_p^{(2)}$ are as in (1.6). There is no loss of generality in assuming that $\mathcal{P}$ consists of orthonormal polynomials. $\mathcal{P}$ is a complete orthonormal system by the Stone-Weierstrass theorem. Define $\hat{f}(p) = \int_H f\bar{p}\, d\sigma$.

Suppose f is a joint eigenfunction of $L^{(1)}$ and $L^{(2)}$ with eigenvalues $\lambda^{(1)}$ and $\lambda^{(2)}$. This means that $\sum_{p\in\mathcal{P}} (|\lambda_p^{(1)}|^2 + |\lambda_p^{(2)}|^2)|\hat{f}(p)|^2 < \infty$ and

$$\sum_{p\in\mathcal{P}} \lambda_p^{(j)} \hat{f}(p) p = L^{(j)} f = \lambda^{(j)} f = \lambda^{(j)} \sum_{p\in\mathcal{P}} \hat{f}(p) p, \quad j = 1,2.$$

Thus $(\lambda^{(j)} - \lambda_p^{(j)})\hat{f}(p) = 0$, whence $\hat{f}(p) = 0$ unless $\lambda^{(j)} = \lambda_p^{(j)}$ for $j = 1,2$. Hence by (1.10) there is a unique $q \in \mathcal{P}$ such that $\hat{f}(q) \neq 0$, whence $f = \hat{f}(q)q$ as required.

3. Examples. In this section we discuss five examples of algebraically complete families of bivariate polynomials related to the normalized Jacobi polynomials (a more extensive discussion of these can be found in [**11**]:

$$R_n^{(\alpha,\beta)}(x) = \frac{P_n^{(\alpha,b)}(x)}{P_n^{(\alpha,\beta)}(1)} = F\bigg(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\bigg).$$

We make a few observations that are common to all five examples.

1. Each example is actually a category of examples with parameters named α, β, γ or δ. For certain values of the parameters each family is a set of orthogonal polynomials.

2. Each example satisfies a strong product formula (indeed a hypergroup product formula) for certain values of the parameters. This guarantees the existence of a pair of linear partial differential operators by Theorem 1. The differential operators and eigenvalues are given explicitly.

3. In all five examples, the polynomials, the eigenvalues, and the coefficients in the linear partial differential operators are all rational functions of the parameters α, β, γ and δ. Thus the validity of the partial differential equations will extend to a larger range of parameters (including complex values; see [**18**, Section 4.22] for a discussion of this in the case of the Jabobi polynomials) where there may not be a strong product measure. Thus the condition that an algebraically complete family of polynomials have a strong product formula is a sufficient, but not necessary, condition for the existence of linear partial differential operators with the polynomials as eigenfunctions.

4. In each case it is easy to check that (1.10) holds so Theorem 2 can be invoked to assert that the pair of linear partial differential operators determine the family of polynomials.

5. For many values of the parameters, the families of polynomials are orthogonal, so Theorem 3 guarantees that the family of polynomials is the complete set of joint eigenfunctions of the linear partial differential operators.

3.1 *Products of Jacobi polynomials.* Let $\alpha, \beta, \gamma, \delta \in (-1, \infty)$ and define

$$p_{n,k}(x,y) = p_{n,k}^{(\alpha,\beta,\gamma,\delta)}(x,y) = R_{n-k}^{(\alpha,\beta)}(x) R_k^{(\gamma,\delta)}(y).$$

These are orthogonal polynomials and are the eigenfunctions of a pair of linear partial differential operators which are obtained from the standard differential equations for Jacobi polynomials [**18**]:

$$J^{(j)}[p_{n,k}] = j_{n,k}^{(j)} p_{n,k}, \quad j = 1, 2 \text{ and } 0 \le k \le n, \tag{3.1}$$

with eigenvalues

$$j_{n,k}^{(1)} = (n-k)(n-k+\alpha+\beta+1)$$
$$j_{n,k}^{(2)} = k(k+\gamma+\delta+1)$$

and linear differential operators

$$J^{(1)} = (x^2-1)D_x^2 + [(\alpha+\beta+2)x+\alpha-\beta]D_x$$
$$J^{(2)} = (y^2-1)D_y^2 + [(\gamma+\delta+2)+\gamma-\delta]D_y.$$

Remarks. 1. It follows from Gasper's product formula [**6**] that this algebraically complete family of polynomials has a hypergroup product formula on the square $Q = [-1,1]\times[-1,1]$ if and only if (α,β) and (γ,δ) belong to

$$E_J = \{(\alpha,\beta) : \alpha \geq \beta > -1 \text{ and either } \beta \geq -1/2 \text{ or } \alpha+\beta \geq 0\}.$$

There are values of the parameters outside this region where the polynomials have a product formula, but the product measures are not always positive (and the polynomials may not even be orthogonal), yet (3.1) still holds.

2. Q has a cusp of order 1 at (1,1) and the eigenvalues above are $2+2\alpha$ times the eigenvalues in (1.9), so $J^{(1)}$ and $J^{(2)}$ are essentially the linear partial differential operators guaranteed by Theorem 1.

3. $(J^{(1)}, J^{(2)})$ is a deterministic pair by Theorem 2.

4. These polynomials play a special role with respect to the square Q. If $\mathcal{P}$ is an algebraically complete family of polynomials with a strong product formula on a set H which has infinitely many points in each of the set $\{1\}\times[-1,1]$ and $[-1,1]\times\{1\}$, and if $\mathcal{P}_1 = \{1, a(x-1)+1, b(y-1)+1\}$ then, up to an affine change of variables

$$\mathcal{P} = \{p_{n,k}^{(\alpha,\beta,\gamma,\delta)} : 0 \leq k \leq n\}$$

for some (α,β) and (γ,δ) belonging to E_J [**4**, Theorem 4.4].

3.2 *Disk polynomials.* These are defined and dealt with extensively in [**4, 7, 17**] and the references cited there. Let $\gamma > -1$ and identify (x,y) with $x+iy = re^{i\theta}$. Define the disk polynomials by

$$R_{n,m}^{\gamma}(x,y) = r^{|n-m|}e^{i(n-m)\theta}R_{n\wedge m}^{(\gamma,|n-m|)}(2r^2-1),$$

where $n \wedge m$ is the minimum of n and m. These functions are polynomials on the unit disk which are orthogonal with respect to the measure $(1-x^2-y^2)^\gamma \,dx\,dy$. If $\gamma \geq 0$, they have a hypergroup product formula so they are the eigenfunctions for a pair of differential operators:

$$U^{(j)}[R^\gamma_{n,m}] = u^{(j)}_{n,m} R^\gamma_{n,m}, \quad j = 1,2 \text{ and } n,m \in \mathbf{N}_0 \tag{3.2}$$

with eigenvalues

$$\begin{aligned} u^{(1)}_{n,m} &= D_x[R^\gamma_{n,m}] = m+n+\frac{2mn}{\gamma+1} \\ u^{(2)}_{n,m} &= D_y[R^\gamma_{n,m}] = i(n-m) \end{aligned}$$

and linear differential operators

$$\begin{aligned} U^{(1)} &= \frac{1}{2(\gamma+1)}(x^2+y^2-1)(D_x^2+D_y^2)+xD_x+yD_y \\ U^{(2)} &= xD_y - yD_x. \end{aligned}$$

Remarks. 1. These operators were first given explicitly in [**4**] in which a form of Theorem 1 with a weaker condition than cusps of order 1 was proven.

2. Theorem 1 guarantees the existence of a pair of linear partial differential operators of order not exceeding 2; in fact, the pair of linear partial differential operators, which we believe were first obtained in [**4**] have order 2 and 1.

3. $(U^{(1)}, U^{(2)})$ is a deterministic pair by Theorem 2.

4. The product formula holds for $\gamma \geq 0$, but the differential equations hold as long as $\gamma > -1$.

5. These polynomials play a special role for the disk. Suppose $\mathcal{P}$ is an orthogonal family of polynomials. Suppose $x+iy \in \mathcal{P}$ and assume also that, for each $p \in \mathcal{P}$, $p(1,0)=1$, $p(x-y)=\overline{p(x,y)}$, $\bar{p} \in \mathcal{P}$. Assume further that the polynomials in $\mathcal{P}$ are joint eigenfunctions of a pair of second order linear partial differential operators $L^{(1)}$ and $L^{(2)}$, and that

$$L^{(1)}(p) = p_x(1,0)p \quad \text{and} \quad L^{(2)}(p) = p_y(1,0)p, \quad p \in \mathcal{P}_2.$$

Then $\mathcal{P} = \{R^{\gamma}_{n,m} : 0 \le n, m\}$ and $L^{(j)} = U^{(j)}$ for some $\gamma > -1$ [**17**, Theorems 3.2 and 3.3].

3.3 *Parabolic biangle polynomials.* Let

$$R^{\alpha,\beta}_{n,k}(x,y) = R^{(\alpha,\beta+k+1/2)}_{n-k}(2x-1)\cdot x^{k/2} R^{(\beta,\beta)}_{k}(x^{-1/2}y)$$

where $\alpha, \beta > -1$ and n and k are integers such that $0 \le k \le n$. These functions are polynomials in x and y and they are orthogonal on the parabolic biangular region

$$B = \{(x,y) : 0 \le y^2 \le x \le 1\}$$

with respect to the measure $(1-x)^{\alpha}(x-y^2)^{\beta}\,dx\,dy$.

Theorem 4. *If $\alpha, \beta > -1$ the parabolic biangle polynomials is a complete set of eigenfunctions for a deterministic pair of second order linear partial differential operators. That is,*

$$B^{(j)}[R^{\alpha,\beta}_{n,k}] = \beta^{(j)}_{n,k} R^{\alpha,\beta}_{n,k}, \quad j = 1,2 \text{ and } 0 \le k \le n$$

with eigenvalues

$$\begin{aligned}\beta^{(1)}_{n,k} = [D_x R^{\alpha,\beta}_{n,k}](1,1) = \frac{n^2}{(\alpha+1)} + \frac{(2\alpha+2\beta+3)n}{2(\alpha+1)} - \frac{k^2}{4(\beta+1)} \\ - \frac{(4\alpha\beta+4\beta^2+3\alpha+10\beta+5)k}{4(\alpha+1)(\beta+1)} - \frac{nk}{\alpha+1} \\ \beta^{(2)}_{n,k} = [D_y R^{\alpha,\beta}_{n,k}](1,1) = \frac{k(k+2\beta+1)}{2(\beta+1)}\end{aligned}$$

and linear partial differential operators

$$\begin{aligned}B^{(1)} &= \frac{x^2-x}{\alpha+1}D_x^2 + \frac{xy-x}{\alpha+1}D_xD_y \\ &\quad + \frac{(\alpha+\beta+2)x-(\alpha+1)y^2-\beta-1}{4(\alpha+1)(\beta+1)}D_y^2 \\ &\quad + \frac{(2\alpha+2\beta+5)x-2\beta-3}{2(\alpha+1)}D_x \\ B^{(2)} &= \frac{y^2-x}{2(\beta+1)}D_y^2 + yD_y.\end{aligned}$$

Proof. When $\alpha \geq \beta + 1/2 \geq 0$, these polynomials satisfy a hypergroup product formula with $e = (1, 1)$ [**14**]. The region B has a cusp of order 1 at e so Theorem 1 and Lemma 1.1 predict that these polynomials are the joint eigenfunctions for a pair of linear second order partial differential operators. The eigenvalues given by (1.9) are computed with *Mathematica* or using the expression for the derivative of Jacobi polynomials given in [**18**]. The differential operators were computed with the aid of *Mathematica*, though the recursion given in the proof of Lemma 6.1 in [**10**] or in the proof of Lemma 1.1 would also do the job. The resulting differential equations are still valid in the full range of the parameters $\alpha, \beta > -1$ and, since the polynomials are orthogonal for these values, Theorem 3 applies. To check that $(B^{(1)}, B^{(2)})$ is a deterministic pair suppose $\beta^{(j)}_{(n_1,k_1)} = \beta^{(j)}_{(n_2,k_2)}$ for $j = 1, 2$. When $j = 2$, it follows that $k_1 = k_2$; to show $n_1 = n_2$ observe that for $n > 0$, $\partial\beta^{(1)}_{n,k}/\partial n > 0$. □

3.4 *Triangle polynomials.* Let

$$R^{\alpha,\beta,\gamma}_{n,k}(x, y) = R^{\alpha,\beta+\gamma+2k+1}_{n-k}(2x - 1) \cdot x^k R^{\beta,\gamma}_k(2x^{-1}y - 1)$$

where $\alpha, \beta, \gamma > -1$ and n and k are integers such that $0 \leq k \leq n$. These functions are polynomials in x and y and they are orthogonal on the triangular region

$$S = \{(x, y) : 0 \leq y \leq x \leq 1\}$$

with respect to the measure $(1 - x)^\alpha (x - y)^\beta y^\gamma \, dx \, dy$.

Theorem 5. *If $\alpha, \beta, \gamma > -1$, the triangle polynomials is a complete set of eigenfunctions of a deterministic pair of second order linear partial differential operators. That is,*

$$S^{(j)}[R^{\alpha,\beta,\gamma}_{n,k}] = \sigma^{(j)}_{n,k} R^{\alpha,\beta,\gamma}_{n,k}, \quad j = 1, 2, \ \ \textit{and } 0 \leq k \leq n$$

with eigenvalues

$$\sigma_{n,k}^{(1)} = [D_x R_{n,k}^{\alpha,\beta,\gamma}](1,1) = \frac{(n-k)(n+k+\alpha+\beta+\gamma+2)}{\alpha+1} - \frac{k(k+\beta+\gamma+1)}{\beta+1} + k$$

$$\sigma_{n,k}^{(2)} = [D_y R_{n,k}^{\alpha,\beta,\gamma}](1,1) = \frac{k(k+\beta+\gamma+1)}{\beta+1}$$

and linear partial differential operators

$$\begin{aligned} S^{(1)} &= \frac{x^2-x}{\alpha+1} D_x^2 + \frac{2(xy-y)}{\alpha+1} D_x D_y \\ &\quad + \frac{y((\alpha+\beta+2)x-(\alpha+1)y-\beta-1)}{(\alpha+1)(\beta+1)} D_y^2 \\ &\quad + \frac{(\alpha+\beta+\gamma+3)x-\beta-\gamma-2}{\alpha+1} D_x \\ &\quad + \frac{(\gamma+1)((\alpha+\beta+2)x-(\alpha+1)y-\beta-1)}{(\alpha+1)(\beta+1)} D_y \\ S^{(2)} &= \frac{(y-x)y}{\beta+1} D_y^2 + \frac{(\beta+\gamma+2)y-(\gamma+1)x}{\beta+1} D_x. \end{aligned}$$

Proof. When $\alpha \geq \beta+\gamma+1 \geq 0$ and $\beta \geq \gamma \geq -1/2$, these polynomials satisfy a hypergroup product formula with $e = (1,1)$ [**14**]. The rest of the argument is very similar to the proof of Theorem 4. □

3.5 *Parabolic triangle polynomials.* The polynomials

$$\frac{1}{2}[R_n^{(\alpha,\beta)}(x) R_k^{(\alpha,\beta)}(y) + R_k^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(y)]$$

where $\alpha, \beta > -1$ and n and k belong to $\mathbf{N}_0$ are orthogonal on the square region $[-1,1] \times [-1,1]$ with respect to the measure $(1-x)^\alpha (1+x)^\beta (1-y)^\alpha (1+y)^\beta \, dx \, dy$. These are symmetric polynomials in x and y so they are also polynomials in the elementary symmetric functions $u = x+y$ and $v = xy$, see [**10**, Section 3]. The resulting polynomials

$$Q_{n,k}^{(\alpha,\beta)}(u,v) = \frac{1}{2}[R_n^{(\alpha,\beta)}(x) R_k^{(\alpha,\beta)}(y) + R_k^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(y)], \quad 0 \leq k \leq n$$

are orthogonal on the parabolic triangular region

$$T = \{(u, v) : |u| - 1 \le v \le u^2/4 \text{ and } |u| \le 2\},$$

with respect to the measure $(1-u+v)^\alpha(1+u+v)^\beta(u^2-4v)^{-1/2}\,du\,dv$. These differ only slightly from Koornwinder's polynomials $p_{n,k}^{\alpha,\beta,\gamma}$ with $\gamma = -1/2$. When $(\alpha, \beta) \in E_J$, see Example 3.1, these polynomials satisfy a hypergroup product formula with $e = (2, 1)$ [**4**, Example 4].

Koornwinder showed [**10**, II (4.4), (5.14)] that the polynomials $p_{n,k}^{\alpha,\beta,\gamma}$ are eigenfunctions of a pair of differential operators; indeed, when $\gamma = -1/2$, his result becomes

$$D^{(j)}[Q_{n,k}^{(\alpha,\beta)}] = \delta_{n,k}^{(j)} Q_{n,k}^{(\alpha,\beta)}$$

where

$$\begin{aligned} D^{(1)} &= (-u^2+2v+2)D_u^2 + (2u-2uv)D_uD_v + (u^2-2v^2-2v)D_v^2 \\ &\quad + [-(\alpha+\beta+2)u + (2\beta-2\alpha)]D_u \\ &\quad + [(\beta-\alpha)u - (2\alpha+2\beta+4)v]D_v \\ D^{(2)} &= D_+D_- \end{aligned}$$

with

$$\begin{aligned} D_- &= D_u^2 + uD_uD_v + vD_v^2 + D_v \\ D_+ &= (1-u+v)^{-\alpha}(1+u+v)^{-\beta}D_-(1-u+v)^{\alpha+1}(1+u+v)^{\beta+1} \end{aligned}$$

and

$$\begin{aligned} \delta_{n,k}^{(1)} &= -n(n+\alpha+\beta+1) - k(k+\alpha+\beta+1) \\ \delta_{n,k}^{(2)} &= k(k+\alpha+\beta+1)n(n+\alpha+\beta+1). \end{aligned}$$

T has a cusp of order 2 at e as can be seen by considering the two boundary curves ending at e. Thus, by Theorem 1,

$$T^{(j)}[Q_{n,k}^{(\alpha,\beta)}] = \tau_{n,k}^{(j)} Q_{n,k}^{(\alpha,\beta)}, \quad j = 1, 2, \text{ and } 0 \le k \le n,$$

where $T^{(1)}$ and $T^{(2)}$ are linear partial differential operators of orders at most 2 and 4, respectively, and

$$\begin{aligned}
\tau_{n,k}^{(1)} &= \frac{-1}{\sqrt{2}}([D_u Q_{n,k}^{(\alpha,\beta)}](e) + [D_v Q_{n,k}^{(\alpha,\beta)}](e)) \\
&= \frac{-1}{4\sqrt{2}(\alpha+1)}[n(n+\alpha+\beta+1) + k(k+\alpha+\beta+1)] \\
\tau_{n,k}^{(2)} &= -\frac{1}{8}([D_u Q_{n,k}^{(\alpha,\beta)}](e) - [D_v Q_{n,k}^{(\alpha,\beta)}](e)) \\
&= \frac{-1}{32(\alpha+1)(\alpha+2)}[n(n+\alpha+\beta+1)(n(n+\alpha+\beta+1)-\beta) \\
&\quad + k(k+\alpha+\beta+1)(k(k+\alpha+\beta+1)-\beta)] \\
&\quad + \frac{1}{16(\alpha+1)^2} n(n+\alpha+\beta+1)k(k+\alpha+\beta+1).
\end{aligned}$$

Now the relation between the two sets of eigenvalues is

$$\begin{aligned}
\tau_{n,k}^{(1)} &= \frac{1}{4\sqrt{2}(\alpha+1)}\delta_{n,k}^{(1)} \\
\tau_{n,k}^{(2)} &= \phi(\delta_{n,k}^{(1)}, \delta_{n,k}^{(2)})
\end{aligned}$$

where the polynomial ϕ is given by

$$\phi(x,y) = \frac{(2\alpha+3)y}{16(\alpha+1)^2(\alpha+2)} - \frac{\beta x + x^2}{32(\alpha+1)(\alpha+2)}. \tag{3.3}$$

Thus

$$T_1 = \frac{D^{(1)}}{4\sqrt{2}(\alpha+1)} \quad \text{and} \quad T^{(2)} = \phi(D^{(1)}, D^{(2)})$$

are the second and fourth order operators as expected.

It is elementary to check that $(D^{(1)}, D^{(2)})$ is a deterministic pair; hence, so is $(T^{(1)}, T^{(2)})$ which holds for (1.10) because of the injectivity of the mappings $(n,k) \mapsto (\delta_{n,k}^{(1)}, \delta_{n,k}^{(2)})$ and $(x,y) \mapsto (x/(4\sqrt{2}(\alpha+1)), \phi(x,y))$.

4. The algebra of linear partial differential operators with polynomial eigenfunctions $\mathcal{P}$.

Definition. Let $\mathcal{A}$ be an algebra of linear partial differential operators with polynomial coefficients. A pair of operators $L^{(1)}$ and $L^{(2)}$ in $\mathcal{A}$ are *generators* of $\mathcal{A}$ if every differential operator in $\mathcal{A}$ has a unique representation as a polynomial of $L^{(1)}$ and $L^{(2)}$.

Our main concern in this section is to show that if $\mathcal{P}$ is any of the five examples given above, the two linear partial differential operators obtained in the proof of Theorem 1 are generators of $D(\mathcal{P})$. We begin the discussion in general terms.

Lemma 4.1. *If $\mathcal{P}$ is an algebraically complete family of polynomials, then $D(\mathcal{P})$ is a commutative algebra.*

Theorem 6. *Suppose the algebraically complete family of polynomials $\mathcal{P}$ are all eigenfunctions of one of the following pairs of partial differential operators*

$$L^{(1)} = a(x)D_x^2 + b(x,y)D_xD_y + c(x,y)D_y^2 + \text{l.o.t.}$$
$$L^{(2)} = e(x,y)D_y^2 + \text{l.o.t.}$$

or

$$L^{(1)} = a(x^2+y^2)(D_x^2 + D_y^2) + \text{l.o.t.}$$
$$L^{(2)} = xD_y - yD_x$$

where none of $a(x)$, $e(x,y)$ and $a(x)e(x,y)$ is the square of a polynomial. Let $\{\lambda_p^{(1)}, \lambda_p^{(2)} : p \in \mathcal{P}\}$ be the eigenvalues associated with $L^{(1)}$ and $L^{(2)}$, and assume that $\Lambda = \{(\lambda_p^{(1)})^n(\lambda_p^{(2)})^k : n,k \in \mathbf{N}_0\}$ is a linearly independent set of functions on $\mathcal{P}$. Then $L^{(1)}$ and $L^{(2)}$ generate $D(\mathcal{P})$.

Proof. The method here is inspired by [**10**, II, Theorem 6.5]. Suppose first that $L^{(1)}$ and $L^{(2)}$ are the first pair of operators. Assume by way of contradiction that $M \in D(\mathcal{P})$ cannot be expressed as a polynomial of $L^{(1)}$ and $L^{(2)}$, suppose that m is the lowest order of any such linear partial differential operator. Thus, M can be assumed to have order m

and have the form

$$M = \sum_{j=k}^{m} d_j(x,y) D_x^{m-j} D_y^j + \text{l.o.t.}, \quad d_k \neq 0,$$

where $d_j(x,y)$ are polynomials. Let M be chosen so that, when represented as above, k is as large as possible.

Now $ML^{(1)} - L^{(1)}M$ and $ML^{(2)} - L^{(2)}M$ are linear partial differential operators of order $m+1$, thus

$$ML^{(1)} - L^{(1)}M = \sum_{j=k}^{m+1} f_j(x,y) D_x^{m+1-j} D_y^j + \text{l.o.t.}$$

$$ML^{(2)} - L^{(2)}M = \sum_{j=k}^{m+1} g_j(x,y) D_x^{m+1-j} D_y^j + \text{l.o.t.}$$

Thus we obtain with the use of Lemma 4.1

$$\begin{aligned} f_k &= d_k(m-k)a^{(1,0)} + kd_k b^{(0,1)} \\ -2ad_k^{(1,0)} - bd_k^{(0,1)} &= 0 \\ g_{k+1} &= k\, d_k e^{(0,1)} - 2ed_k^{(0,1)} = 0. \end{aligned}$$

The solution of the second equation is

$$d_k(x,y) = e(x,y)^{k/2} h(x)$$

for an arbitrary differentiable function h; substitution of this expression in the first equation yields

$$\text{(4.1)} \quad [2(m-k)ea^{(1,0)} + 2keb^{(0,1)} - kbe^{(0,1)} - 2kae^{(1,0)}]h - 4aeh^{(1,0)} = 0.$$

Now the coefficient of $w^{(1,2)}$ in $L^{(1)}(L^{(2)}(w)) - L^{(2)}(L^{(1)}(w))$ is $be^{(0,1)} + 2ae^{(1,0)} - 2eb^{(0,1)}$ which vanishes by Lemma 4.1.

This can be used to simplify (4.1) to obtain

$$(m-k)a'h - 2ah' = 0$$

whence $h = Ka^{(m-k)/2}$, so

$$d_k(x,y) = Ke(x,y)^{k/2}a(x)^{(m-k)/2}.$$

Thus $d_k = 0$ if k is odd or m is odd since a, e and ae are not perfect squares, while if k and m are even

$$M - KL^{(1)^{(m-k)/2}}L^{(2)^{k/2}}$$

belongs to $D(\mathcal{P})$ but cannot be expressed as a polynomial of $L^{(1)}$ and $L^{(2)}$; this violates our choice of M.

Now we turn to the second pair of linear partial differential operators which we express in polar coordinates:

$$\begin{aligned} L^{(1)} &= a(r^2)(D_r^2 + r^{-2}D_\theta^2) + \text{l.o.t.} \\ L^{(2)} &= D_\theta. \end{aligned}$$

Assume by way of contradiction that $M \in D(\mathcal{P})$ cannot be expressed as a polynomial of $L^{(1)}$ and $L^{(2)}$, suppose that m is the lowest order of any such linear partial differential operator. M given in polar coordinates by

$$M = \sum_{j=k}^{m} g_j(r,\theta)D_r^{m-j}D_\theta^j + \text{l.o.t.}$$

We assume M to be chosen so that, when represented as above, k is as large as possible. Now

$$ML^{(2)} - L^{(2)}M = \sum_{j=k}^{m} D_\theta[g_j(r,\theta)]D_r^{m-j}D_\theta^j + \text{l.o.t.};$$

hence, by Lemma 4.1 each $g_j(r,\theta) = g_j(r)$ is a function of r only. Now in the same way we examine the coefficient of $D_r^{m+1-k}D_\theta^k$ in $ML^{(1)} - L^{(1)}M$ to obtain

$$2(m-k)ra'(r^2)g_k(r) - 2a(r^2)g_k'(r) = 0, \quad 0 \le k < m,$$

and

$$2a(r^2)g_m'(r) = 0.$$

When $0 \le k < m$, we obtain $g_k(r) = Ka(r^2)^{(m-k)/2}$ where K is a constant so that $m-k$ must be even and $M - KL^{(1)^{(m-k)/2}}$ violates our choice of M. When $k = m$, we get a similar violation from $M - KL^{(2)^m}$ for an appropriate constant K.

To establish uniqueness, it suffices to show that the only polynomial $\phi(x,y)$ such that $\phi(L^{(1)}, L^{(2)}) = 0$ is $\phi(x,y) = 0$. Now $\phi(L^{(1)}, L^{(2)})$ has eigenvalues $\phi(\lambda_p^{(1)}, \lambda_p^{(2)})$, and by the linear independence assumption, this can only vanish if $\phi(x,y) = 0$. □

The following technical lemma gives us a simple way to establish the linear independence of Λ in the examples. In each example except the disk polynomials, $\mathcal{P} = \{p_{n,k} : 0 \le k \le n\}$. In the disk case, let $k = n - m$ so in that case $\mathcal{P} = \{p_{n,k} : n \in \mathbf{N}_0, k \in \mathbf{Z}\}$. For $\nu = 1, 2$, let $\lambda_{n,k}^{(\nu)} = \lambda_{p_{n,k}}^{(\nu)}$. In the examples $\lambda_{n,k}^{(\nu)}$ are polynomials in n and k.

Lemma 4.2. *Suppose that* $\{(\lambda_{n,0}^{(1)})^m : m \in \mathbf{N}_0\}$ *is a linearly independent set of polynomials, assume* $\lambda_{n,0}^{(2)} = 0$ *but* $\lambda_{n,k}^{(2)} \ne 0$ *for* $k \ne 0$. *Then* $\Lambda = \{(\lambda_{n,k}^{(1)})^m (\lambda_{n,k}^{(2)})^j : m, j \in \mathbf{N}_0\}$ *is a linearly independent set of polynomials in* (n,k).

Proof. Suppose $A(n,k) = \sum_{m,j=0}^N \alpha_{m,j} (\lambda_{n,k}^{(1)})^m (\lambda_{n,k}^{(2)})^j = 0$. We can write

$$A(n,k) = \sum_{j=0}^N \left[\sum_{m=0}^N \alpha_{m,j} (\lambda_{n,k}^{(1)})^m \right] (\lambda_{n,k}^{(2)})^j,$$

so $0 = A(n,0) = \sum_{m=0}^N \alpha_{m,0} (\lambda_{n,0}^{(1)})^m$, hence $\alpha_{m,0} = 0$ for $0 \le m \le N$.

Now assume $\alpha_{m,j} = 0$ for $0 \le j \le l-1$ and $0 \le m \le N$, then if $k \ne 0$

$$0 = \frac{A_{n,k}}{(\lambda_{n,k}^{(2)})^l} = \sum_{j=l}^N \left[\sum_{m=0}^N \alpha_{m,j} (\lambda_{n,k}^{(1)})^m \right] (\lambda_{n,k}^{(2)})^{j-l}.$$

Since the expression on the right is a polynomial in k for each fixed n, it must also vanish when $k = 0$, whence $\alpha_{m,l} = 0$ for $0 \le m \le N$. □

The result alluded to at the beginning of the section can now be stated as

Theorem 7. *Let $\mathcal{P}$ denote one of the five examples discussed above. Then the pair of linear partial differential operators obtained in Theorem 1 generate $D(\mathcal{P})$.*

Proof. Theorem 6 includes as special cases all of the examples except the parabolic triangle polynomials.

We now present an argument for the parabolic triangle polynomials. It is clear that $D^{(1)}$ and $D^{(2)}$ are polynomials ψ_1 and ψ_2 in $T^{(1)}$ and $T^{(2)}$. Koornwinder shows [**10**, II, Theorem 6.5] that each linear partial differential operator M which has the polynomials $\{Q_{n,k}^{(\alpha,\beta)}\}$ as eigenfunctions can be expressed uniquely is a polynomial in $D^{(1)}$ and $D^{(2)}$. Uniqueness follows, since if $\psi(T^{(1)},T^{(2)})=0$ for some polynomial ψ, then

$$\eta(D^{(1)},D^{(2)})=\psi\left(\frac{D^{(1)}}{4\sqrt{2}(\alpha+1)},\phi(D^{(1)}D^{(2)})\right)=\psi(T^{(1)},T^{(2)})=0,$$

where ϕ is given by (3.3) which contradicts Koornwinder's result. □

5. Admissibility. In this section we investigate which of the five examples admit an admissible-2 linear partial differential operator; two do and three do not. We begin with two lemmas, the first of which requires no proof.

Lemma 5.1. *Suppose $\mathcal{P}$ is an algebraically complete family and suppose $L^{(1)}$ and $L^{(2)}$ generate $D(\mathcal{P})$. Suppose further that*

$$L^{(j)}p=\lambda_p^{(j)}p,\quad p\in\mathcal{P},\quad j=1,2,$$

then $\mathcal{P}$ has an admissible-2 linear partial differential operator if and only if there is a polynomial $\phi(x,y)$ such that $\phi(\lambda_p^{(1)},\lambda_p^{(2)})$ is a function which depends only on deg (p). *In this case the linear partial differential operator $\phi(L^{(1)},L^{(2)})$ is admissible-2.*

Lemma 5.2. *Suppose $\phi(x,y)$ is a polynomial, and suppose either*

(5.1)
$$A(n,k)=an^2+ck^2+\text{l.o.t.}\ \textit{and}\ B(n,k)=dn^2k^2+\text{l.o.t.}$$

or

(5.2)
$$A(n,k) = an^2 + bnk + ck^2 + \text{l.o.t} \textit{ and } B(n,k) = dk^2 + \text{l.o.t.}$$

where a, b, c and d are nonzero. Then $\phi(A(n,k), B(n,k))$ is a function of n alone if and only if $\phi(x,y)$ is a constant polynomial.

Proof. Suppose A and B are given by (5.1), that $\phi(x,y)$ is a polynomial such that $\phi(A(n,k), B(n,k))$ is a function of n only, and assume by way of contradiction that ϕ has minimal bivariate degree, say (m,j), thus

$$\phi(x,y) = \sum_{(r,s)\le(m,j)} \alpha_{r,s} x^{r-s} y^s$$

with $m > 0$ and $\alpha_{m,j} \neq 0$. The coefficient of k^{2m} in $\phi(A(n,k), B(n,k))$ is

$$\sum_{s=0}^{j} (c^{m-s} d^s) \alpha_{m,s} n^{2s}.$$

Thus, since $\phi(A(n,k), B(n,k))$ depends only on n, this must vanish, so $\alpha_{m,j} = 0$ which contradicts ϕ having minimal bivariate degree.

Now suppose A and B are given by (5.2) and assume by way of contradiction that ϕ is as above. $\phi(B(n,k), A(n,k))$ is a polynomial in n and k of degree $2m$ and the portion of it which is degree $2m$ is

$$\sum_{i=0}^{j} \alpha_{m,i} (dk^2)^{m-i} (an^2 + bnk + ck^2)^i. \tag{5.3}$$

Case 1. $j = m$. The coefficient of $n^{2m-1}k$ in (5.3) is $(ma^{m-1}b)\alpha_{m,m}$, thus since $\phi(B(n,k), A(n,k))$ depends only on n, $\alpha_{m,m} = 0$, which contradicts the minimal degree of ϕ.

Case 2. $j < m$. Since we are assuming $\alpha_{m,i} = 0$ for $i > j$ the only term in (5.3) which involves $n^{2j}k^{2m-2j}$ is $(d^{m-j}a^j)\alpha_{m,j} n^{2j}/k^{2m-2j}$, thus since $\phi(B(n,k), A(n,k))$ depends only on n, $\alpha_{m,j} = 0$, which again contradicts the minimal degree of ϕ. □

Combining the last two lemmas, we have

Theorem 8. *Suppose* $\mathcal{P} = \{p_{n,k} : 0 \le k \le n\}$ *is an algebraically complete family with* $\deg p_{n,k} = m$*, and suppose* $L^{(1)}$ *and* $L^{(2)}$ *belong to* $D(\mathcal{P})$ *such that*

$$L^{(1)} p_{n,k} = A(n,k) p_{n,k} \quad \text{and} \quad L^{(2)} p_{n,k} = B(n,k) p_{n,k}, \quad 0 \le k \le n$$

where $A(n,k)$ *and* $B(n,k)$ *are given by* (5.1) *or* (5.2)*. Suppose that every linear partial differential operator in* $D(\mathcal{P})$ *can be expressed as a polynomial in* $L^{(1)}$ *and* $L^{(2)}$*. Then* $\mathcal{P}$ *does not have an admissible-*2 *linear partial differential operator.*

We now turn to discussions of the five examples.

5.1 *Products of Jacobi polynomials.* There is no admissible 2-linear partial differential operator because the eigenvalues satisfy (5.2).

5.2 *Disk polynomials.* First observe that the disk polynomials of degree n are $R^{\gamma}_{n-k,k}$ for $-n \le k \le n$. Krall and Sheffer show [**13**, (4.22), (4.23), (5.14)] that the disk polynomials have an admissible-2 linear partial differential operator

$$L = (x^2 - 1)D_x^2 + 2xyD_{xy} + (y^2 - 1)D_y^2 + (2\gamma + 3)(xD_x + yD_y)$$

with eigenvalue $\lambda_n = n(n+2\gamma+2)$ for all disk polynomials of degree n. (There is a small misprint in equation (5.14) of Krall and Sheffer.) By Theorem 7 we know L can be expressed as a polynomial of $U^{(1)}$ and $U^{(2)}$; the easiest way to do this is first to express λ_n as a polynomial in $u^{(1)}_{n-j,j}$ and $u^{(2)}_{n-j,j}$. It is easy to discover that the desired relation is $\lambda_n = (2\gamma + 2)u^{(1)}_{n-j,j} - [u^{(2)}_{n-j,j}]^2$, thus $L = (2\gamma + 2)U^{(1)} - [U^{(2)}]^2$.

5.3 *Parabolic biangle polynomials.* There is no admissible-2 linear partial differential operator because the eigenvalues satisfy (5.2).

5.4 *Triangle polynomials.* Let

$$\lambda_n = (\alpha+1)\sigma_{n,k}^{(1)} + (\alpha+\beta+2)\sigma_{n,k}^{(2)} = n(n+\alpha+\beta+\gamma=2),$$

then $\lambda_n \neq \lambda_m$ when $m \neq n$, and so $(\alpha+1)S^{(1)} + (\alpha+\beta+2)S^{(2)}$ is an admissible linear partial differential operator for the triangle polynomials with eigenvalues λ_n by Lemma 5.1.

Remark. $-\lambda_n$ agrees with the eigenvalues corresponding to [**13**, (5.51)] by Theorem 3.1 so the partial differential equation [**13**, (5.52)] coincides with $(\alpha+1)S^{(1)} = (\alpha+\beta+2)S^{(2)}$ following the change of variables $(x,y) \mapsto (1-x, 1-x-y)$.

5.5 *Parabolic triangle polynomials.* Since the eigenvalues $\delta_{n,k}^{(1)}$ and $\delta_{n,k}^{(2)}$ satisfy (5.1), Theorem 8 implies that the parabolic biangle polynomials do not have an admissible linear partial differential operator.

6. Some open questions. We conclude the article with a number of questions. We assume $\mathcal{P}$ is an algebraically complete family of polynomials which satisfy a strong product formula with identity element e. Let $L^{(1)}$ and $L^{(2)}$ be the differential operators with eigenvalues $\lambda_p^{(1)}$ and $\lambda_p^{(2)}$ guaranteed by Theorem 1.

Question 1. Must the pair $(L^{(1)}, L^{(2)})$ be deterministic and must (1.10) hold? When H has a cusp of order 1 at e, this is equivalent to asking whether the mapping $p \mapsto (p^{(1,0)}(e), p^{(0,1)}(e))$ must be injective.

Question 2. If $M \in D(\mathcal{P})$, must M have a unique representation as a polynomial in $L^{(1)}$ and $L^{(2)}$?

Question 3. Is there $M \in D(\mathcal{P})$ so that every polynomial in $\mathcal{P}$ has a distinct eigenvalue? In particular, this asks in the case of a cusp of order 1 where there is a polynomial $\phi(x,y)$ such that the mapping $p \mapsto \phi(\lambda_p^{(1)}, \lambda_p^{(2)})$ is injective. One solution would be $L^{(1)} + iL^{(2)}$ if the pair $(L^{(1)}, L^{(2)})$ determines $\mathcal{P}$ provided both linear partial differential operators and the members of $\mathcal{P}$ are real.

Question 4. In the disk example, $L^{(2)}$ has order less than 2. Is this because the disk has a smooth boundary at e?

REFERENCES

1. G.E. Andrews, R. Askey and R. Roy, *Special functions*, Cambridge University Press, 1999.

2. S. Bochner, *Über Sturm-Liouvillische Polynomesysteme*, Math. Z. **29** (1929), 730–736.

3. W.C. Connett and A.L. Schwartz, *Analysis of a class of probability preserving measure algebras on a compact interval*, Trans. Amer. Math. Soc. **320** (1990), 371–393.

4. ———, *Continuous* 2*-variable polynomial hypergroups*, in *Applications of hypergroups and related measure algebras*, O. Gebuhrer, W.C. Connett and A.L. Schwartz, editors, American Mathematical Society, 1995.

5. ———, *Subsets of R which support hypergroups with polynomial characters*, J. Comput. Appl. Math. **65** (1995), 73–84.

6. G. Gasper, *Banach algebras for Jacobi series and positivity of a kernel*, Ann. Math. **95** (1972), 261–280.

7. O. Gebuhrer and A.L. Schwartz, *Sidon sets and Riesz sets for some measure algebras on the disk*, Colloq. Math. **72** (1997), 269–279.

8. ———, *Harmonic analysis on compact commutative hypergroups*: *the role of the maximum group*, J. Analyse Math. (2000)

9. K.H. Kwon, D.W. Lee and L.L. Littlejohn, *Differential equations having orthogonal polynomial solutions*, J. Comput. Appl. Math. **80** (1997), 1–16.

10. T.H. Koornwinder, *Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent partial differential operators*, I, II, III, IV, Indag. Math. **36** (1974), 48–58, 59–66, 357–369, 370–381.

11. ———, *Two-variable analogues of the classical orthogonal polynomials*, in *Theory and applications of special functions*, Richard A. Askey, editor, Academic Press, Inc., New York, 1975.

12. H.L. Krall and I.M. Sheffer, *Differential equations of infinite order for orthogonal polynomials*, Ann. Mat. Pura Appl. **74** (1966), 135–172.

13. ———, *Orthogonal polynomials in two variables*, Ann. Mat. Pura Appl. **76** (1967), 325–376.

14. T.H. Koornwinder and A.L. Schwartz, *Product formulas and associated hypergroups for orthogonal polynomials on the simplex and on a parabolic biangle*, Constr. Approx. **13** (1997), 537–567.

15. W. Rudin, *Principles of mathematical analysis*, McGraw Hill, New York, 1976.

16. A.L. Schwartz, *Three lectures on hypergroups*, in *Harmonic analysis and hypergroups*, K.A. Ross et al., editor, Birkhäuser, Boston, 1998.

17. ———, *Partial differential equations and bivariate orthogonal polynomials*, J. Symb. Comput. **28** (1999), 827–845.

18. G. Szegö, *Orthogonal polynomials*, Volume 23 of Colloquium Publications, second edition, American Math. Society, Providence, 1967.

Department of Mathematics and Computer Science, University of Missouri-St. Louis, St. Louis, Missouri 63121
Email address: connett@arch.umsl.edu

Department of Mathematics and Computer Science, University of Missouri-St. Louis, St. Louis, Missouri 63121
Email address: schwartz@arch.umsl.edu