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LINEAR RATIONAL INTERPOLATION AND ITS APPLICATION IN APPROXIMATION AND BOUNDARY VALUE PROBLEMS

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ABSTRACT. We consider the case that a function with large gradients in the interior of an interval has to be approximated over this interval or that the pseudospectral method is used to compute a similar solution of an ordinary boundary value problem. In both cases we assume that the function has minimal continuity properties but can be evaluated anywhere in the given interval. The key idea is then to attach poles to the polynomial interpolant, respectively solution of the collocation problem to obtain a special rational function with poles whose location has been optimized suitably. In the first case, the max norm of the error is minimized while in the second, the same norm is minimized of the residual of the given differential equation. The algorithms are presented and discussed. Their effectiveness is demonstrated with numerical results.

1. Introduction. In this paper we address two problems which are not necessarily related but for both of which we propose in principle the same basic approach. The first problem is that of interpolating a given continuous function f between $N+1$ distinct points $x_0, x_1, \ldots, x_N$ in an interval $[a,b]$. We can choose $[a,b] = [-1,1]$ without loss of generality. The second problem is that of solving on the same interval the boundary value problem (BVP)

$$u''(x) + p(x)u'(x) + q(x)u(x) = f(x)$$
$$u(-1) = u_l, \quad u(1) = u_r,$$

where all arising functions belong to $C^\infty[-1,1]$ and where u_l and u_r are given real numbers. For more details on both problems, see [**16, 17**].

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Our goal is to improve on the classical method of polynomial interpolation for the first and that of polynomial collocation for the second problem in the case that the interpolated function f, respectively, the solution u of the boundary value problem, has steep gradients in the interior of the domain.

In Section 2 we will outline the method proposed for interpolation and in Section 3 that for collocation. Both lead to an optimization problem or a sequence of optimization problems which are nondifferentiable and of which a global optimum is sought. In Section 4 we use essentially the same example in that $f = u$, to demonstrate the effectiveness of our method on both problems. A conclusion and outlook to future work is given in the last section. A referee has suggested that Section 2 was an "implementation" of a method given in [**8**]. However this earlier work deals with the Padé approximation of analytic functions with known branch points. No interpolation and no pole optimization is involved as in the present work. In [**9**], then, the authors suggest a certain choice of poles, again nonoptimal, in the context of improved calculations of critical indices from series expansion for Ising models.

2. The interpolation problem.

2.1 *Attaching poles to a rational interpolant.* Let $\mathcal{P}_m$ and $\mathcal{R}_{m,n}$, respectively, denote the linear space of all polynomials of degree $\leq m$ and the set of all rational functions with numerator degree $\leq m$ and denominator degree $\leq n$; furthermore, denote by f_k the interpolated values $f(x_k)$, $k = 0(1)N$, of f. Then the unique polynomial $p \in \mathcal{P}_N$ that interpolates f between the x_ks can be written in its *barycentric form* [**25**]

$$p(x) = \sum_{k=0}^{N} \frac{w_k}{x - x_k} f_k \Bigg/ \sum_{k=0}^{N} \frac{w_k}{x - x_k}, \tag{1}$$

where the so-called *weight* w_k corresponding to the point x_k is given by

$$w_k := 1 \Bigg/ \prod_{\substack{i=0 \\ i \neq k}}^{N} (x_k - x_i).$$

The barycentric formula has several advantages [**18**, page 357]. One of them is the fact that the weights appear in the numerator and in the denominator, so that they can be divided by any common factor. For example, simplified weights for equidistant points are given by

$$w_k^* = (-1)^k \binom{N}{k}$$

[**19**], while for the Čebyšev points of the second kind $\cos\phi_k$, $\phi_k := k(\pi/N)$, one simply has [**26**]

$$w_k^* = (-1)^k \delta_k, \quad \delta_k = \begin{cases} 1/2 & k = 0 \text{ or } k = n, \\ 1 & \text{otherwise.} \end{cases}$$

As explained in the introduction, we now want to improve the quality of approximation of the interpolant, for instance for functions with very large derivatives. For that purpose, we will divide the interpolant by an optimized denominator, while maintaining interpolation.

Let $P, P \leq N$, be the number of the poles z_i, $i = 1(1)P$, we want to attach to the polynomial. If some rational interpolant $r \in \mathcal{R}_{N,P}$ exists with poles at the z_is and only there, then its denominator takes the values

$$d_k := a \prod_{i=1}^{P} (x_k - z_i), \quad a \neq 0 \in \mathbf{C} \text{ arbitrary}, \tag{2}$$

at the interpolation points x_k. (The fact that it does not exist may mean that attaching the poles is not advisable from an approximation point of view, see [**10**].) To ensure interpolation, the values of the numerator at the same points will be $f_k d_k$. Writing the numerator and the denominator as interpolating polynomials in their barycentric form (1) and simplifying, one gets

$$r(x) := \sum_{k=0}^{N} \frac{w_k \prod_{i=1}^{P}(x_k - z_i)}{x - x_k} f_k \Bigg/ \sum_{k=0}^{N} \frac{w_k \prod_{i=1}^{P}(x_k - z_i)}{x - x_k}. \tag{3}$$

Equation (3) is the barycentric representation of r with weights $v_k := w_k \prod_{i=1}^{P}(x_k - z_i)$. In the present case, with all poles prescribed, the

weights are unique up to a constant [**10**]. Barycentric representations exist for every rational interpolant in $\mathcal{R}_{N,N}$ [**12, 15**]: Every $r \in \mathcal{R}_{N,N}$ interpolating a continuous function f between interpolation points $x_0, \ldots, x_N$ can be written in its barycentric form

$$r(x) = \sum_{j=0}^{N} \frac{\beta_j}{x - x_j} f(x_j) \Bigg/ \sum_{j=0}^{N} \frac{\beta_j}{x - x_j} \tag{4}$$

for some (nonunique) numbers β_j, one per node. This form can be used to solve the classical rational interpolation problem also when only a subset of the poles are prescribed [**10**].

In order to stay with real interpolants, we will assume here that the poles z_i with $\Re z_i \in [-1, 1]$ and $\Im z_i \neq 0$ arise in complex conjugate pairs.

2.2 *Optimizing the location of the poles.* In order to find a best denominator, we will solve the optimization problem of minimizing $\|r - f\|_\infty$ with respect to the z_is, where r is given by (3).

It should be noted that it is not possible that a pole z_i thereby comes to lie in the interval $[-1, 1]$: The corresponding r could never be a best approximation to the continuous f. In particular, no z_i can be zero.

The *existence* of an optimum is easily seen. For that purpose, write r as

$$r(x) := \sum_{k=0}^{N} \frac{w_k \prod_{i=1}^{P} (1 - (x_k/z_i))}{x - x_k} f_k \Bigg/ \sum_{k=0}^{N} \frac{w_k \prod_{i=1}^{P} (1 - (x_k/z_i))}{x - x_k} \tag{5}$$

and consider every z_i on its Riemann sphere $\overline{\mathbf{C}}$. For the polynomial, every pole is at infinity, i.e., at the north pole of its sphere. And a set of z_is must exist for which $\|r - f\|_\infty$ is minimal, since the latter is a continuous function over the cross product of the P spheres, which is compact.

The *unicity* question is more involved. We refer to [**16**] for a discussion. This question can also be narrowed to an interesting one, to which we do not have the answer. It is obvious from the above construction that nonvanishing of the numerator at z_i is a sufficient

condition for r to have a pole there. We have checked this condition in its equivalent form [**10**]

$$c_i := \sum_{k=0}^{N} w_k f_k \sum_{\substack{j=1 \\ j\neq i}}^{P} (x_k - z_j) \neq 0. \tag{6}$$

Do the conditions (6)–one for every z_i–imply that the corresponding optimal r is the only one minimizing $\|r - f\|_\infty$?

We want to point to another representation of (6). Indeed, $\sum_{k=0}^{N} w_k g(x_k)$ is the leading coefficient of the polynomial interpolating a function g between the x_k's, and this coefficient is the divided difference of g with respect to all x_k's [**3**]. Condition (6) can therefore be written as

$$g_i[x_0, x_1, \ldots, x_N] \neq 0, \quad \text{where } g_i(x) := f(x) \prod_{\substack{j=1 \\ j\neq i}}^{P} (x - z_j). \tag{7}$$

A nice property of the suggested interpolation deserves special notice. The approximation error cannot increase with the number of poles, this in sharp contrast with classical rational interpolation. Indeed, as a new unknown, say z_P, is added to the set of variables, $\{z_1, \ldots, z_{P-1}\}$, the optimal value of the latter is a feasible vector for the higher dimensional optimization–simply set $z_P = \infty$ in (5). In particular, attaching poles to the interpolating polynomial can never worsen the quality of the approximation.

3. The collocation problem.

3.1 *The linear rational collocation method for boundary value problems.* This generalization, suggested in [**13**] and [**14**], of the now classical polynomial pseudospectral method is the application to BVPs of the corresponding method for time evolution problems [**6**]. In view of the presence in the problem of the boundary values we will restrict ourselves here to sets of nodes containing the extremities -1 and 1, i.e., Lobatto points (and in particular to Čebyšev points of the second kind in numerical computations).

For fixed $\beta := [\beta_0, \dots, \beta_N]^T$ the set of all interpolants (4) is a linear space, which we denote by $\mathcal{R}_N^{(\beta)}$. The functions

$$L_j^{(\beta)}(x) := \frac{\beta_j}{x - x_j} \Bigg/ \sum_{k=0}^{N} \frac{\beta_k}{x - x_k}, \quad j = 0, 1, \dots, N$$

make up a basis for this space, and they satisfy the Lagrange property

$$L_j^{(\beta)}(x_i) = \delta_{ij}. \tag{8}$$

The *linear rational collocation method* (in barycentric form) for the nodes x_j tries to find u as an interpolant

$$\tilde{u}(x) = \sum_{j=0}^{N} \tilde{u}_j L_j^{(\beta)}(x) \in \mathcal{R}_N^{(\beta)} \tag{9}$$

for some *given* weights β and some unknown values $\tilde{u}_j$ at the x_j's, inserts $\tilde{u}$ into the differential equation and collocates at the same interior x_j's, for simplicity (collocation points different from the interpolation points, as in [**23**], are equally possible). This yields the following linear system of equations for the $\tilde{u}_j$:

(10)
$$\sum_{j=0}^{N} \tilde{u}_j L_j^{(\beta)''}(x_i) + p(x_i) \sum_{j=0}^{N} \tilde{u}_j L_j^{(\beta)'}(x_i) + q(x_i) \sum_{j=0}^{N} \tilde{u}_j L_j^{(\beta)}(x_i) = f(x_i),$$
$$i = 1, \dots, N-1, \quad \tilde{u}_0 = u_r, \quad \tilde{u}_N = u_l.$$

In order to write this in a more concise way, we introduce the following vectors and matrices in $\mathbf{R}^{N-1}$, respectively $\mathbf{R}^{(N-1)\times(N-1)}$:

$$\begin{aligned}
\tilde{\mathbf{u}} &:= [\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_{N-1}]^T, \\
\mathbf{D}^{(1)} &= (D_{ij}^{(1)}), \quad D_{ij}^{(1)} := L_j^{(\beta)'}(x_i), \\
\mathbf{D}^{(2)} &= (D_{ij}^{(2)}), \quad D_{ij}^{(2)} := L_j^{(\beta)''}(x_i), \\
\mathbf{P} &:= \operatorname{diag}(p(x_i)), \quad \mathbf{Q} := \operatorname{diag}(q(x_i)), \\
\mathbf{f} &:= [f(x_i) - u_r(L_0^{(\beta)''}(x_i) + p(x_i) L_0^{(\beta)'}(x_i)) \\
&\qquad - u_l(L_N^{(\beta)''}(x_i) + p(x_i) L_N^{(\beta)'}(x_i))]^T, \\
&\qquad\qquad i, j = 1, \dots, N-1.
\end{aligned}$$

In view of (8), the system (10) for the unknown values $\tilde{\mathbf{u}}$ of the approximant then reads $\mathbf{A}\tilde{\mathbf{u}} = \mathbf{f}$, with

$$\mathbf{A} := \mathbf{D}^{(2)} + \mathbf{P}\mathbf{D}^{(1)} + \mathbf{Q}.$$

Despite its large condition number for N large, it can be solved very precisely by Gaussian elimination [**11, 27**] for only the differentiation operator $\mathbf{A}$ is ill-conditioned, not the integration operator $\mathbf{A}^{-1}$. The system can often also be solved efficiently via iterative methods [**11**], although the ill-conditioned $\mathbf{A}$ then slows down the convergence. (A conformal shift of the points can improve on this, see [**13**].) In our calculations we have alleviated the instability by using the modified Schneider-Werner formulae [**5, 4**]

$$D_{ij}^{(1)} = \begin{cases} [(\beta_j/\beta_i)/x_i - x_j] & i \neq j, \\ -\sum_{k\neq i} D_{ik}^{(1)} & i = j, \end{cases}$$

and

$$D_{ij}^{(2)} = \begin{cases} 2D_{ij}^{(1)}[D_{ii}^{(1)} - 1/(x_i - x_j)] & i \neq j, \\ -\sum_{k\neq i} D_{ik}^{(2)} & i = j, \end{cases}$$

for the differentiation matrices.

In the polynomial case ($\beta_j = w_j$, all j) and with the interpolation points used here, the convergence of $\tilde{u}$ toward the exact solution u is *exponential* if p, q and f are analytic in an ellipse containing $[-1, 1]$. This can be seen through subtraction (and use of the exponential convergence of the interpolant of f) if p and q are constant, by more elaborate theorems [**20**] in general cases. However, this fast convergence may show only after too large an N for practical purposes if u has huge gradients, see the introduction in [**16**]. For error bounds through estimates of the norm of the inverse operator, see the work by Wright and collaborators, [**1, 22**].

We will now make use of the pole attachment in Section 2.1 for improving upon the polynomial pseudospectral solution of two-point boundary value problems. As in Section 2.2 we suggest to move them from infinity toward an optimal position where they minimize some error functional, which we take here as the norm

$$J(\mathbf{z}) := \|r'' + pr' + qr - f\|_\infty, \quad \mathbf{z} := [z_1, \ldots, z_P]^T, \tag{11}$$

of the *residual* of the differential equation for the approximation r with given values $\tilde{\mathbf{u}}$ of the solution u at the x_js.

Note that the interpolated values $\tilde{u}_j$ at the nodes do not change as one displaces the poles: Interpolation is warranted by the barycentric formula [**12, 15, 28**]. And *the optimization can only decrease the value of* J, since the interpolating polynomial belongs to the feasible set.

Optimizing the poles z_i is a nonlinear problem to be solved by iteration. There is always an optimal $\mathbf{z}$ but, at least in special cases, there can be several of them. Nevertheless, in every undetermined case among our many tests, the optimal set was a continuum and the multiplicity could easily be detected from the divergence of the optimization procedure.

3.2 *The linear rational collocation method with iteratively optimized poles.* The algorithm we suggest here for solving the boundary value problem improves iteratively on the polynomial pseudospectral method. It consists in recursively performing the linear collocation method and the optimal placement of poles.

Let the $N+1$ interpolation points $x_0, \ldots, x_N$ be given, as well as the number P of poles to be optimized, which are first supposed at infinity (if no information on their final location is known at the onset). For $k = 1, 2, \ldots$, repeat

Step 1) compute the approximate solution $\tilde{\mathbf{u}}^{(k)} = [\tilde{u}_1^{(k)}, \ldots, \tilde{u}_{N-1}^{(k)}]^T$ by the linear rational collocation method with $\beta_j = w_j d_j$, d_j from (2), $d_j \equiv 1$ for $k = 1$. This modifies $\tilde{\mathbf{u}}$, for $k > 1$, but not the poles $\mathbf{z}$ and the weights β.

Step 2) for the $\tilde{\mathbf{u}}^{(k)}$ inherited from Step 1), optimize the location of the poles $\mathbf{z}$ by minimizing $J(\mathbf{z})$. This changes β, but not $\tilde{\mathbf{u}}^{(k)}$ and yields a new interpolant $\hat{u}^{(k)}(x)$ of the latter values.

When to stop? Roughly speaking, when the decrease in J becomes too small in comparison with the cost of one step of the algorithm.

The recurrence of Step 2) makes the algorithm costly. Note, however, that at the outcome, when β and $\tilde{\mathbf{u}}$ have been computed, evaluating $\tilde{u}$ by the formula (9) is exactly as expensive as evaluating the polynomial solution. The algorithm presented here therefore aims at such cases in which, e.g., the time for finding the solution is not very relevant, but

the latter must be evaluated a great many times.

4. Numerical results.

4.1 *An interpolation example.* In this section, we consider essentially the same example to demonstrate the effectiveness of the proposed methods in both cases. As interpolation/collocation points we have chosen the Čebyšev points of the second kind $x_j := \cos(j\pi/N)$, $j = 0, \ldots, N$. The maximum norm of the interpolation error, respectively the residual in (11), has been estimated by evaluating at the equally spaced points

$$\hat{x}_l = -\frac{5}{4} + \frac{l-1}{L-1} \cdot \frac{10}{4}, \quad l = 1(1)L$$

with $L = 1000$ for interpolation and $L = 100$ for collocation and then by computing the maximal absolute value at those $\hat{x}_l$ lying in $[-1, 1]$. Due to the nondifferentiable and global nature of the optimization problems, the simulated annealing algorithm of [**21**] was chosen. The computations were performed in Fortran 77 on HP workstations.

We consider a case with a large derivative in the interior of the interval of interpolation, as motivated by the introduction. With erf denoting the standard error function and for given positive ε, the function to approximate is chosen as [**24**]

$$f(x) = \cos \pi x + \frac{\operatorname{erf}(\delta x)}{\operatorname{erf}(\delta)}, \quad \delta = \sqrt{.5\varepsilon}. \tag{12}$$

This function has values -2 at $x = -1$ and 0 at $x = 1$ and has a steep gradient near $x = 0$ for large ε. Figure 1 shows the graph of f for $\varepsilon = 10,000$.

For not too large values of ε, the cases of moderate numbers P of poles could be relatively easily solved, while the problem is getting harder with increasing ε due to the steep gradient near $x = 0$. For example, with $\varepsilon = 100$ everything works perfectly: with two pairs of poles, the error decreases exponentially with N, from $4.0 \cdot 10^{-3}$ for $N = 7$ to $4.1 \cdot 10^{-14}$ for $N = 63$, whereas the polynomial error decreases merely

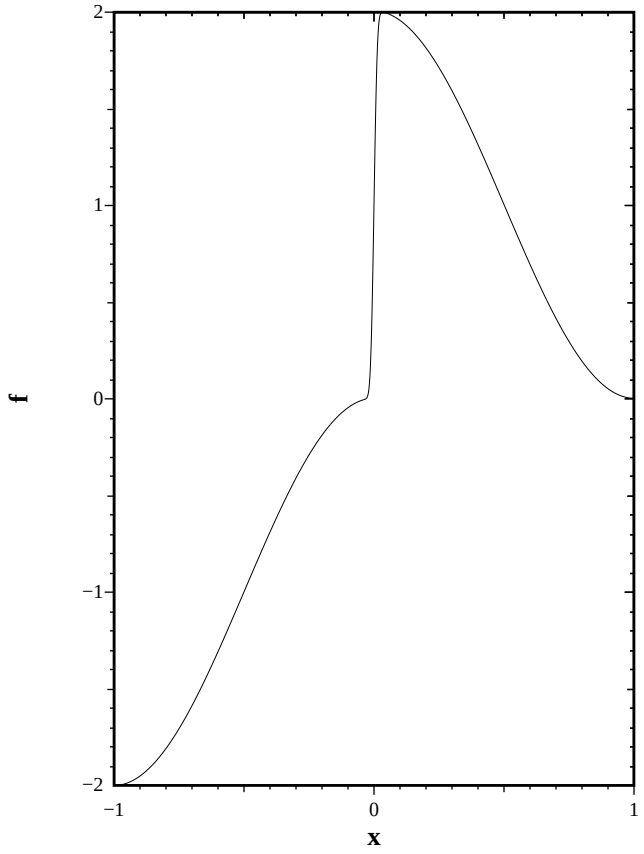

FIGURE 1.

from $1.3 \cdot 10^{-1}$ to $2.1 \cdot 10^{-6}$. For $\varepsilon = 10,000$ and Čebyšev points of the second kind, in two of the cases the algorithm used failed to produce the desired results. In all other cases, however, the numbers in Table 1 show again that the attachment of a small number of poles leads to a significant improvement of the approximation properties of the interpolant.

Note the decreasing values of $\mathrm{abs}\,(c_i)$, the test for the presence of the poles, as N grows. This stems from the fact, noted above, that this quantity is a divided difference of order N: if the derivatives do not increase as fast as the corresponding factorials, divided differences become smaller as their order increases.

For small N, classical rational interpolation, Table 2, has a hard time with the latter example. Poles occur in the interpolation interval at least until $N = 15$, where our r has already decreased the error to $5.5 \cdot 10^{-3}$, a value the classical interpolant does not even reach with 64 points for as small a denominator degree.

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Example 4. We present results for a problem whose solution is given by (12):

$$u''(x) + \varepsilon x u'(x) = -\pi^2 \cos(\pi x) - \varepsilon \pi x \sin(\pi x),$$
$$u(-1) = -2, \quad u(1) = 0.$$

We have solved the problem for ε up to 10,000. Some of our results with the larger ε are summarized in Table 3. Since we have seen in Section 4.1 that for too small an N the optimization procedure may fail to converge, we give numbers only for $N \geq 128$. They share some common features. For instance, for given ε and N, the imaginary parts of the optimal poles are quite close to one another. Moreover, if four poles are optimized, they have the tendency to gather as the vertices of a rectangle about the origin, where the maximum gradient arises.

As for the errors, the optimization improves the residuum by four to six digits, much more than it does with the maximum error, less than three digits–see the comment on the condition of the problem in the conclusion. Nevertheless, with $\varepsilon = 5'000$ and $N = 64$ or $\varepsilon = 10'000$ and $N = 128$ or $N = 256$, attaching poles decreases the maximum error, more than doubling N!

TABLE 1. Interpolation example, Čebyšev points of the second kind

N	P	max error	Re(z_i)	Im(z_i)	abs(c_i)
7	0	.860929			
	2	.585487	.498187E-01	.855217E-01	.9510E-01
	4	.250594	.963782	.195789	.1388
			.173890E-02	-.519644E-01	.3671E-01
	6	.136934	.381364	-5.03092	.4074
			-1.490736	.527043	.1002
			.218230E-02	-.363523E-01	.1622E-01
15	0	.731061			
	2	.152567	.445251E-07	.386686E-01	.1594E-01
	4	.129811E-01	.691377E-05	-.226144E-01	.1851E-03
			.714004E-04	-.148341	.2528E-02
	6	.550262E-02	.178129E-01	.334510	.1433E-03
			.199111E-02	-.105384	.1369E-03
			.180070E-04	.209114E-01	.2179E-04
31	0	.527525			
	2	.347874E-01	-.303433E-12	-.251649E-01	.2897E-03
	4	.609649E-02	-.899892E-12	.994387E-01	.1162E-03
			-.279530E-12	.207341E-01	.2197E-04
	6				
63	0	.269966			
	2	.612221E-02	.378870E-09	-.208431E-01	.3708E-09
	4				
	6	.808776E-03	.628162E-02	-.190003E-01	.3167E-10
			-.118652E-08	.694756E-01	.1117E-09
			-.628161E-02	.190003E-01	.3167E-10
127	0	.102178			
	2	.2822739E-02	-.263671E-03	.217792E-01	.2285E-15
	4	.584158E-03	-.674335E-02	-.204741E-01	.1388E-14
			.674335E-02	.204741E-01	.1110E-14
	6	.143965E-04	.156860E-01	-2.49369E-01	.1221E-14
			-.156860E-01	.249370E-01	.7772E-15
			-.106757E-07	.263960E-01	.2224E-15

TABLE 2. Interpolation example, $\varepsilon = 10^4$, errors for classical rational interpolation

N	P	max error
7	2	pole
	4	1.398
	6	pole
15	2	.6274
	4	pole
	6	pole
	8	pole
31	2	.3719
	4	.3154
	6	.2833
	8	.2659
63	2	.1388
	4	.1991E-01
	6	.5993
	8	.9756E-01

4.2 *The collocation example.*

Our results are not quite as good as those obtained in [**2**] with the same example. We recall, however, that they are not comparable, for our method yields C^∞-approximations of the C^∞-solutions of the problems considered here.

Finally, we give in Figures 2 and 3 error curves for a fixed ε-N-pair and an increasing number of attached poles: even with the large gradient the error behaves nicely as P increases.

5. Conclusion and future work. In the present work we have discussed rational interpolants with guaranteed interpolation, no poles in the interval of interpolation, and an error which usually decreases, and never increases, with the degree of the denominator. The error is consistently smaller than that of classical rational interpolation with the same denominator degree.

Simple comparison is not fair, however: our r has total degree

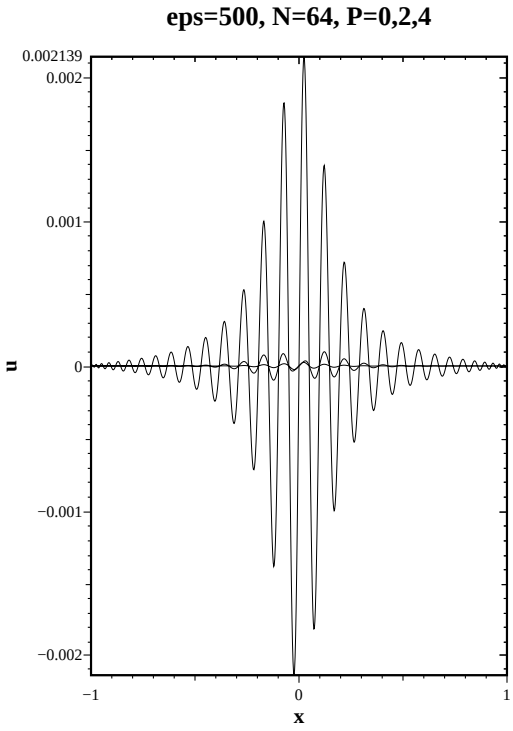

FIGURE 2.

$N + P$, as opposed to N for the classical rational interpolant. More importantly, it requires knowledge of the interpolated function in the entire interpolation interval, so that it is more an "interpolative approximant" than an interpolant in the classical sense, but with interesting applications.

One of these is the solution of two-point boundary value problems. Our approach consists of an iterative improvement of the polynomial pseudospectral method, which is known to converge exponentially for good interpolation points and highly differentiable problems. After having obtained the solution at some (collocation) points by the polynomial method, we compute (one of the) rational interpolant(s) of these same values with a denominator of given degree by minimizing the residuum of the differential equation. This defines the new linear space of all rationals interpolating between these same points and having that same denominator. We then have only to start again with the solution

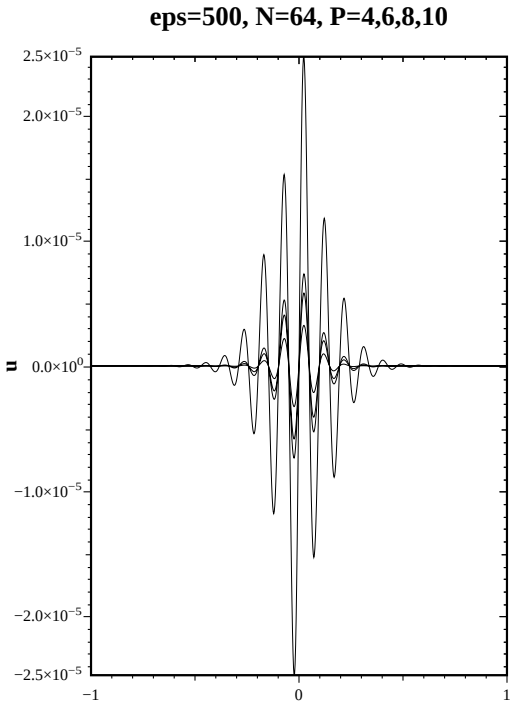

FIGURE 3.

of the original equation in the new space, and so on. Although we can prove the effectiveness of a Galerkin version of the method [**17**], in practice we solve the problem with the much simpler collocation method.

The computed examples show the somewhat surprising result that one can usually gain between three and five digits of accuracy in comparison with classical polynomial collocation and this almost independently of N. This is especially significant in cases where the precision obtained with the latter method is low and one does not want to increase the number of points to keep consequent evaluation of the solution as cheap as possible.

The placement of the poles is a very well-conditioned problem in the sense that many of their locations around the optimal one yield merely slightly larger residua. The tables show that the gain in the residual error is usually much larger than the improvement in the precision

TABLE 3. Collocation example.

ε	N	P	res. norm	max. error	$\mathrm{Re}\,(z_i)$	$\mathrm{Im}\,(z_i)$
5e3	128	0	7.592e+03	4.804e-02		
		2	5.890e+01	3.106e-03	-.209478e-11	.354525e-01
		4	3.347e+00	1.575e-03	.119518e-01	.332938e-01
					-.119518e-01	.332938e-01
		6	8.687e-01	4.942e-04	.307586e-04	.405216e-01
					-.206948e-01	.382752e-01
					.207135e-01	.383201e-01
	256	0	1.079e+02	1.212e-04		
		2	5.306e-02	7.308e-06	.137751e-09	.548452e-01
		4	2.779e-03	1.074e-06	-.135939e-01	.56742e-01
					.135939e-01	.567424e-01
		6	1.397e-04	1.285e-07	.234374e-01	.576278e-01
					-.882100e-05	.578770e-01
					-.234495e-01	.576437e-01
	512	0	7.823e-07	1.019e-13		
1e4	128	0	3.443e+04	.1591		
		2	1.695e+02	8.342e-03	.303713e-08	.207640e-01
		4	7.610e+01	1.579e-02	.652847e-02	.201166e-01
					-.652852e-02	.201166e-01
		6	1.057e+00	3.371e-03	.172391e-01	.250201e-01
					-.488556e-06	.251768e-01
					-.172387e-01	.250215e-01
	256	0	5.677e+03	5.680e-03		
		2	2.985e+00	4.197e-04	.144070e-06	.294745e-01
		4	1.197e-01	1.146e-04	.934422e-02	.296229e-01
					-.934516e-02	.296229e-01
	512	0	5.254e-01	8.860e-08		

of $\tilde{u}$. This is probably due in part to the fact that the computation of the residuum is smeared by the ill-conditioning of the differentiation matrices, despite the improvement by the methods in [**5**].

A further improvement in both methods presented above will be achieved through a conformal mapping of the nodes towards an equidistant distribution, see [**7**]. In particular, the derivatives of the functions will be better approximated. We plan on investigating this in the near future.

REFERENCES

1. A.H. Ahmed and K. Wright, *Error estimation for collocation solution of linear ordinary differential equations*, Comp. Math. Appl. **12** (1986), 1053–1059.

2. U. Ascher, J. Christiansen and R.D. Russell, *A collocation solver for mixed order systems of boundary value problems*, Math. Comp. **33** (1979), 659–679.

3. K.E. Atkinson, *An introduction to numerical analysis*, 2nd ed., John Wiley, New York, 1978.

4. R. Baltensperger, *Improving the accuracy of the matrix differentiation method for arbitrary collocation points*, Appl. Numer. Anal. **33** (2000), 143–149.

5. R. Baltensperger and J.-P. Berrut, *The errors in calculating the pseudospectral differentiation matrices for Čebyšev-Gauss-Lobatto points*, Comput. Math. Appl. **37** (1999), 41–48.

6. ———, *The linear rational collocation method*, J. Comput. Appl. Math., J. Comp. Appl. Math. 134 (2001), 243–258.

7. R. Baltensperger, J.-P. Berrut and B. Noël, *Exponential convergence of a linear rational interpolant between transformed Čebyšev points*, Math. Comp. **68** (1999), 1109–1120.

8. R.T. Baumel, J.L. Gammel and J. Nuttall, *Placements of cuts in Padé-like approximation*, J. Comp. Appl. Math. **7** (1981), 135–140.

9. R.T. Baumel, J.L. Gammel, J. Nuttall and D.C. Power, *The calculation of critical indices by the rational approximation method*, J. Phys. **15** (1982), 3233–3240.

10. J.-P. Berrut, *The barycentric weights of rational interpolation with prescribed poles*, J. Comp. Appl. Math. **86** (1997), 45–52.

11. ———, *A pseudospectral Čebyšev method with preliminary transform to the circle*: *Ordinary differential equations*, Report No. 252, Math. Institut, Technische Universität München, 1990; revised Université de Fribourg, 1995.

12. ———, *Linear rational interpolation of continuous functions over an interval*, in *Mathematics of computation* 1943–1993: *A half-century of computational mathematics*, W. Gautschi, ed., American Math. Soc., Providence, (1994) 261-264.

13. J.-P. Berrut and R. Baltensperger, *The linear rational collocation method for boundary value problems*, BIT 41 (2001), 868–879.

14. R. Baltensperger, J.-P. Berrut and Y. Dubey, *The linear rational collocation method with preassigned poles*,submitted.

15. J.-P. Berrut and H.D. Mittelmann, *Lebesgue constant minimizing linear rational interpolation of continuous functions over the interval*, Comp. Math. Appl. **33** (1997), 77–86.

16. ———, *Rational interpolation through the optimal attachment of poles to the interpolating polynomial*, Numer. Algorithms **23** (2000), 315–328.

17. ———, *The linear rational collation method with iteratively optimized poles for two-point boundary value problems*, SIAM J. Sci. Comp. 23 (2001), 961–975.

18. ———, *Matrices for the direct determination of the barycentric weights of rational interpolation*, J. Comp. Appl. Math. **78** (1997), 355–370.

19. R. Bulirsch and H. Rutishauser, *Interpolation und genäherte Quadratur*, R. Sauer, I. Szabó, Hsg., Mathematische Hilfsmittel des Ingenieurs, Grundlehren der math. Wissenschaften Bd. 141, Springer, Berlin, 1968.

20. C. Canuto and A. Quarteroni, *Variational methods in the theoretical analysis of spectral approximations*, in *Spectral methods for partial differential equations*, R.J. Voigt, D. Gottlieb and M.Y. Hussaini, eds., SIAM, Philadelphia, 1984.

21. A. Corana, M. Marchesi, C. Martini and S. Ridella, *Minimizing multimodal functions of continuous variables with the "simulated annealing" algorithm*, ACM Trans. Math. Software **13** (1987), 262–280.

22. D.M. Cruickshank and K. Wright, *Computable error bounds for polynomial collocation methods*, SIAM J. Numer. Anal. **15** (1978), 134–151.

23. D. Funaro, *Spectral elements for transport-dominated equations*, Springer, Berlin, 1997.

24. P. Hemker, *A numerical study of stiff two-point boundary problems*, Math. Centrum, Amsterdam, 1977.

25. P. Henrici, *Essentials of numerical analysis*, John Wiley, New York, 1982.

26. H.E. Salzer, *Lagrangian interpolation at the Chebyshev points* $x_{n,\nu} = \cos(\nu\pi/n)$, $\nu = 0(1)n$; *some unnoted advantages*, The Computer J. **15** (1972), 156–159.

27. T. Tang and M.R. Trummer, *Boundary layer resolving pseudospectral methods for singular perturbation problems*, SIAM J. Sci. Comput. **17** (1996), 430–438.

28. W. Werner, *Polynomial interpolation*: *Lagrange versus Newton*, Math. Comp. **43** (1984), 205–217.

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