ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 2, Summer 2002

PICK FUNCTIONS RELATED TO THE GAMMA FUNCTION

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ABSTRACT. We show that the function

$$f(z) = \frac{\log \Gamma(z+1)}{z \mathrm{Log}\, z},$$

holomorphic in the complex plane cut along the negative real axis, is a Pick function and we find its integral representation. We also show that various other related functions are Pick functions.

1. Introduction. The function

$$f(x) = \frac{\log \Gamma(x+1)}{x \log x}, \quad x > 0,$$

has attracted the attention of several authors, see [**3, 2**] and [**7**]. In [**4**], we proved that the reciprocal function $1/f$ has a holomorphic extension to the cut plane

$$\mathcal{A} = \mathbf{C} \setminus \left]-\infty, 0\right],$$

and that this extension is a Stieltjes transform. We also found its Stieltjes representation. As a corollary, we obtained that the restriction of f' to the positive real axis is completely monotone, thereby answering a question raised by Dimitar Dimitrov at the Fifth International Symposium on Orthogonal Polynomials, Special Functions and Applications held in Patras in September 1999. This result was thus obtained by considering the reciprocal function, and in the course of the proof we had to establish that the only zeros of the function $\log \Gamma$, defined below, in $\mathcal{A}$ are those at $z = 1$ and $z = 2$. The reciprocal of a Stieltjes function is a Pick function, so the result of [**4**] implies that f is a Pick function.

2000 AMS *Mathematics subject classification.* Primary 33B15, Secondary 30E20, 30E15.
Keywords and phrases. Gamma function, completely monotone function.
Received by the editors on August 17, 2000.

We feel that it is worthwhile to show directly that f is a Pick function, and this is the main goal of the paper. Our result easily implies that f' is completely monotone. It also implies that $\log\Gamma(z)$ is zero free in $\mathbf{C}\setminus\mathbf{R}$. This fact can be obtained in a more elementary way by showing that the function

$$z \longrightarrow \frac{\log\Gamma(z+1)}{z}$$

is a Pick function. This we verify in the last section.

The function Log denotes the principal logarithm, holomorphic in the cut plane $\mathcal{A}$ and defined in terms of the principal argument Arg. The function $\log\Gamma = \log|\Gamma| + i\arg\Gamma$ denotes the holomorphic branch in $\mathcal{A}$ that is real on the positive real axis. Such a branch exists, since Γ is holomorphic in the simply connected domain $\mathcal{A}$ and has no zeros there.

We recall that a *Pick function* is a holomorphic function φ in the upper half-plane $\mathbf{H} = \{z \in \mathbf{C} \mid \Im z > 0\}$ with $\Im\varphi(z) \geq 0$ for $z \in \mathbf{H}$. Pick functions are extended by reflection to holomorphic functions in $\mathbf{C}\setminus\mathbf{R}$, and they have the following integral representation

$$\varphi(z) = az + b + \int_{-\infty}^{\infty}\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)d\mu(t), \tag{1}$$

where $a \geq 0$, $b \in \mathbf{R}$ and μ is a nonnegative Borel measure on $\mathbf{R}$ satisfying

$$\int_{-\infty}^{\infty}\frac{d\mu(t)}{t^2+1} < \infty,$$

see, e.g. [**6**]. It is known that

$$\begin{aligned} a &= \lim_{y\to\infty}\varphi(iy)/(iy), \quad b = \Re\varphi(i), \\ \mu &= \lim_{y\to 0+}\frac{\Im\varphi(t+iy)\,dt}{\pi}, \end{aligned} \tag{2}$$

where the last limit is in the *vague* topology, and finally that φ has a holomorphic extension to $\mathcal{A}$ if and only if $\operatorname{supp}(\mu) \subseteq\,]-\infty, 0]$.

Our main results are the following:

Theorem 1.1. *The function*

$$f(z) = \frac{\log\Gamma(z+1)}{z\operatorname{Log} z}, \quad z \in \mathcal{A}$$

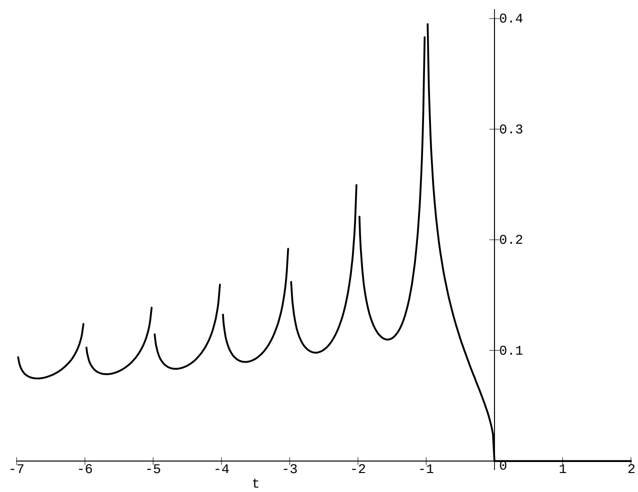

FIGURE 1. The graph of d.

is a Pick function of the form (1) *with* $a = 0$, $b = (1/\pi)\log((\sinh\pi)/\pi)$ *and* $\mu = d(t)\,dt$ *where* $d : \mathbf{R} \to [0,\infty]$ *is defined as* $d(t) = 0$ *for* $t \geq 0$, $d(t) = \infty$, *for* $t \in \{-1,-2,\dots\}$ *and for* $k \geq 1$,

$$d(t) = -\frac{\log|\Gamma(t+1)| + (k-1)\log|t|}{t((\log|t|)^2+\pi^2)} \quad \textit{for} \quad t \in \,]-k,-k+1[\,. \tag{3}$$

Theorem 1.2. *We have*

$$\frac{\log\Gamma(z+1)}{z\mathrm{Log}\,z} = 1 - \int_0^\infty \frac{d(-t)}{z+t}\,dt,$$

where d *is defined in* (3).

In Figure 1 we have drawn the graph of the function d. There are vertical asymptotes at $t = -1, -2, \dots$, and there is a vertical tangent (from the left) at $t = 0$. The minimum of d on the interval $]-k,-k+1[$ behaves asymptotically as $1/(\log k)^2$ as k tends to infinity, see Remark 3.6.

2. Properties of the gamma function. We give some preliminary relations involving the Γ-function. We have the relation

$$\log \Gamma(z+1) = \log \Gamma(z+k+1) - \sum_{l=1}^{k} \mathrm{Log}\,(z+l) \tag{4}$$

for $z \in \mathcal{A}$ and for any $k \geq 1$. Indeed, both sides of this relation are holomorphic in $\mathcal{A}$ and they agree on the positive real axis because of the functional equation of Γ.

Taking real and imaginary parts on both sides of (4) we obtain

$$\log |\Gamma(z+1)| = \log |\Gamma(z+k+1)| - \sum_{l=1}^{k} \log |z+l| \tag{5}$$

and

$$\arg \Gamma(z+1) = \arg \Gamma(z+k+1) - \sum_{l=1}^{k} \mathrm{Arg}\,(z+l). \tag{6}$$

We put, for $k \in \mathbf{Z}$,

$$R_k = \{z = x + iy \in \mathbf{C} : -k \leq x < -k+1, 0 < y \leq 1\}$$

and $R = \cup_{k=0}^{\infty} R_k$. We note that $\overline{R_0}$ is a compact subset of the domain of holomorphy of $\log \Gamma(z+1)$. Therefore, a positive constant c exists such that

$$-c \leq \log |\Gamma(z+1)| \leq c$$

and

$$-c \leq \arg \Gamma(z+1) \leq c$$

for all $z \in \overline{R_0}$.

Lemma 2.1. *We let c denote the constant mentioned above and let $z \in R_k$. For $k = 1$ we have*

$$-c - \log 2 \leq \log |\Gamma(z+1)| \leq c - \log |z+1|,$$

while for $k \geq 2$ we have

$$-c - k \log k \leq \log |\Gamma(z+1)| \leq c - \log |z+k| - \log |z+k-1|.$$

For $k \geq 1$ we have

$$-c - k\pi \leq \arg \Gamma(z+1) \leq c - (k-1)\pi/2.$$

We stress that the constant c does *not* depend on k.

Proof. From the relation (5) and the observation above,

$$-c - \sum_{l=1}^{k} \log |z+l| \leq \log |\Gamma(z+1)| \leq c - \sum_{l=1}^{k} \log |z+l|.$$

This readily gives the first assertion in the lemma. For $k \geq 2$, we note that

$$c - \sum_{l=1}^{k} \log |z+l| \leq c - \log |z+k-1| - \log |z+k|,$$

since, clearly, $|z+l| \geq 1$ for $l = 1, \dots, k-2$. Furthermore, $|z+l| \leq 1 + (k-l)$ for $l = 1, \dots, k-1$ and $|z+k| \leq 2$. Therefore,

$$\begin{aligned} -\sum_{l=1}^{k} \log |z+l| &\geq -\sum_{l=1}^{k-1} \log(k-l+1) - \log 2 \\ &= -\sum_{j=1}^{k-1} \log(j+1) - \log 2 \\ &\geq -k \log k. \end{aligned}$$

This proves the second assertion about $\log |\Gamma(z+1)|$. The assertion about $\arg \Gamma(z+1)$ is obtained in the same way, using (6) and the estimates

$$(k-1)\pi/2 \leq \sum_{l=1}^{k} \operatorname{Arg}(z+l) \leq k\pi$$

for $z \in R_k$. $\square$

We also need a formula going back to Stieltjes, see [**4**, Proof of Proposition 2.4]:

$$\log \Gamma(z+1) = \log \sqrt{2\pi} + (z+1/2)\mathrm{Log}\, z - z + \mathcal{J}(z). \tag{7}$$

Here

$$\mathcal{J}(z) = \frac{1}{2}\int_0^\infty \frac{Q(t)}{(z+t)^2}\, dt,$$

where Q is periodic with period 1 and $Q(t) = t - t^2$ for $t \in [0,1[$. Furthermore,

$$|\mathcal{J}(z)| \leq \frac{\pi}{8} \tag{8}$$

for $z \in \mathbf{H} \setminus R$.

3. Estimates of a harmonic function. Throughout the paper f denotes the function in Theorem 1.1. We shall study the function $V : \mathcal{A} \to \mathbf{R}$ defined by $V = \Im f$. It is harmonic in $\mathcal{A}$ and we find that with as usual, $z = x + iy$,

$$\begin{aligned} V(z) = & \frac{(x \log|z| - y \mathrm{Arg}\, z) \arg \Gamma(z+1)}{|z \mathrm{Log}\, z|^2} \\ & - \frac{(y \log|z| + x \mathrm{Arg}\, z) \log|\Gamma(z+1)|}{|z \mathrm{Log}\, z|^2}. \end{aligned} \tag{9}$$

We shall prove that V is a positive function in $\mathbf{H}$, and to do that we first investigate the boundary behavior of V on the real line. We also estimate its behavior near infinity.

To determine the boundary behavior of V, we recall [**4**, Lemma 2.1]:

Lemma 3.1. *We have, for any $k \geq 1$,*

$$\lim_{\substack{z \to t \\ \Im z > 0}} \log \Gamma(z) = \log|\Gamma(t)| - i\pi k$$

for $t \in \,]-k, -k+1[$ *and*

$$\lim_{\substack{z \to t \\ \Im z > 0}} |\log \Gamma(z)| = \infty$$

for $t = 0, -1, -2, \dots$.

We recall that $d : \mathbf{R} \to \mathbf{R} \cup \{\infty\}$ was defined as $d(t) = 0$ for $t \geq 0$, $d(t) = \infty$, for $t \in \{-1, -2, \dots\}$ and for $k \geq 1$,

$$d(t) = -\frac{\log|\Gamma(t+1)| + (k-1)\log|t|}{t((\log|t|)^2 + \pi^2)} \quad \text{for} \quad t \in \,]-k, -k+1[\,.$$

We notice that d is a nonnegative function. For $k = 1$, it follows from the fact that $t + 1 \in \,]0, 1[$ so that $\Gamma(t+1) > 1$. For $k \geq 2$, we use

$$\begin{aligned}\log|\Gamma(t+1)| + (k-1)\log|t| &= \log|\Gamma(t+k)| + (k-1)\log|t| \\ &\quad - \sum_{l=1}^{k-1} \log|t+l| \\ &= \log|\Gamma(t+k)| + \sum_{l=1}^{k-1} \log\frac{|t|}{|t+l|},\end{aligned}$$

where all terms are positive.

Lemma 3.2. *We have that* $V(z) \to \pi d(t)$ *for* $t \in \mathbf{R}$ *as* $z \to t$ *within* $\mathbf{H}$. *In particular,* V *has nonnegative boundary values.*

Proof. Since f is real on the positive axis, $V(z) \to 0$ as $z \to t > 0$. For $t \in \,]-k, -k+1[$ the limit is straightforward to compute, using Lemma 3.1. To find the limit when $t = -k$ for $k = 1, 2, \dots$, we use Lemma 2.1. The lemma tells us that $y \log|\Gamma(z+1)| \to 0$ as

$z = x + iy \to -k$, and that $\arg \Gamma(z+1)$ remains bounded. Hence,

$$\begin{aligned}\liminf_{z\to -k} V(z) &= \liminf_{z\to -k} \left\{ \frac{(x\log|z| - y\operatorname{Arg} z)\arg\Gamma(z+1)}{|z|^2|\operatorname{Log} z|^2} \right. \\ &\qquad \left. - \frac{(y\log|z| + x\operatorname{Arg} z)\log|\Gamma(z+1)|}{|z|^2|\operatorname{Log} z|^2} \right\} \\ &\geq \mathrm{Const} - \lim_{z\to -k} \frac{y\log|z|\log|\Gamma(z+1)|}{|z|^2|\operatorname{Log} z|^2} \\ &\quad + \lim_{z\to -k} \frac{-x\operatorname{Arg} z\log|\Gamma(z+1)|}{|z|^2|\operatorname{Log} z|^2} \\ &= \mathrm{Const} + 0 + \infty = \infty.\end{aligned}$$

To handle the behavior at the origin, we estimate V as

$$|V(z)| \leq \left|\frac{\log\Gamma(z+1)}{z}\right| \left|\frac{1}{\operatorname{Log} z}\right| \leq \mathrm{Const}\, \frac{1}{|\log|z||}, \tag{10}$$

where we have used the fact that $(\log\Gamma(z+1))/z$ has a removable singularity at $z = 0$. Therefore, $\lim_{z\to 0} V(z) = 0$. □

Proposition 3.3. *There is a constant $C > 0$ such that*

$$V(z) \geq -C$$

for all $z \in \mathbf{H}$ of large absolute value.

Proof. We have, from (7) and (8),

$$\frac{\log\Gamma(z+1)}{z\operatorname{Log} z} = 1 + \frac{\log\sqrt{2\pi} + (\operatorname{Log} z)/2 - z + \mathcal{J}(z)}{z\operatorname{Log} z} \longrightarrow 1, \tag{11}$$

and hence $V(z) \to 0$ as $|z| \to \infty$ within $\mathbf{H} \setminus R$. Thus $|V(z)| \leq 1$, say, for all $z \in \mathbf{H} \setminus R$ of sufficiently large absolute value.

We shall consider next the situation where $z \in R$. Suppose that $z \in R_k$ for some $k \geq 2$. Then, in particular, $-k \leq x < 0$, $\log|z| > 0$, $y > 0$ and $\operatorname{Arg} z > 0$. Therefore we have, by Lemma 2.1,

$$\begin{aligned}L_1(z) &\equiv (x\log|z| - y\operatorname{Arg} z)\arg\Gamma(z+1) \\ &\geq c(x\log|z| - y\operatorname{Arg} z) \\ &\geq -c(k\log|z| + \pi).\end{aligned}$$

From that lemma we also get

$$\begin{aligned}L_2(z) &\equiv -(y\log|z| + x\operatorname{Arg} z)\log|\Gamma(z+1)| \\ &= -y\log|z|\log|\Gamma(z+1)| \\ &\quad - x\operatorname{Arg} z\log|\Gamma(z+1)| \\ &\geq -cy\log|z| \\ &\quad + (\log|z+k| + \log|z+k-1|)y\log|z| \\ &\quad - x\operatorname{Arg} z(-c-k\log k).\end{aligned}$$

Furthermore, we have

$$(\log|z+k| + \log|z+k-1|)y \geq 2y\log y \geq -1.$$

Hence,

$$L_2(z) \geq -(c+1)\log|z| - k(c+k\log k)\pi.$$

Furthermore,

$$\log|z| \leq \log(k+1) \leq \log k + \log 2,$$

so we get

$$L_1(z) + L_2(z) \geq -\text{Const}\, k^2 \log k,$$

where the constant is independent of k. Using finally that

$$\log|z| \sim \log k, \qquad |z\operatorname{Log} z| \sim k\log k$$

for $z \in R_k$, $k \to \infty$, we obtain

$$V(z) \geq -\frac{\text{Const}}{\log|z|}, \quad z \in R_k,$$

when k is sufficiently large. □

After these preliminary results we are able to find a useful estimate of V in the upper half plane.

Theorem 3.4. *The harmonic function V is nonnegative in the upper half plane.*

Proof. We apply the theorem in [**8**, page 27], with $A = 0$, to the subharmonic function $u(z) = -V(z)$. $\square$

Remark 3.5. The relation (11) can be used to give a better estimate of V in $\mathbf{H} \setminus R$. If we multiply by $(\log |z|)^2$ on both sides of (11), we get

$$(\log |z|)^2 \frac{\log \Gamma(z+1)}{z \mathrm{Log}\, z} = (\log |z|)^2 - \frac{(\log |z|)^2}{\mathrm{Log}\, z} + \frac{\log |z|}{\mathrm{Log}\, z} \times \frac{(\log |z|)(\log \sqrt{2\pi} + (\mathrm{Log}\, z)/2 + \mathcal{J}(z))}{z}.$$

If we now take imaginary parts, we obtain

$$(\log |z|)^2 V(z) - \mathrm{Arg}\, z \longrightarrow 0$$

as $|z| \to \infty$ within $\mathbf{H} \setminus R$. This is because

$$\Im((\log |z|)^2/\mathrm{Log}\, z) = \frac{-\mathrm{Arg}\, z}{1 + (\mathrm{Arg}\, z)^2/(\log |z|)^2}$$

and

$$\frac{\log |z|}{\mathrm{Log}\, z} \frac{(\log |z|)(\log \sqrt{2\pi} + (\mathrm{Log}\, z)/2 + \mathcal{J}(z))}{z} \longrightarrow 0$$

as $|z| \to \infty$.

Remark 3.6. The asymptotic behavior of V in R is less regular. As we have seen in Lemma 3.2, V tends to infinity as z tends to a negative integer.

We shall find the asymptotic behavior of the minimum of d on the interval $]-k, -k+1[$ as k tends to infinity. On the interval $]-k, -k+1[$, we have

$$d(t) = \frac{A(t)}{B(t)},$$

where

$$A(t) = \log \Gamma(t+k) + \sum_{l=1}^{k-1} \log \frac{t}{t+l}$$

and

$$B(t) = -t((\log |t|)^2 + \pi^2).$$

We estimate the numerator as follows: since $\Gamma(t+k) \geq 1$ for $-k < t < -k+1$,

$$A(t) \geq \sum_{l=1}^{k-1} \log \frac{t}{t+l}.$$

Here, the righthand side is increasing for $-k < t < -k+1$, and hence

$$A(t) \geq (k-1)\log k - \log(k-1)! = (k-1)\log k - \log\Gamma(k).$$

On the other hand,

$$\min\{A(t) \mid -k < t < -k+1\} \leq A(-k+1/2).$$

We find

$$\begin{aligned} A(-k+1/2) &= \log\Gamma(1/2) + (k-1)\log(k-1/2) \\ &\quad - \sum_{l=1}^{k-1} \log(k-1/2-l) \\ &= 2\log\Gamma(1/2) + (k-1)\log(k-1/2) \\ &\quad - \log\Gamma(k-1/2). \end{aligned}$$

Stirling's formula gives us that

$$\log\Gamma(t) = (t-1/2)\log t - t + \log\sqrt{2\pi} + o(1),$$

as $t \to \infty$, and therefore

$$\begin{aligned} k - (1/2)\log k - \log\sqrt{2\pi} + o(1) &\leq \min A(t) \\ &\leq k - 1/2 + 2\log\Gamma(1/2) \\ &\quad - \log\sqrt{2\pi} + o(1). \end{aligned}$$

Since

$$(k-1)((\log(k-1))^2 + \pi^2) \leq B(t) \leq k((\log k)^2 + \pi^2),$$

we obtain

$$\frac{\min\{d(t) \mid -k < t < -k+1\}}{1/(\log k)^2} \longrightarrow 1$$

as $k \to \infty$.

4. The representation as a Pick function.

Theorem 4.1. *We have*

$$f(z) = \frac{1}{\pi} \log \left(\frac{\sinh \pi}{\pi} \right) + \int_{-\infty}^{0} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d(t)\, dt,$$

where d is the function in (3).

Proof. From Theorem 3.4, we know that f is a Pick function. It thus has a representation of the form

$$f(z) = az + b + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{t^2+1} \right) d\mu(t),$$

where μ is a positive measure, a is nonnegative and b is a real number. Our problem is to find these constants and this measure.

From the general theory we know that $a = \lim_{y\to\infty} f(iy)/(iy)$ and that $b = \Re f(i)$. We find, using the relation (7),

$$a = \lim_{y\to\infty} \frac{1}{iy} \left(1 + \frac{\log \sqrt{2\pi} + (\operatorname{Log} iy)/2 - iy + \mathcal{J}(iy)}{iy \operatorname{Log} iy} \right) = 0,$$

since by (8), $|\mathcal{J}(iy)| \leq \pi/8$ for all $y \geq 1$. The constant b is equal to

$$\Re f(i) = -i(2/\pi) \log |\Gamma(i+1)| = \frac{1}{\pi} \log \frac{\sinh(\pi)}{\pi}.$$

Here the last equality sign follows from Weierstrass's product expansion of the function $\Gamma(z+1)$.

We know that μ is the vague limit of the sequence of positive measures $\Im f(t+i/n)\, dt/\pi = V(t+i/n)\, dt/\pi$ as n tends to infinity.

Let h be a nonnegative continuous function on the real line and suppose that its support is compact and hence is contained in $[-K, K]$ for some K.

We shall show that

$$(12)\qquad \int_{-K}^{K} h(t)V(t+i/n)\,dt \longrightarrow \pi \int_{-K}^{K} h(t)\,d(t)\,dt,$$

as n tends to infinity by appealing to Lebesgue's theorem on dominated convergence.

We claim that, for any $k \geq 1$,

$$V(t+i/n) \leq \text{Const}\,(1+|\log|t+k||),$$

for all t satisfying $|t+k| \leq 1/2$ and all $n \geq 1$. The constant depends on k but not on n. In the following, we assume $k \geq 2$, but the proof is easily adapted to the case $k=1$.

From (9), we have, with $z = t+i/n$,

$$\begin{aligned}
|z\mathrm{Log}\,z|^2 V(z) &= (x\log|z| - y\mathrm{Arg}\,z)\arg\Gamma(z+1)\\
&\quad - (y\log|z| + x\mathrm{Arg}\,z)\log|\Gamma(z+1)|\\
&\leq (|t|\log\sqrt{t^2+1}+\pi)\ \text{Const}\\
&\quad + (\log\sqrt{t^2+1}+|t|\pi)\ \text{Const}\\
&\quad + (\log\sqrt{t^2+1}+|t|\pi)\\
&\quad \times (|\log|z+k-1|| + |\log|z+k||),
\end{aligned}$$

because of Lemma 2.1. We have

$$\log|t+k| \leq \log|z+k| \leq \log(3/2)$$

for all $|t+k| \leq 1/2$ and $n \geq 1$. Since $|\log|z+k-1||$ is also bounded for $|t+k| \leq 1/2$ and all $n \geq 1$, we consequently obtain

$$|z\mathrm{Log}\,z|^2 V(z) \leq \text{Const} + \text{Const}\,|\log|t+k|| \leq \text{Const}\,(1+|\log|t+k||)$$

for all t, such that $|t+k| \leq 1/2$, and all n. The constants depend on k, but not on n. The member on the righthand side is integrable on $[-k-1/2, -k+1/2]$ because it has a logarithmic singularity at k.

If $|t| \leq 1/2$ and $n \geq 2$, we have

$$|\mathrm{Log}\,(t+i/n)| \geq |\log\sqrt{t^2+1/4}|.$$

Thus, as used in (10),

$$V(t+i/n) \leq \frac{\text{Const}}{|\log\sqrt{t^2+1/4}|}.$$

The constant is independent of n and the righthand side is again integrable. Furthermore,

$$|(t+i/n)\operatorname{Log}(t+i/n)|^2 \longrightarrow t^2((\log|t|)^2+\pi^2)$$

uniformly on compact subsets of $]-\infty,0[$. Therefore, we see that Lebesgue's theorem is applicable. We conclude that (12) holds, and hence, that

$$\mu(t) = d(t)\,dt. \qquad \square$$

We shall now simplify the representation in Theorem 4.1.

By a change of variables we have

$$f(z) = \frac{1}{\pi}\log\left(\frac{\sinh\pi}{\pi}\right) - \int_0^\infty \left(\frac{1}{t+z} - \frac{t}{t^2+1}\right) d(-t)\,dt.$$

It is tempting to rewrite the integral as a sum of two integrals

$$\int_0^\infty \frac{d(-t)}{t+z}\,dt - \int_0^\infty \frac{td(-t)}{t^2+1}\,dt.$$

This can be done if we can verify that

$$\int_0^\infty \frac{td(-t)}{t^2+1}\,dt < \infty. \tag{13}$$

This property can be verified directly, but we prefer to argue as follows:

By differentiating under the integral sign, we find the following corollary to the theorem above.

Corollary 4.2. *We have, for $n \geq 1$,*

$$f^{(n)}(z) = (-1)^{n+1} n! \int_0^\infty \frac{d(-s)}{(s+z)^{n+1}}\,ds.$$

In particular, f' is completely monotone on $]0,\infty[$.

The holomorphic function $z \mapsto 1 - f(z)$ in $\mathcal{A}$ has the properties

1. $\Im(1 - f(z)) \le 0$ for $z \in \mathbf{H}$ (by Theorem 3.4).

2. $1 - f(x) \ge 0$ for $x > 0$ (by Corollary 4.2, $1 - f(x)$ is decreasing, and from Stirling's formula we see that $1 - f(x) \to 0$ as $x \to \infty$).

These properties ensure that $1-f$ is a Stieltjes transform and thus has a representation of the form

$$1 - f(z) = \alpha + \int_0^\infty \frac{d\sigma(t)}{z+t}.$$

Here $\alpha = \lim_{x\to\infty}(1 - f(x)) = 0$ and $\int_0^\infty (1/(t+1))\sigma(t) < \infty$. Concerning these facts, see, e.g., [**4**]. The measure σ is given as the vague limit for $n \to \infty$ of

$$-\frac{1}{\pi}\Im(1 - f(-t + i/n))\, dt = \frac{1}{\pi} V(-t + i/n)/dt,$$

hence $d\sigma(t) = d(-t)\, dt$. Therefore, (13) holds and we find

$$\int_0^\infty \frac{td(-t)}{t^2+1}\, dt = 1 - \frac{1}{\pi}\log\left(\frac{\sinh \pi}{\pi}\right).$$

We conclude:

Theorem 4.3. *We have*

$$\frac{\log\Gamma(z+1)}{z\mathrm{Log}\, z} = 1 - \int_0^\infty \frac{d(-t)}{z+t}\, dt,$$

where d is defined in (3).

5. Some related Pick functions. We shall show that the holomorphic functions in $\mathcal{A}$

$$(14) \qquad f_1(z) = \frac{\log\Gamma(z+1)}{z}, \qquad f_2(z) = z - \frac{\log\Gamma(z+1)}{\mathrm{Log}\, z}$$

are Pick functions and we shall find their integral representations.

The first of these results is more elementary than the main result of the paper. It has, however, the interesting consequence that $\log\Gamma(z)$ does not vanish in $\mathbf{C}\setminus\mathbf{R}$.

Theorem 5.1. *The function*

$$f_1(z)=\frac{\log\Gamma(z+1)}{z},\quad z\in\mathcal{A}$$

is a Pick function of the form (1) *with* $a=0$,

$$b=-\gamma+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\arctan\frac{1}{k}\right)$$

and $\mu=d_1(t)\,dt$, *where* $d_1:\mathbf{R}\to[0,\infty[$ *is defined as* $d_1(t)=0$ *for* $t\ge -1$ *and*

$$d_1(t)=\frac{k-1}{-t}\quad \text{for } t\in[-k,-k+1[\,,\quad k=2,3,\dots\,.$$

The proof can be done in analogy with the proof of Theorem 1.1, but we shall give another proof based on the following lemma.

Lemma 5.2. *The function* $-(\operatorname{Log}(1+z))/z$ *is a Pick function with the representation*

$$-\frac{\operatorname{Log}(1+z)}{z}=-\frac{\pi}{4}+\int_{-\infty}^{-1}\left(\frac{1}{t-z}-\frac{t}{t^2+1}\right)\frac{dt}{-t},\quad z\in\mathbf{C}\setminus\,]-\infty,-1]\,. \tag{15}$$

Proof. The righthand side is a Pick function, and if we integrate term by term, it is easy to establish the formula. $\square$

Proof of Theorem 5.1. By Weierstrass's product expansion, we get

$$\frac{\log\Gamma(z+1)}{z} = -\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{\operatorname{Log}(1+z/k)}{z}\right), \quad z \in \mathcal{A},$$

and since the partial sum

$$\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{\operatorname{Log}(1+z/k)}{z}\right)$$

is a Pick function by the lemma, it is clear that f_1 is a Pick function. To find the integral representation, we replace z by z/k and substitute $t = s/k$ in formula (15) and get

$$\begin{aligned}-\frac{\operatorname{Log}(1+z/k)}{z} &= -\frac{\pi}{4k} + \int_{-\infty}^{-k}\left(\frac{1}{s-z} - \frac{s}{s^2+k^2}\right)\frac{ds}{-s}\\ &= \arctan k - \frac{\pi}{2} + \int_{-\infty}^{-k}\left(\frac{1}{s-z} - \frac{s}{s^2+1}\right)\frac{ds}{-s}.\end{aligned}$$

Using $\pi/2 - \arctan k = \arctan(1/k)$, we find

$$\begin{aligned}\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{\operatorname{Log}(1+z/k)}{z}\right) &= \sum_{k=1}^{n}\left(\frac{1}{k} - \arctan\frac{1}{k}\right)\\ &\quad + \int_{-\infty}^{-1}\left(\frac{1}{s-z} - \frac{s}{s^2+1}\right)\varphi_n(s)\,ds,\end{aligned}$$

where

$$\varphi_n(s) = \begin{cases} -(k-1)/s & \text{for } s \in [-k, -k+1[,\ k = 2, \dots, n,\\ -n/s & \text{for } s < -n.\end{cases}$$

For $n \to \infty$, we get the assertion of the theorem. $\square$

Remark 5.3. From the unicity of the integral representation of a Pick function, we get that

$$\arg\Gamma(i+1) = -\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \arctan\frac{1}{k}\right).$$

This follows of course also directly from the Weierstrass product.

Theorem 5.4. *The function*

$$f_2(z) = z - \frac{\log \Gamma(z+1)}{\operatorname{Log} z}, \quad z \in \mathcal{A}$$

is a Pick function of the form (1), *with* $a = 0$, $b = -(2/\pi)\arg\Gamma(1+i)$ *and* $\mu = -td(t)\,dt$, *where* $d(t)$ *is defined in Theorem* 1.1.

Proof. By Theorem 1.2, we have

$$\Phi(z) \equiv 1 - \frac{\log \Gamma(z+1)}{z\operatorname{Log} z} = \int_0^\infty \frac{d(-t)}{t+z}\,dt,$$

so $\Phi(-z)$ is a Pick function, continuous and positive on $]-\infty, 0[$. By a result in [**1**, page 127], it follows that $z\Phi(-z)$ is a Pick function with representing measure $td(-t)\,dt$ concentrated on $[0,\infty[$, i.e.,

$$z\Phi(-z) = \alpha z + \beta + \int_0^\infty \left(\frac{1}{t-z} - \frac{t}{t^2+1}\right) td(-t)\,dt,$$

and by the relations (2) and (11), we see that $\alpha = 0$, $\beta = (2/\pi)\arg\Gamma(1+i)$. Replacing z by $-z$ and t by $-t$ leads to

$$f_2(z) = -\beta + \int_{-\infty}^0 \left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)(-t)\,d(t)\,dt,$$

which is the desired representation. □

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