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FROM NON-HERMITIAN OSCILLATOR-LIKE OPERATORS TO FREUD POLYNOMIALS AND SOME CONSEQUENCES

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ABSTRACT. Non-Hermitian quantum Hamiltonians dealing with oscillator-like interactions are discussed when realized in terms of creation and annihilation operators that are no longer adjoint to each other. Specific differential realizations are exploited and lead to real spectra and typical eigenfunctions including (unexpected) Freud orthogonal polynomials. Hermiticity is finally revisited with respect to new scalar products of specific Hilbert spaces.

Subnormality [**14**] of linear operators is a relatively recent mathematical property. An operator S in H is said to be *subnormal* if it has a normal extension N (recall that N is normal on K including H if $\|Nf\| = \|N^\dagger f\|$ for $f \in D(N) = D(N^\dagger)$). This subnormality has not yet been exploited in physics up to several remarks on the (bosonic) *creation operator* (denoted by $a^\dagger$) in the context of (one-dimensional) quantum harmonic oscillators. Such an operator, considered as the *best representative* of subnormal (unbounded) operators, generates with its companion, a, i.e., the (bosonic) annihilation operator, the Lie-Heisenberg commutation relations [**4**] associated to this study. Acting on a Hilbert space, currently called the Fock-Bargmann space [**3**], characterized by orthonormalized state-vectors $|n>$, $n = 0, 1, 2, \dots$, the corresponding Hamiltonian

$$H_{H.O.} = a^\dagger a + \frac{1}{2} = \frac{1}{2}(a^\dagger a + a a^\dagger) \equiv \frac{1}{2}\{a^\dagger, a\} \tag{1}$$

is directly expressed in terms of these operators and appears trivially as a *self-adjoint* operator (with respect to the well-known scalar product of the above-mentioned Hilbert space) admitting trivially a *real* spectrum as expected.

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Very recent developments [**6, 8**] have exploited the subnormality of $a^\dagger$ by λ-deforming its expression, λ being a real parameter, without deforming the corresponding Heisenberg algebra. Indeed, we have considered the new Hamiltonian

$$H_\lambda = a_\lambda^\dagger a + \frac{1}{2}, \tag{2}$$

where

$$a_\lambda^\dagger \equiv a^\dagger + \lambda I, \quad \lambda \in R, \tag{3}$$

leading once again to a *real* spectrum, (see hereafter), while H_λ is no more self-adjoint as it appears immediately from its expression, $a_\lambda^\dagger$ being no more the adjoint of a.

Such an oscillator-like context characterized by $(a_\lambda^\dagger)^\dagger \neq a$ simultaneously with $H^\dagger \neq H$ has been visited [**6, 8**] with the hope that it could give new properties in the *coherence* [**11**] and (or) squeezing [**18**] domains due to the nontrivial presence of the parameter λ. Let us only recall that in coherence *both* squares of dispersions on the position and the momentum do reach their minimal value 1/2 while in squeezing only *one* of such squares is less than 1/2.

Two kinds of results have already been obtained depending on the explicit realizations we are considering for the original operators a and $a^\dagger$, both contexts leading to *non-Hermitian* Hamiltonians (a reasonable property actually visited in different physical contexts [**7**]).

First we take care of the usual realization

$$a = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right) \tag{4}$$

leading to

$$H_\lambda = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{\lambda}{\sqrt{2}}\frac{d}{dx} + \frac{1}{2}x^2 + \frac{\lambda}{\sqrt{2}}x + \frac{1}{2}. \tag{5}$$

Taking a look at the equation

$$H_\lambda \psi_{n,\lambda}(x) = E_{n,\lambda}\psi_{n,\lambda}(x), \tag{6}$$

we obtain the (normalized) eigenfunctions

$$\psi_{n,\lambda}(x) = \langle x \mid n \rangle_\lambda = \frac{2^{-(n/2)}\pi^{-1/4}}{\sqrt{n!}\sqrt{L_n^{(0)}(-\lambda^2)}} e^{-x^2/2} H_n\left(x + \frac{\lambda}{\sqrt{2}}\right), \tag{7}$$

where H_n and $L_n^{(0)}$ refer, respectively, to Hermite and generalized Laguerre polynomials [**12**] as usual. Rather surprisingly, despite the nonself-adjointness of the Hamiltonian

$$H_\lambda^\dagger \neq H_\lambda, \tag{8}$$

the spectrum is real

$$E_{n,\lambda} = E_{n,\lambda=0} = n + \frac{1}{2}, \quad n = 0, 1, 2, \dots . \tag{9}$$

Moreover, the state vectors $|n >_\lambda$ are such that

$$a \mid n >_\lambda = \sqrt{n}\left(\frac{L_{n-1}^{(0)}(-\lambda^2)}{L_n^{(0)}(-\lambda^2)}\right)^{1/2} |n-1 >_\lambda, \tag{10}$$

$$a_\lambda^\dagger \mid n >_\lambda = \sqrt{n+1}\left(\frac{L_{n+1}^{(0)}(-\lambda^2)}{L_n^{(0)}(-\lambda^2)}\right)^{1/2} \mid n+1 >_\lambda, \tag{11}$$

implying

$$[a, a_\lambda^\dagger] = I, [H_\lambda, a] = -a, [H_\lambda, a_\lambda^\dagger] = a_\lambda^\dagger. \tag{12}$$

While it is well known that the $\lambda = 0$ context does not permit *squeezing*, we are now able to show the interest of nonzero λ-values. These results [**6**] have been obtained through evaluations of mean values, dispersions and constraints coming from the Heisenberg uncertainty relations. Let us only mention here the presence of this parameter $\lambda \neq 0$ *reduces* the indetermination and leads to new squeezed states [**6**] compared to the old ones [**18**].

Second, we now indicate another realization for a and $a^\dagger$ ensuring the Lie-Heisenberg relations but relaxing the demand of a and $a^\dagger$ to be adjoint operators. Indeed, we take one of the Ushveridze proposal [**15**], i.e.,

$$a = \frac{d}{dx} + cx^3 \quad \text{and} \quad a^\dagger = x, \quad c > 0, \tag{13}$$

leading here to the *non-Hermitian* Hamiltonian (1), but also to the real spectrum $E_{n,\lambda} = (9)$. In fact, such a study deals with new eigenvectors written on the form

$$\psi_n(x) = \frac{\sqrt{2}(c/2)^{2n+1/8}}{\sqrt{\Gamma(2n+1/4)}} e^{-(c/4)x^4}, \quad n = 0, 1, 2, \dots , \tag{14}$$

which are still square-integrable on the real line even if they have been obtained via nonadjoint operators a and $a^\dagger$. By noticing the lost of the orthogonality of such eigenfunctions, we can restore it through the Schmidt procedure and get eigenfunctions directly proportional to (unexpected) Freud orthogonal polynomials [**9**], recently exploited elsewhere [**16**]. Let us recall here only the first ones given by

$$\begin{aligned} p_0(x) &= 1, \quad p_1(x) = x \\ p_2(x) &= \sqrt{2c}x^2 - 2\frac{\Gamma(3/4)}{\Gamma(1/4)}, \\ p_3(x) &= \sqrt{2c}x^3 - \frac{1}{2}\frac{\Gamma(1/4)}{\Gamma(3/4)}x. \end{aligned} \tag{15}$$

Inside the new (complete) basis, given by orthonormal eigenfunctions of the form

$$\Lambda_n(x) = M_n e^{-(c/4)x^4} p_n(x), \tag{16}$$

we can show that once again squeezing is possible in this context characterized by non-adjoint operators, (see (13)). Important open questions arise here. In particular, we propose to study the implications of the choice (13) on new "position and momentum operators" that could be defined as

$$\bar{x} = \frac{1}{\sqrt{2}}(a + a^\dagger) = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x + cx^3\right) = \operatorname{Re}\bar{x} + i\operatorname{Im}\bar{x} \tag{17}$$

and

$$\bar{p} = \frac{i}{\sqrt{2}}(-a + a^\dagger) = \frac{i}{\sqrt{2}}\left(-\frac{d}{dx} + x - cx^3\right) = \operatorname{Re}\bar{p} + i\operatorname{Im}\bar{p}, \tag{18}$$

where we associate the following identifications

$$\operatorname{Re}\bar{x} = \frac{1}{\sqrt{2}}(x + cx^3),\ \operatorname{Im}\bar{x} = \frac{-i}{\sqrt{2}}\left(\frac{d}{dx}\right) \tag{19}$$

and

$$\operatorname{Re}\bar{p} = \frac{-1}{\sqrt{2}}\left(\frac{d}{dx}\right),\ \operatorname{Im}\bar{p} = \frac{1}{\sqrt{2}}(x - cx^3) \tag{20}$$

such that

$$[\bar{x}, \bar{p}] = iI = [\operatorname{Re}\bar{x}, \operatorname{Re}\bar{p}] - [\operatorname{Im}\bar{x}, \operatorname{Im}\bar{p}]. \tag{21}$$

These developments are generalizations of the usual Heisenberg context and lead to the extended Heisenberg relation

$$\Delta(\operatorname{Re}\bar{x})\Delta(\operatorname{Re}\bar{p}) + \Delta(\operatorname{Im}\bar{x})\Delta(\operatorname{Im}\bar{p}) \geq \frac{1}{2}, \tag{22}$$

giving back the original one when $\operatorname{Im}\bar{x} = \operatorname{Im}\bar{p} = 0$. These *non-Hermitian* "position and momentum operators" effectively lead to a new approach of the harmonic oscillator by maintaining nice properties such as the real spectrum of the corresponding Hamiltonian, the square-integrability of its eigenfunctions, the orthonormal and complete characters of the basis and, moreover, they open the possibility of squeezing and the consideration of generalized Heisenberg relations [**8**].

Third, we want to add some remarks concerning the two steps previously discussed. For example, let us point out that, in strong connection with the so-called *intertwining relations* already pointed out in equations (5) as well as in equations (12), super-symmetry [**17**] and the factorization method [**1, 10**] are included in our developments, although we do not require that the meaningful creation and annihilation operators included in our (non-Hermitian) Hamiltonian are adjoint ones.

Another remark has recently been formulated concerning more precisely, the loss of Hermiticity of the Hamiltonians that we are considering here. In [**5**], we have decided to restore the adjoint character of H by modifying the scalar product of the Hilbert space generated by the new basis implied by our operators. Let us illustrate such a new point of view by requiring the existence of a fundamental groundstate–let us call it ϕ_0– and of creation ($A^\dagger$) and annihilation (A) operators such that

$$A\phi_0 = 0 \tag{23}$$

$$A\phi_n = \sqrt{n}\phi_{n-1} \tag{24}$$

$$A^\dagger\phi_{n-1} = \sqrt{n}\phi_n, \tag{25}$$

satisfying once again the Lie-Heisenberg relation

$$[A, A^\dagger] = I. \tag{26}$$

Let us then consider that these $\phi_0, \phi_1, \ldots, \phi_n, \ldots$ form a basis of our Hilbert space characterized by an inner product by means of a bilinear form such that

$$\langle \phi_n, \phi_m \rangle = \delta_{nm}. \tag{27}$$

Inside the context of a λ-family of creation operators $A^\dagger_\lambda$, as proposed in equation (3), we can construct for all n the functions

$$\phi_{n,\lambda} = \frac{1}{\sqrt{n!}}(A^\dagger_\lambda)^n\phi_0 \tag{28}$$

and require that

$$\langle A^\dagger_\lambda\phi_{n,\lambda}, \phi_{m,\lambda}\rangle = \langle \phi_{n,\lambda}, (A^\dagger_\lambda)^\dagger\phi_{m,\lambda}\rangle = \langle \phi_{n,\lambda}, a\phi_{m,\lambda}\rangle, \tag{29}$$

identifying $(A^\dagger_\lambda)^\dagger$ with the old annihilation operator a:

$$(A^\dagger_\lambda)^\dagger = a. \tag{30}$$

The property (29) can be made explicit in connection with the old standard scalar product of the harmonic oscillator. Indeed, we have

$$\langle \phi_{n,\lambda}, \phi_{m,\lambda}\rangle = \int_{-\infty}^{+\infty} \phi_n^-(x)\phi_m(x)\rho(\lambda, x)\, dx \tag{31}$$

where $\rho(\lambda, x)$ is the factor characterizing the new measure, which is necessary for ensuring that a is the adjoint of $A_\lambda^\dagger$. Direct calculations lead to the following factor [**5**]:

$$\rho(\lambda, x) = e^{-\lambda^2/2} e^{-\sqrt{2}\lambda x}, \tag{32}$$

the first exponential being determined by requiring that

$$\langle \phi_{0,\lambda}, \phi_{0,\lambda} \rangle = \langle \phi_0, \phi_0 \rangle = 1. \tag{33}$$

Inside this well-defined Hilbert space, characterized by a, (complete), orthonormal basis and the typical inner product (31)–(32), the Hamiltonian

$$H_\lambda = \frac{1}{2}\{A_\lambda^\dagger, A_\lambda\} = \frac{1}{2}\{A_\lambda^\dagger, a\} = A_\lambda^\dagger a + \frac{1}{2} \tag{34}$$

is now a Hermitian operator with the real spectrum (9), and the evaluation of all mean-values and constraints coming from the usual Heisenberg relations can now be reconsidered. As a result in these λ-contexts we have constructed new coherent states that satisfy the DOCS (coherent states being obtained through the action of a displacement operator on the vacuum), AOCS (coherent states being eigenstates of the annihilation operator a) and MUCS (coherent states minimizing the uncertainty relation) points of view [**13**].

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