ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 2, Summer 2002

ALGEBRA OF DIFFERENTIAL FORMS WITH EXTERIOR DIFFERENTIAL $d^3 = 0$ IN DIMENSIONS ONE AND TWO

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ABSTRACT. In this paper, we construct the algebra of differential forms with exterior differential satisfying $d^3 = 0$ over an associative algebra with one and n generators satisfying quadratic relations. Supposing $d^2 \neq 0$, we introduce the second order differentials d^2x^i. We also assume that the homomorphism defining a first order differential calculus is linear in variables, and that there are no relations between the terms $(dx^i)^2$ and d^2x^j. A graded q-differential algebra with $d^3 = 0$ is constructed by means of the Wess-Zumino method. The commutation relations between generators x^i, dx^j, d^2x^k of the algebra of differential forms in pairs and themselves are found. In the case of the algebra with n generators, the commutation relations between noncommutataive derivatives ∂_i and generators d^2x^j also are found, and the consistency conditions are described.

1. Introduction. An idea to generalize the classical exterior differential calculus with $d^2 = 0$ to the case $d^N = 0$, $N > 2$, arises in a recent series of papers [**2–4, 6**], where the different approaches to this idea are developed, and these generalizations have been proposed and studied. In the paper [**5**] such a generalization is provided by the notion of graded q-differential algebra which is, according to the definition given in [**2**], an associative unital **N**-graded algebra endowed with a linear endomorphism d (q-differential) of degree 1 satisfying $d^N = 0$ and the graded q-Leibniz rule

$$d(\omega\tau) = d(\omega)\tau + q^{\mathrm{gr}(\omega)}\omega\, d(\tau), \tag{1}$$

where ω, τ are arbitrary elements of the algebra; $\mathrm{gr}(\omega)$ is the grade of an element ω; q is a primitive cubic root of unity.

In the paper [**5**], a q-differential calculus with $d^3 = 0$ is constructed on a classical smooth n-dimensional manifold. We construct the q-differential calculus on an associative algebra generated by one variable

Received by the editors on December 14, 2000, and in revised form on July 2, 2001.
This work was supported by Estonian Science Foundation grant No. 1134.

x and on an associative algebra generated by n variables $x^1, x^2, \dots, x^n$ satisfying quadratic relations $x^i x^j = B^{ij}_{kl} x^k x^l$, $i, j, k, l = 1, \dots, n$. Thus we deal with noncommutative geometry.

For construction of generalized differential calculus with $d^3 = 0$, we apply the Wess-Zumino method [**8**]. Assuming that $d^2 \neq 0$, we have to introduce the second order differentials $d^2 x^i$. Supposing that there are no relations between the terms $(dx^i)^2$ and $d^2 x^i$ and differentiating the commutation relations relating the right and left structures of a bimodule of first order differential forms ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$, we get the commutation relations between x^i and $d^2 x^j$, and between dx^k. The assumed and obtained commutation relations completely determine a multiplication law of the constructed algebra. Furthermore, in the case of algebra with quadratic relations, diffrentiating the commutation relations between dx^i and differentiating three times the quadratic relations between generators x^i, we obtain the Wess-Zumino-like consistency conditions. The Yang-Baxter equation appears as a solution of the obtained quadratic consistency conditions [**8**].

In Section 2, we consider the case of an algebra with one generator and show that in this case we can construct the two different graded q-differential algebras with $d^3 = 0$.

In Section 3, we consider the case where the homomorphism ξ defining a first order differential calculus is linear in variables. Our construction of an exterior differential calculus with $d^3 = 0$ naturally includes the exterior calculus on the quantum plane, with $d^2 = 0$, obtained by Wess and Zumino in [**8**].

2. Algebra of differential forms in dimension one. In this section, we construct a graded q-differential algebra with exterior differential d satisfying $d^3 = 0$ over an associative algebra with one generator. Let $\mathcal{A}$ be the associative unital algebra generated by one variable x. An arbitrary element of $\mathcal{A}$ is a polynomial in x of some degree p, $p \in \mathbf{N}$

$$f = \alpha_0 + \alpha_1 x^1 + \cdots + \alpha_p x^p, \quad \alpha_0, \alpha_1, \dots, \alpha_p \in \mathcal{C}. \tag{2}$$

Let ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ be the bimodule freely generated as a right module by the differential dx, i.e., every form $\omega \in {}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ has a unique representation $\omega = dx\, h$, $h \in \mathcal{A}$. Let $d : \mathcal{A} \to {}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ be a linear map satisfying the ordinary Leibniz rule

$$d(fg) = d(f)g + fd(g), \quad \forall f, g \in \mathcal{A}. \tag{3}$$

Following [**1**], we define a coordinate differential calculus over $\mathcal{A}$ by

$$df = dx\partial(f), \quad \forall f \in \mathcal{A}, \tag{4}$$

where the partial derivative ∂ is the linear map $\mathcal{A} \to \mathcal{A}$ such that

$$\begin{aligned} \partial(x) &= 1, \\ \partial(fg) &= \partial(f)g + \xi(f)\partial(g), \quad \forall f, g \in \mathcal{A}, \end{aligned} \tag{5}$$

where $\xi : \mathcal{A} \to \mathcal{A}$ is a homomorphism defining a left module structure of the bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ by means of the right module structure

$$x\,dx = dx\,\xi(x). \tag{6}$$

Assuming that $d^2 \neq 0$, we introduce the second order differential d^2x. Let $\Omega_\xi(\mathcal{A})$ be the right free module over the algebra $\mathcal{A}$ generated by the monomials $(d^2x)^k(dx)^l$, $k, l = 0, 1, \ldots$. Further, we shall call the elements from $\Omega_\xi(\mathcal{A})$ differential forms. The module $\Omega_\xi(\mathcal{A})$ becomes an **N**-graded module if we introduce the grade one and two to the differentials of first and second order respectively and the grade zero to the elements of algebra $\mathcal{A}$. Then the module $\Omega_\xi(\mathcal{A})$ splits into direct sum $\Omega_\xi(\mathcal{A}) = \oplus_{\mu=0}^{\infty}\Omega_\xi^\mu(\mathcal{A})$, where $\Omega_\xi^\mu(\mathcal{A})$ is a submodule of homogeneous differential forms of degree μ, and $\mu = 2k + l$.

Now we find commutation relations between x and d^2x and between dx and d^2x. If we differentiate the commutation relation (6) supposing that (1) holds, we get

$$d(x\,dx - dx\xi(x)) = dx\,dx + x\,d^2x - d^2x\xi(x) - q\,dx\,dx\partial(\xi(x)) = 0.$$

Since we assume that the terms $dx\,dx$ and $x\,d^2x$ must cancel separately, it follows that

$$dx\,dx = q\,dx\,dx\partial(\xi(x)), \tag{7}$$

$$x\,d^2x = d^2x\xi(x). \tag{8}$$

The relation (7) has two solutions: $(dx)^2 = 0$ and $\partial(\xi(x)) = q^{-1}$. Consider the second solution. Since $\partial(x) = 1$, from this solution we get

$$\xi(x) = q^{-1}x.$$

Thus ξ is completely determined and we can rewrite (6) and (8) in the form

$$x\,dx = q^{-1}\,dx\,x, \tag{9}$$

$$x\,d^2x = q^{-1}\,d^2x\,x. \tag{10}$$

The relation between dx and d^2x can be obtained immediately from (10) by the differentiation under the assumption that $d^3x = 0$:

$$dx\,d^2x = qd^2x\,dx. \tag{11}$$

Further, since ∂ and ξ are related by (5), the direct calculation gives us that $\partial(x^3) = 0$. It implies $x^3 = 0$. Then an arbitrary element of algebra $\mathcal{A}$ is a polynomial in x of degree ≤ 2, that is,

$$f = \alpha_0 + \alpha_1 x + \alpha_2 x^2, \tag{12}$$

and we have $\partial^3(f) = 0$, for all $f \in \mathcal{A}$. By direct calculation, we get $d^3f = 0$ for all $f \in \mathcal{A}$ and $d^3\omega = 0$ for all $\omega \in \Omega_\xi(\mathcal{A})$. In fact,

$$d^3f = d^3x\partial f + [3]_q d^2x\,dx\partial^2 f + q^3\,dx\,dx\,dx\partial^3 f = 0.$$

Here and later, we use the notion $[k]_q = 1 + q + q^2 + \cdots + q^{k-1}$. In our case, $[3]_q = 0$.

In order to prove $d^3\omega = 0$, we calculate d^3 of the form $\omega_\mu \in \Omega^\mu_\xi(\mathcal{A})$

$$\omega_\mu = \sum_{2k+l=\mu} (d^2x)^k(dx)^l f_{kl}.$$

Using the equality

$$1 + q^2 + \cdots + q^{2l} = \frac{1 - q^{2l+2}}{1 - q^2},$$

we get

$$\begin{aligned} d^3((d^2x)^k(dx)^l f_{kl}) = {} & \frac{q^{3(2k+1)}}{(1-q^2)^3}(1-(q^3)^{2(l-1)} \\ & - q^{2l}(1-q^{2(l-1)})[3]_q)(d^2x)^{k+3}(dx)^{l-3} f_{kl} \\ & + q^{2k+l+2}\frac{(1-q^{2l})^2}{1-q^2}[3]_q(d^2x)^{k+2}(dx)^{l-1}\partial f_{kl} \\ & + q^{2k+2l+1}\frac{1-q^{2l+l}}{1-q^2}[3]_q(d^2x)^{k+1}(dx)^{l+1}\partial^2 f_{kl} \\ & + q^{2k+3l+3}(d^2x)^k(dx)^{l+3}\partial^3 f_{kl} = 0. \end{aligned}$$

Extending d^3 on the whole $\Omega_\xi(\mathcal{A})$, we get $d^3\omega = 0$ for all $\omega \in \Omega_\xi(\mathcal{A})$.

Thus, if we define a multiplication law on the right-module $\Omega_\xi(\mathcal{A})$ by the formulae

$$\begin{gathered} x^3 = 0, \quad xd^2x = q^{-1}d^2x\,x, \\ x\,dx = q^{-1}\,dx\,x, \quad dx\,d^2x = qd^2x\,dx, \end{gathered} \tag{13}$$

then $\Omega_\xi(\mathcal{A})$ becomes a graded q-differential algebra with $d^3 = 0$ generated by x, dx, d^2x.

Consider the other solution of (7): $(dx)^2 = 0$. This solution does not give any restriction on the homomorphism ξ.

The differentiation of the relations (8) gives us the equation

$$(d^2x)^2 = q^4(d^2x)^2\partial\xi(x),$$

from which two solutions follow. If we assume that $(d^2x)^2 \neq 0$, then we get the same condition on the homomorphism ξ:

$$\xi(x) = q^{-1}x,$$

as we already obtained under the assumption $(dx)^2 \neq 0$ above. The second solution, $(d^2x)^2 = 0$, does not give any restriction on ξ.

Under the assumption $(dx)^2 = (d^2x)^2 = 0$ mentioned above, we get

$$d^3 f = q^2 d^2x\;dx\partial^2(f). \tag{14}$$

Since we require that $d^3 f = 0$ for all $f \in \mathcal{A}$, then $\partial^2 f = 0$. Hence we obtain the additional condition $x^2 = 0$, i.e., the algebra $\mathcal{A}$ contains only linear functions

$$f = \alpha_0 + \alpha_1 x.$$

By direct calculation, we get $d^3 = 0$ for all $\omega \in \Omega_\xi(\mathcal{A})$. Every form from $\Omega_\xi(\mathcal{A})$ can be written under the conditions $(dx)^2 = (d^2x)^2 = 0$ in the following term

$$\omega = d^2x\, dx f_{11} + d^2x f_{10} + dx\, f_{01}, \quad f_{ij} \in \mathcal{A}, \quad i, j = 0, 1.$$

Then we get

$$\begin{aligned} d^3(\omega) &= d^3(d^2x\, dx f_{11} + d^2x f_{10} + dx\, f_{01}) \\ &= d^2(0 + q^2 d^2x\, dx \partial f_{10} + d^2x f_{01}) \\ &= d(0 + q^2 d^2x\, dx\, \partial f_{01}) = 0 \end{aligned}$$

for arbitrary form $\omega \in \Omega_\xi(\mathcal{A})$. Hence, on the bimodule of the differential forms, we can define the second multiplication law by the relations

$$\begin{aligned} x^2 = (dx)^2 = (d^2x)^2 = 0, &\quad x\, dx = dx\, \xi(x), \\ x\, d^2x = d^2x\, \xi(x), &\quad dx\; d^2x = d^2x\; dx\; \xi(x). \end{aligned} \tag{15}$$

Then the bimodule $\Omega_\xi(\mathcal{A})$ becomes the algebra.

3. Algebra of differential forms on the quantum plane. At the beginning of this section, we find all commutation relations and consistency conditions in the case of algebra with n generators satisfying the relations "$x^i x^j = B^{ij}_{kl} x^k x^l$. Then these relations and conditions are specified for the case of n-dimensional quantum plane. Finally, we find a restriction which allows us to construct a graded q-differential algebra with $d^3 = 0$ in the case of a two-dimensional quantum plane.

3.1 *Commutation relations and consistency conditions.* Now let $\mathcal{A}$ be a unital associative **C**-algebra generated by the variables x^i, $i = 1, \ldots, n$, satisfying the commutative relation

$$x^i x^j = B^{ij}_{kl} x^k x^l \quad \text{or} \quad (\delta^i_k \delta^j_l - B^{ij}_{kl}) x^k x^l = 0. \tag{16}$$

Let $({}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}, d)$ be a first order coordinate differential calculus over algebra $\mathcal{A}$. Here ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ is the bimodule over $\mathcal{A}$ generated by the first order differentials dx^i, $i = 1, \dots, n$; the differential d is the linear map $\mathcal{A} \to {}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ obeying the ordinary Leibniz rule

$$d(fg) = d(f)g + fd(g), \quad \forall f, g \in \mathcal{A}. \tag{17}$$

The word "coordinate" means that the differential d is defined on the elements of algebra by

$$d(f) = dx^i \partial_i(f), \quad \forall f \in \mathcal{A}, \quad i = 1, \dots, n, \tag{18}$$

where the partial derivatives ∂_i are linear maps $\mathcal{A} \to \mathcal{A}$ such that

$$\partial_i(fg) = \partial_i(f)g + \xi_i^k(f)\partial_k(g), \quad \forall f, g \in \mathcal{A}, \quad i, k = 1, \dots, n, \tag{19}$$

where $\xi : \mathcal{A} \to \mathcal{A}_{n\times n}$ is a homomorphism to the algebra of $(n \times n)$-matrices over $\mathcal{A}$ acting linearly on the generators of $\mathcal{A}$, that is,

$$\xi_i^k(x^j) = C_{il}^{jk} x^l, \tag{20}$$

where C_{il}^{jk} are numerical coefficients, $i, j, k, l = 1, \dots, n$, [**7**].

As follows from (17), (18) and (19), the left structure of the bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ is defined by the formula

$$x^i dx^j = C_{kl}^{ij} dx^k x^l, \quad i, j, k, l = 1, \dots, n, \tag{21}$$

by means of the right free module structure.

We suppose that two Wess-Zumino consistency conditions [**8**] have to be satisfied; namely, the linear condition

$$(\delta_k^i \delta_l^j - B_{kl}^{ij})(\delta_{st}^{kl} + C_{st}^{kl})\, dx^s x^t = 0, \tag{22}$$

in the tensor form: $(E_{12} - B_{12})(E_{12} + C_{12}) = 0$, appearing by the differentiation of (16), and the quadratic condition

$$(\delta_k^i \delta_l^j - B_{kl}^{ij}) C_{st}^{lr} C_{uv}^{ks} dx^u x^v x^t = 0, \tag{23}$$

in the tensor form: $(E_{12} - B_{12})C_{23}C_{12} dx_1 x_2 x_3 = 0$ appearing under the multiplication the relations (16) by dx^r from the right and the commutation dx^r through to the left by the commutation relations (21).

In order to construct a consistent differential calculus with the properties $d^3 = 0$ and $d^2 \neq 0$, we have to introduce the second order differentials d^2x^i of the generators x^i [**5**].

Let $\Omega_C(\mathcal{A})$ be the free right unital associative module over the algebra $\mathcal{A}$ generated by all monomials composed from powers of dx^i, d^2x^i, $i = 1, \ldots, n$. If we attribute the grade 0 to any element of $\mathcal{A}$ and the grade 1 and 2 respectively to the differentials dx^i and d^2x^i, then the module $\Omega_C(\mathcal{A})$ becomes an **N**-graded module.

The commutation relations between x^i and dx^j are defined by (21). Now we find four sorts of commutation relations: among the first order differentials dx^i; between x^i and d^2x^j; between dx^i and d^2x^j; among the second order differentials d^2x^i.

Assume that the differential d satisfies the graded q-Leibniz rule (1) and that there are no relations between dx^i and d^2x^j. Differentiating (21), we get commutation relations among the first order differentials and between x^i and d^2x^j at once. In fact,

$$\begin{aligned} d(x^i dx^j - C^{ij}_{kl} dx^k x^l) &= dx^i\, dx^j + x^i d^2 x^j \\ &\quad - C^{ij}_{kl} d^2 x^k x^l - q C^{ij}_{kl}\, dx^k\, dx^l \\ &= 0. \end{aligned}$$

From this equality we have two kinds of commutation relations, supposing that the terms $dx^i\, dx^j$ and $x^i\, d^2 x^j$ must cancel separately

$$dx^i\, dx^j = q C^{ij}_{kl}\, dx^k\, dx^l, \tag{24}$$

$$x^i\, d^2\, x^j = C^{ij}_{kl}\, d^2\, x^k\, x^l. \tag{25}$$

The first differentiation of (25) gives, at once, the commutation relations between the first and second order differentials

$$dx^i\, d^2\, x^j = q^2 C^{ij}_{kl}\, d^2\, x^k\, dx^l. \tag{26}$$

The second differentiation of (25) gives the commutation relations among second order differentials (26)

$$d^2\, x^i\, d^2\, x^j = q C^{ij}_{kl}\, d^2\, x^k\, d^2\, x^l. \tag{27}$$

Now we have five sorts of commutation relations: (21), (24), (25), (26) and (27), where the last four are obtained from the relations (21) by the differentiation. If we define a multiplication law on the $\Omega_C(\mathcal{A})$ by these commutation relations, then the module $\Omega_C(\mathcal{A})$ becomes a unital associative algebra generated by x^i, dx^i, d^2x^i, $i = 1, \dots, n$. Obviously, the bimodule ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ is embedded into the algebra $\Omega_C(\mathcal{A})$.

Consider the commutation relations (24). By the differentiation of (24), we get the Wess-Zumino-like consistency condition for the operator C

$$d(dx^i\,dx^j - qC^{ij}_{kl}\,dx^k\,dx^l) = (\delta^i_k\delta^j_l - qC^{ij}_{kl})\,(\delta^k_s\delta^l_t + C^{kl}_{st})\,d^2x^s\,dx^t = 0,$$

supposing that the relations (26) hold, or in the tensor form

$$(E_{12} - qC_{12})(E_{12} + C_{12}) = 0. \tag{28}$$

The third differentiation of (16) gives the consistency condition

$$(E_{12} - B_{12})(E_{12} + C_{12})(-E_{12} + qC_{12}) = 0. \tag{29}$$

Now, one can see that the conditions (28) and (22) imply the condition (29).

As it follows from the paper [**4**], another two sorts of commutation relations exist: between the derivatives and the variables

$$\partial_j x^i = \delta^i_j + C^{ik}_{jl}x^l\partial_k, \tag{30}$$

and between the derivatives and the first order differentials

$$\partial_j\,dx^i = (C^{-1})^{ik}_{jl}\,dx^l\partial_k. \tag{31}$$

The relations (30) follow from the Leibniz rule (17) if we consider both ∂_j and x^i as operators. The relations (31) can be obtained from the assumption $\partial_j dx^i - D^{ik}_{jl}dx^l\partial_k = 0$, where the tensor D is to be determined. Multiplying the lefthand side of the last equation by x^r from the right side and using (21) and (31), we see that the equality

$$(\partial_j\,dx^i - D^{ik}_{jl}\,dx^i\partial_k)x^r = D^{ir}_{st}C^{su}_{jv}x^v(\partial_u\,dx^t - D^{tm}_{up}dx^p\partial_m) = 0$$

requires $D = C^{-1}$.

For construction of the consistent differential calculus with $d^3 = 0$, we add the relations between the derivatives and the second order differentials

$$\partial_j d^2 x^i = (C^{-1})^{ik}_{jl}\, d^2 x^l \partial_k \tag{32}$$

to the commutation relations obtained.

These relations can be obtained if we assume that

$$\partial_j d^2 x^i - K^{ik}_{jl}\, d^2 x^l \partial_k = 0.$$

Then we find the tensor K by multiplying this equation by x^r from the right side and commuting x^r through to the left by the commutation relations (25) and (30). Then we have

$$\begin{aligned}(\partial_j\, d^2x^i - K^{ik}_{jl}\, d^2x^l\partial_k)x^r & \\ &= (C^{-1})^{ir}_{jt}\, d^2x^t + (C^{-1})^{ir}_{st} C^{su}_{jv} x^v \partial_u d^2 x^t \\ &\quad - K^{ir}_{jt}\, d^2x^t - K^{ir}_{st} C^{su}_{jv} (C^{-1})^{tp}_{ub} x^v\, d^2x^b \partial_p \\ &= (C^{-1})^{ir}_{st} C^{su}_{jv} x^v (\partial_j\, d^2x^t - (C^{-1})^{tp}_{ub}\, d^2x^b \partial_p) = 0,\end{aligned}$$

if and only if $K = C^{-1}$.

Finally, in the paper [**4**], the authors show that the commutation relations among the derivatives

$$\partial_i \partial_j = F^{lk}_{ji} \partial_k \partial_l$$

lead to the two conditions of consistency

$$(E_{12} + C_{12})(E_{12} - F_{12}) = 0, \tag{33}$$

$$C_{12} C_{23} F_{12} = F_{23} C_{12} C_{23}. \tag{34}$$

Comparing (22) and (33), we can easily see that if F is equal to B, then (33) holds.

The equation (34) is a Yang-Baxter equation. Another two Yang-Baxter equations appear if we multiply the commutation relations (16) and (24) from the right side by dx^r and d^2x^r, respectively, and using the corresponding commutation relations commute dx^r and d^2x^r through to the left

$$(35)\ (\delta^i_k\delta^j_l - B^{ij}_{kl})\colon x^k\, x^l\, dx^r = (\delta^i_k\delta^j_l - B^{ij}_{kl})\colon C^{l\,r}_{st} C^{ks}_{uv}\, dx^u\, x^v\, x^t = 0,$$

$$(36)\ (\delta^i_k\delta^j_l - C^{ij}_{kl})\, dx^k\, dx^l\, d^2\, x^r = (\delta^i_k\delta^j_l - C^{ij}_{kl}) q^2 C^{lr}_{st} C^{ks}_{uv}\, d^2x^u\, dx^v\, dx^t = 0.$$

We get the Yang-Baxter equation

$$B_{12}C_{23}C_{12} = C_{23}C_{12}B_{23}, \tag{37}$$

as a solution of (35). In detail,

$$\begin{aligned}
&\delta^i_k\delta^j_l C^{lr}_{st}C^{ks}_{uv}\, dx^u\, x^v\, x^t = B^{ij}_{kl}C^{lr}_{st}C^{ks}_{uv}\, dx^u\, x^v\, x^t \Longrightarrow\\
&C^{jr}_{st}C^{is}_{uv}B^{vt}_{ab}\, dx^u\, x^a\, x^b = B^{ij}_{kl}C^{lr}_{st}C^{ks}_{uv}\delta^v_a\delta^t_b\, dx^u\, x^a\, x^b \Longrightarrow\\
&C^{jr}_{st}C^{is}_{uv}B^{vt}_{ab}\, dx^u\, x^a\, x^b = B^{ij}_{kl}\colon C^{l\,r}_{sb}C^{ks}_{ua}\, dx^u\, x^a\, x^b,
\end{aligned}$$

or in the tensor form

$$C_{23}C_{12}B_{23} = B_{12}C_{23}C_{12}.$$

By the same way as a solution of (36), we get the following Yang-Baxter equation:

$$C_{12}C_{23}C_{12} = C_{23}C_{12}C_{23}. \tag{38}$$

3.2 *Exterior calculus on the quantum plane.* Now we consider all commutation relations and consistency conditions obtained above in the case of the quantum plane, which is the associative unital algebra generated by the variables x^i, $i = 1, \dots, n$, satisfying the commutation relation $x^ix^j = qx^jx^i$, $i < j$, which can be rewritten by means of the $\hat{R}$-matrix

$$x^ix^j = \frac{1}{q}\hat{R}^{ij}_{kl}x^kx^l \quad \text{or} \quad (\delta^i_k\delta^j_l - \frac{1}{q}\hat{R}^{ij}_{kl})x^kx^l = 0, \tag{39}$$

where

$$\hat{R}^{ij}_{kl} = \delta^i_k \delta^j_l (1 + (q-1)\delta^{ij}) + \left(q - \frac{1}{q}\right) \delta^i_k \delta^j_l \theta(j-i),$$
$$\theta(j-i) = \begin{cases} 1 & \text{if } j > i, \\ 0 & \text{if } j \leq i, \end{cases}$$

i.e., we have $B = q^{-1}\hat{R}$. As was shown in [**4**], the consistency condition (22) holds if one chooses the values $q\hat{R}$ or $q^{-1}\hat{R}^{-1}$ for the tensor C, i.e.,

$$(E - q^{-1}\hat{R})(E + q\hat{R}) = 0 \quad \text{or} \quad (E - q^{-1}\hat{R})(E + q^{-1}\hat{R}^{-1}) = 0, \tag{40}$$

respectively, where E is the unit matrix, q^{-1} and q are the eigenvalues of the $\hat{R}$-matrix.

We show that the consistency condition (28) is satisfied only for the value $C = q\hat{R}$. Here we make use of the identities

$$\hat{R}^2 = E + (q - q^{-1})\hat{R} \quad \text{and} \quad \hat{R}^{-1} = \hat{R} + (q^{-1} - q)E. \tag{41}$$

If $C = q\hat{R}$, we have

$$(E - q^2\hat{R})(E + q\hat{R}) = (1 - q^3)E - (q^2 - q + q^4 - q^2)\hat{R}. \tag{42}$$

As q is the cubic root of unity, the coefficients are equal to zero.

However, if $C = q^{-1}\hat{R}^{-1}$, then

$$\begin{aligned} (E - \hat{R}^{-1})(E + q^{-1}\hat{R}^{-1}) &= (-2q^{-1}E - \hat{R})(q^{-2}E + q^{-1}\hat{R}) \\ &= (q-1)(E + q\hat{R}). \end{aligned} \tag{43}$$

Therefore we can only choose C to be $q\hat{R}$.

In our case, the three Yang-Baxter equations (34), (37) and (38) reduce to the single equation

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}, \tag{44}$$

as well as in [**4**].

By means of $\hat{R}$-matrix, we rewrite all commutation relations obtained in section 3.1. Now we have

$$\begin{aligned} x^i dx^j &= q\hat{R}^{ij}_{kl}\, dx^k\, x^l, & x^i d^2x^j &= q\hat{R}^{ij}_{kl} d^2x^k x^l, \\ dx^i\, dx^j &= q^2\hat{R}^{ij}_{kl}\, dx^k\, dx^l, & dx^i d^2x^j &= \hat{R}^{ij}_{kl} d^2\, x^k\, dx^l, \\ \partial_i\partial_j &= \frac{1}{q}\hat{R}^{lk}_{ji}\partial_k\partial_l, & d^2x^i d^2x^j &= q^2\hat{R}^{ij}_{kl} d^2x^k d^2x^l, \\ \partial_j\, dx^i &= \frac{1}{q}(\hat{R}^{-1})^{ik}_{jl}\, dx^l\partial_k, & \partial_j d^2x^i &= \frac{1}{q}(\hat{R}^{-1})^{ik}_{jl} d^2x^l\partial_k. \end{aligned}$$

3.3 *Algebra of differential forms on the two-dimensional quantum plane.* In the case of the two-dimensional quantum plane, we denote $x^1 = x$, $x^2 = y$. Since in the two-dimensional case the $\hat{R}$-matrix is equal to

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix},$$

we rewrite explicitly all commutation relations of Section 3.2:

$$\begin{aligned} x\,dx &= q^2\,dx\,x, & x\,d^2x &= q^2\,d^2x\,x, \\ x\,dy &= q\,dy\,x + (q^2-1)\,dx\,y, & x\,d^2y &= q\,d^2yx + (q^2-1)\,d^2x\,y, \\ y\,dx &= q\,dx\,y, & y\,d^2x &= q\,d^2x\,y, \\ y\,dy &= q^2\,dy\,y, & y\,d^2y &= q^2\,d^2y\,y, \\ dx\,dy &= q\,dy\,dx, & d^2x\,d^2y &= q\,d^2y\,d^2x, \end{aligned}$$

$$\begin{aligned} dx\,d^2x &= qd^2x\,dx, \\ dx\,d^2y &= d^2y\,dx + (q-q^{-1})d^2x\,dy, \\ dy\,d^2x &= d^2x\,dy, \\ dy\,d^2y &= q\,d^2y\,dy, \qquad \partial_x\partial_y = q^{-1}\partial_y\partial_x, \end{aligned}$$

$$\begin{aligned} \partial_x\,dx &= q^{-2}\,dx\partial_x, & \partial_x\,d^2x &= q^{-2}\,d^2x\,x, \\ \partial_x\,dy &= q^{-1}\,dy\partial_x, & \partial_x\,d^2y &= q^{-1}\,d^2y\,\partial_x, \\ \partial_y\,dx &= q^{-1}\,dx\partial_y & \partial_y\,d^2x &= q^{-1}\,d^2x\partial_y \\ &\quad + (1-q^{-2})\,dy\partial_x, & &\quad + (1-q^{-2})\,dy\partial_x, \\ \partial_y\,dy &= q^{-2}\,dy\,\partial_y, & \partial_y\,d^2y &= q^{-2}\,d^2y\partial_y. \end{aligned}$$

The direct calculation of $d^3 f$ shows that the requirements $(dx)^3 = 0$ and $(dy)^3 = 0$ imply $d^3 f = 0$. In fact, all the terms except $(dx)^3$ and $(dy)^3$ cancel by using appropriate commutation relations.

We add these two requirements to the commutation relations obtained by defining the multiplication law on the graded algebra $\Omega_C(\mathcal{A})$. This algebra splits into the direct sum $\Omega_C(\mathcal{A}) = \oplus_{k=0}^{\infty} \Omega_C^k(\mathcal{A})$ of its subspaces of homogeneous elements of grade k.

Let F_{2k}^{μ} be any monomial of grade $2k$ on second order differentials

$$F_{2k}^{\mu} = (d^2x)^{m_1}(d^2y)^{m_2}, \tag{45}$$

where $k \geq 1$, $m_1 + m_2 = k$, μ is the multi-index entirely determined by (m_1, m_2). Then the even form $\omega_e \in \Omega_C^{2k}(\mathcal{A})$ can be written as

$$\begin{aligned}\omega_e = {} & F_{2k}^{\mu} f_{00} + F_{2(k-1)}^{\nu}(dx\, dx\, f_{11} + dx\, dy f_{12} + dy\, dy\, f_{22}) \\ & + F_{2(k-2)}^{\eta}\, dx\, dx\, dy\, dy\, h_{22},\end{aligned}$$

and the odd form $\omega_0 \in \Omega_C^{2k+1}(\mathcal{A})$ as

$$\omega_0 = F_{2k}^{\mu}(dx\, f_{10} + dy\, f_{01}) + F_{2(k-1)}^{\nu}(dx\, dx\, dy\, h_{21} + dx\, dy\, dy\, h_{12}).$$

Differentiating ω_e and ω_0, we get

$$\begin{aligned}d(\omega_e) = {} & q^{2k} F_{2k}^{\mu}(dx\partial_x + dy\partial_y) f_{00} \\ & - q^{2k-1} F_{2(k-1)}^{\nu}((d^2x\, dx + dx\, dx\, dy\partial_y) f_{11} \\ & \qquad - (qd^2x\, dy + d^2y\, dx) f_{12} + (d^2y\, dy + qdx\, dy\, dy\partial_x) f_{22}) \\ & - q^{2(k-2)} F_{2(k-2)}^{\eta}(q^2 d^2x\, dx\, dy\, dy + d^2y\, dx\, dx\, dy) h_{22}, \\ d(\omega_0) = {} & q^{2k} F_{2k}^{\mu}((d^2x + qdx\, dx\partial_x) f_{10} + dx\, dy(q\partial_y f_{10} + \partial_x f_{01}) \\ & \qquad + (d^2y + qdy\, dy) f_{01}) \\ & + q^{2(k-1)} F_{2(k-1)}^{\nu}((-d^2x\, dx\, dyqd^2y\, dx\, dx)\, h_{21} \\ & \qquad + dx\, dx\, dy\, dy(q\partial_x\, h_{12} + \partial_y\, h_{21}) \\ & \qquad + (qd^2x\, dy\, dy - q^2 d^2y\, dx\, dy) h_{12}),\end{aligned}$$

respectively. Hence, one can easily see that the differential d is a linear endomorphism of degree 1.

The direct calculation gives $d^3(\omega) = 0$ for all $\omega \in \Omega_C(\mathcal{A})$.

Thus we have proven the following:

Proposition 1. *The algebra $\Omega_C(\mathcal{A})$ of differential forms with the requirement $(dx)^3 = (dy)^3 = 0$ is a graded differential algebra with respect to the exterior differential d satisfying the q-Leibniz rule, i.e., $d^3 f = 0$ for all $f \in \mathcal{A}$, $d^3\omega = 0$ for all $\omega \in \Omega_C(\mathcal{A})$.*

4. Acknowledgments. The author is very grateful to the organizers of the NATO Advanced Study Institute "Special Functions 2000: Current Perspective and Future Directions" for financial support and hospitality. Also, the author expresses gratitude to Professor A. Borowiec for fruitful discussion of the results obtained the obtained results, and acknowledges the financial support of Estonian Science Foundation under the grants No.1134 and No.4515.

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