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ORTHOGONAL POLYNOMIALS WITH RESPECT TO A DIFFERENTIAL OPERATOR. EXISTENCE AND UNIQUENESS

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ABSTRACT. A new type of orthogonal polynomial connected with linear differential operators, intimately related with Sobolev orthogonal polynomials and Hermite-Padé polynomials, is introduced. We study the question of uniqueness of the sequence of orthogonal polynomials arising from this construction. As we show, this problem is related to the analytic properties of the fundamental system of solutions of the operator. The notion of T-system of Tchebyshev plays a key role in the analysis. Some examples of general classes of operators which produce a unique system of polynomials are given.

1. Introduction.

1.1 *Definition of the main object.* In the last two decades there has been a growing interest in different generalizations of the notion of orthogonal polynomials. To name a few, we have *Hermite-Padé polynomials*, [**2, 10**], and the *matrix orthogonal polynomials*, [**1, 3**]. They arise in a natural way in problems of approximation theory, mathematical physics and number theory. From the theoretical point of view, the study of these constructions pose questions whose solution requires new methods and interrelations of classical techniques of analysis.

In this paper we introduce sequences of polynomials orthogonal with respect to a linear homogeneous differential operator.

Definition. Let $\sigma(x)$ be a positive Borel measure on the real line and $\{\rho_k(x)\}_{k=0}^m$, $\rho_m \equiv 1$, be a set of functions such that $\rho_k(x)\,d\sigma(x)$ has

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finite moments, $k = 0, 1, \dots, m$. Denote

$$L^{(m)} = \sum_{k=0}^{m} \rho_k(x) \frac{d^k}{dx^k}. \tag{1}$$

We say that $\{Q_n\}$, $n \in N$, is a sequence of *orthogonal polynomials with respect to the differential operator* (OPDO) $L^{(m)}$ if $\deg Q_n \leq n$ and

$$\int L^{(m)}[Q_n(x)] P(x)\, d\sigma(x) = 0 \tag{2}$$

for any polynomial $P(x)$ such that $\deg P \leq n - 1$.

We note that the coefficients of the differential operator can be taken as Borel measures on R with finite moments; then the orthogonality relations take the form

$$\sum_{k=0}^{m} \int Q_n^{(k)}(x) P(x)\, d\mu_k(x) = 0, \quad \deg P \leq n - 1. \tag{3}$$

When $m = 0$, we obtain the classical construction of *orthogonal polynomials*:

$$\int Q_n(x) P(x)\, d\mu(x) = 0, \quad \deg P \leq n - 1.$$

1.2 *Relation with Sobolev orthogonal polynomials and Hermite-Padé linear forms.* We recall that *Sobolev orthogonal polynomials* are defined as the sequence of polynomials $\{Q_n(x)\}$ such that $\deg Q_n \leq n$ and

$$\sum_{k=0}^{m} \int_a^b Q_n^{(k)}(x) P^{(k)}(x)\, d\lambda_k(x) = 0, \quad \deg P \leq n - 1. \tag{4}$$

Let us see how such polynomials may be reduced to OPDO. For simplicity, we consider the case $m = 1$ with $d\lambda_1(x) = \omega(x)\, dx$ and $\omega \in C^1[a, b]$. Then (4) reduces to

$$0 = \int_a^b Q_n(x) P(x)\, d\lambda_0(x) + \int_a^b Q_n'(x) P'(x) \omega(x)\, dx.$$

Integrating by parts the second term in the righthand side, one obtains

$$
\begin{aligned}
0 &= \int_a^b Q_n(x)P(x)\,d\lambda_0(x) \\
&\quad - \int_a^b Q_n'(x)P(x)\omega'(x)\,dx + Q_n'(b)P(b)\omega(b) \\
&\quad - Q_n'(a)P(a)\omega(a) - \int_a^b Q_n''(x)P(x)\omega(x)\,dx \\
&= \int_a^b Q_n(x)P(x)\,d\mu_0(x) \\
&\quad + \int Q_n'(x)P(x)\,d\mu_1(x) + \int Q_n''(x)P(x)\,d\mu_2(x),
\end{aligned}
\tag{5}
$$

where

$$
\begin{aligned}
&d\mu_0(x) = d\lambda_0(x) \\
&d\mu_1(x) = \omega(b)\delta(x-b) - \omega(a)\delta(x-a) - \omega'(x)\,dx,
\end{aligned}
$$

and

$$d\mu_2(x) = -\omega(x)\,dx.$$

As we can see, (5) has a form as in (3). For general m, an analogous reduction can be carried out.

Hermite-Padé polynomials are also intimately connected with OPDO. They are defined as follows:

Let $\{n_k\}_{k=0}^m$ be a set of indices in Z_0^{m+1}, $\sum_{k=0}^m n_k = n$. Then $\{Q_{n,k}\}_{k=0}^m$ are the *Hermite-Padé polynomials* (type I) if $\deg Q_{n,k} \le n_k$ and

$$\sum_{k=0}^m \int Q_{n,k}(x)P(x)\,d\mu_k(x) = 0, \quad \deg P \le \sum_{k=0}^m n_k - 1. \tag{6}$$

It is obvious that (6) and (3) have similar expressions. Although it is not possible to reduce one case to the other, as was done with SOP, the methods of investigation of Hermite-Padé polynomials turn out to be effective in the study of OPDO. For this reason, OPDO serve as an intermediate link when trying to apply to SOP the widely developed analytical methods of the theory of Hermite-Padé approximants.

1.3 *Formal properties of* OPDO. The determination of the OPDO sequence $\{Q_n\}$ defined by (2) or (3) can be reduced to the solution of a system of n algebraic linear homogeneous equations on the $n+1$ coefficients of Q_n, thus the existence is guaranteed. Unlike SOP, it is not possible to affirm uniqueness up to a constant factor. This situation also occurs in Hermite-Padé polynomials.

We say that n is a *normal index* if, for a given n, the solution is uniquely determined up to a constant factor.

A *sufficient condition for uniqueness* is that any polynomial satisfying (2) or (3) has exact degree n. In fact, because of the linearity in the construction, the difference of two solutions is also a solution; therefore, two different solutions of equal degree not multiples of each other generate another one of smaller degree contradicting our assumption.

1.4 *Description of results.* In this paper we study the problem of normality of OPDO. Normality for index n reduces to the fact that the fundamental system of solutions of operator $L^{(m)}$ and its derivatives up to n form Tchebyshev T-systems of functions, (see Theorem 2). For each n, this fact can be expressed recurrently in terms of the coefficients in $L^{(m)}$ (see Theorem 3). As for Hermite-Padé polynomials, normality turns out to be a complicated problem. Nevertheless, when $m = 1$, we give a simple sufficient condition on the coefficients of $L^{(1)}$ for normality of all indices (see Theorem 4). In particular, from Theorem 4, it follows:

Let (Q_n) *be defined by*

$$\int_a^b L^{(1)}[Q_n(x)]P(x)\,d\sigma(x) = 0, \quad \deg P \le n-1,$$

where

$$L^{(1)} = \frac{d}{dx} + \int_c^d \frac{d\tau(t)}{x-t},$$

and $[a,b] \cap [c,d] = \varnothing$. *Then, for all* n, $\deg Q_n = n$, *and therefore all indices are normal.*

This result is analogous to the corresponding one for the Nikishin systems of Hermite-Padé polynomials, (see [**9**, Theorem 3]).

2. Uniqueness of orthogonal polynomials with respect to a differential operator.

2.1 *Normality, general case.* As mentioned in the introduction, normality for OPDO is connected with the notion of T-system on the interval $\Delta = [a, b]$. We recall the definition, (see [**6**]).

Definition. A set $\{u_v(x)\}_{\nu=0}^{n}$ of continuous functions on Δ is called a *Tchebyshev system* (T-*system*) on Δ if any linear combination of it

$$\sum_{\nu=0}^{n} \alpha_\nu u_\nu(x)$$

has at most n zeros on this interval. If for each $n' = 0, \dots, n$, the set of functions $\{u_\nu\}_{\nu=0}^{n'}$ forms a T-system it is called a *Markov system* (M-*system*).

Theorem 1. *Given* $L^{(m)}$ *as in* (1), *let us assume that* $\{L^{(m)}(x^\nu)\}_{\nu=0}^{n}$ *is an* M-*system on* Δ. *Then* $\deg Q_n = n$.

Proof. Assume that $\deg Q_n = n' < n$, then

$$L^{(m)}[Q_n(x)] = \sum_{\nu=0}^{n'} \alpha_\nu L^{(m)}(x^\nu). \tag{7}$$

Because of the orthogonality relations (2), $L^{(m)}[Q_n(x)]$ has at least n zeros on Δ, but, by assumption and (7), it follows that $L^{(m)}[Q_n(x)]$ cannot have more than $n' < n$ zeros on Δ, bringing us to a contradiction. Therefore, $\deg Q_n = n$. $\square$

Markov proved, (see [**6**]), that a system $\{u_k(x)\}_{k=0}^{m-1}$ of n times continuously differentiable functions on $[a, b]$ is a T-system if

$$W(u_0, \dots, u_{m-1}) = \begin{vmatrix} u_0(x) & u_1(x) & \cdots & u_{m-1}(x) \\ u_0'(x) & u_1'(x) & \cdots & u_{m-1}'x \\ \vdots & \vdots & & \\ u_0^{(m-1)}(x) & u_1^{(m-1)}(x) & \cdots & u_{m-1}^{(m-1)}(x) \end{vmatrix} \neq 0. \tag{8}$$

This result allows us to establish a correspondence between T-systems and *fundamental systems of solutions* of linear differential equations because any fundamental solution $(u_0, \dots, u_{m-1})$ of $L^{(m)}[u] = 0$ satisfies (8). Therefore, any such solution is a T-system.

Theorem 2. *Let $\{u_0, \dots, u_{m-1}\}$ be a fundamental system of solutions of $L^{(m)}(u) = 0$. Let us assume that $n \in N$ is given and that*

$$\{(u_0^{(\nu)}(x), \dots, u_{m-1}^{(\nu)}(x))\}$$

is a T-system for $\nu = 1, 2, \dots, n+1$. Then $\deg Q_n = n$ where Q_n is the nth orthogonal polynomial with respect to $L^{(m)}$.

In the proof of Theorem 2, we use a well known, (see [**4**]), integral representation of a fundamental system of the operator $L^{(m)}$. If $\{u_0, \dots, u_{m-1}\}$ is a fundamental system, then m functions $\{v_0, \dots, v_{m-1}\}$ exist that can be determined recurrently by means of relations.

$$\begin{aligned} &v_0 = u_0, \\ &v_r = \frac{d}{dx}\left[\frac{1}{v_{r-1}}\frac{d}{dx}\left[\frac{1}{v_{r-2}}\frac{d}{dx}\left[\cdots\frac{d}{dx}\left[\frac{1}{v_1}\frac{d}{dx}\left(\frac{u_r}{v_0}\right)\right]\cdots\right]\right]\right], \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad r = 1, \cdots, m-1. \end{aligned}$$

Therefore,

$$\begin{aligned} &u_0 = v_0, \quad u_1 = v_0 \int v_1\, dx, \\ &u_r = v_0 \int v_1 \int \cdots \int v_r (dx)^r, \quad r = 2, 3, \cdots, m-1, \end{aligned} \tag{9}$$

and the operator $L^{(m)}$ can be expressed as

(10)

$$L^{(m)}[u] = D\left[\frac{1}{v_{m-1}}D\left[\frac{1}{v_{m-2}}\left[\cdots\left[\frac{1}{v_1}D\left[\frac{u}{v_0}\right]\right]\cdots\right]\right]\right], \quad D = \frac{d}{dx}.$$

Proof of Theorem 2. By use of Theorem 1, it is sufficient to prove that under the present assumptions, $\{L^{(m)}[x^\nu]\}_{\nu=0}^n$ forms an M-system. Suppose this is not so. Let us fix $n' = 1, \dots, n$. Then, due to a result of Markov, (see [**5**]), constants $\{\alpha_0, \dots, \alpha_{n'}\}$ exist such that

$$\sum_{\nu=0}^{n'} \alpha_\nu L^{(m)}(x^\nu) = L^{(m)}\left[\sum_{\nu=0}^{n'} \alpha_\nu x^\nu\right]$$

has at some point $x_0 \in \Delta$ a zero of order $n'+1$. Using the representation of the operator $L^{(m)}$ in form (10), we have

$$L^{(m)}\left[\sum_{\nu=0}^{n'} \alpha_\nu x^\nu\right] = D\left[\frac{1}{v_{m-1}} D\left[\frac{1}{v_{m-2}}\left[\cdots D\left[\frac{\sum_{\nu=0}^{n'} \alpha_\nu x^\nu}{v_0}\right]\cdots\right]\right]\right].$$

From this it follows that a constant c_{m-1} exists such that

$$\frac{1}{v_{m-1}} D\left[\frac{1}{v_{m-2}}\left[\cdots D\left[\frac{\sum_{\nu=0}^{n'} \alpha_\nu x^\nu}{v_0}\right]\cdots\right]\right] + c_{m-1},$$

and therefore

$$D\left[\frac{1}{v_{m-2}}\left[\cdots D\left[\frac{\sum_{\nu=0}^{n'} \alpha_\nu x^\nu}{v_0}\right]\cdots\right]\right] + c_{m-1} v_{m-1}$$

has, at the point x_0, a zero of order $n'+2$. Repeating this process m times, we obtain that constants $c_0, c_1, \dots, c_{m-1}$ exist for which

$$\sum_{\nu=0}^{n'} \alpha_\nu x^\nu + c_0 v_0 + c_1 v_0 \int v_1\, dx + \cdots + c_{m+1} v_0 \int v_1 \int \cdots \int v_{m-1} (dx)^{m-1}$$

has, at the point x_0, a zero of order $n'+m+1$. Taking into account (9), one obtains that

$$\sum_{\nu=0}^{n'} \alpha_\nu x^\nu + c_0 u_0(x) + c_1 u_1(x) + \cdots + c_{m-1} u_{m-1}(x)$$

has, at the point x_0, a zero of order $n'+m+1$. Differentiating this expression $(n'+1)$ times, one has that

$$c_0 u_0^{(n'+1)}(x) + c_1 u_1^{(n'+1)}(x) + \cdots + c_{m-1} u_{m-1}^{(n'+1)}(x)$$

has, at the point x_0, a zero of order m which contradicts our assumption that $\{u_k^{(n'+1)}(x)\}_{k=0}^{m-1}$ is a T-system and it cannot have more than $m-1$ zeros. This concludes the proof. $\square$

Theorem 2 gives us a condition of normality in terms of a fundamental system of solutions. In order to reformulate this condition in terms of the coefficients of the differential operator, we state the following theorem:

Theorem 3. *Assume that $L^{(m)}$ has infinitely differentiable coefficients $\{\rho_k\}_{k=0}^m$ on Δ. Define recurrently the system of functions $\{\rho_{k,n'}\}_{k=0}^m$, $n' = 1, 2, \dots,$ as follows*

$$\{\rho_{k,0} := \rho_k\}_{k=0}^m$$

and

$$\rho_{k,n'+1} = \rho_{k,n'} + \rho_{0,n'} \left(\frac{\rho_{k+1,n'}}{\rho_{0,n'}} \right)',$$
$$k = 0, \dots, m-1, \quad n' \in N,$$
$$\rho_{m,n'} \equiv 1, \quad n' \in N.$$

Then $\deg Q_n = n$ *if, for all* $n' = 0, \dots, n$, $\rho_{0,n'}(x) \neq 0$, $x \in \Delta$.

Proof of Theorem 3. The last coefficient ρ_0 of the differential operator

$$L^{(m)}[u] = u^{(m)}(x) + \rho_{m-1}(x) u^{(m-1)}(x) + \cdots + \rho_1(x) u'(x) + \rho_0(x) u(x) \tag{11}$$

and the Wronskian of a fundamental system of solutions $\{u_0, \dots, u_{m-1}\}$ are connected by the following formula, (see [**4**]),

$$\rho_0(x) = (-1)^m \frac{W(u_0', \dots, u_{m-1}')}{W(u_0, \dots, u_{m-1})}.$$

Therefore, in order that $\{u_k'\}_{k=0}^{m-1}$ be a T-system, it is sufficient that

$$\rho_0(x) \neq 0, \quad x \in \Delta.$$

We wish to derive an analogous condition for the system $\{u_k^{(n')}\}_{k=0}^{m-1}$, $0 \le n' \le n+1$. We must find a differential operator for which $\{u_k^{(n')}\}_{k=0}^{m-1}$ is a fundamental solution and look at its last coefficient.

To this end, we divide (11) by ρ_0, differentiate once, and multiply by ρ_0. As a result, one has

$$\begin{aligned} L_1^{(m)}[u^{(1)}] = {} & \frac{d^m}{dx^m} u^{(1)}(x) \\ & + \left\{ \rho_{m-1}(x) + \rho_0(x) \left(\frac{1}{\rho_0(x)} \right)' \right\} \frac{d^{m-1}}{dx^{m-1}} u^{(1)}(x) \\ & + \left\{ \rho_{m-2}(x) + \rho_0(x) \left(\frac{\rho_{m-1}(x)}{\rho_0(x)} \right)' \right\} \\ \times {} & \frac{d^{m-2}}{dx^{m-2}} u^{(1)}(x) + \cdots \\ & + \left\{ \rho_0(x) + \rho_0(x) \left(\frac{\rho_1(x)}{\rho_0(x)} \right)' \right\} u^{(1)}(x). \end{aligned}$$

We obtain, using the same arguments as above, that $\{u_k''\}_{k=0}^{m-1}$ is a T-system if

$$\rho_{0,1} = \rho_0 + \rho_0 \left(\frac{\rho_1}{\rho_0} \right)' = (-1)^m \frac{W(u_0'', \dots, u_{m-1}'')}{W(u_0', \dots, u_{m-1}')} \ne 0, \quad x \in \Delta.$$

Continuing this process for $n' = 2, \dots, n-1$, one arrives at the statement of the theorem. $\square$

Remark. The process of the proof leads to the following expression for the ratio of the last coefficients of two consecutive operators $L_n^{(m)}$,

$$\begin{aligned} \frac{\rho_{0,n+1}}{\rho_{0,n}} = {} & 1 \\ & + \sum_{k=1}^{p=\min(m,n+1)}{}' \sum_{\{\nu_j^k\}_{j=0}^n \in \vee} F\left(\left\{ \frac{\rho_{0,j}}{\rho_{0,j+1}} \right\}_{j=0}^{n-1}, \left\{ \frac{\rho_{k,0}}{\rho_{0,0}} \right\}, \{\nu_j^k\}_{j=0}^n \right) \end{aligned}$$

where

$$F\left(\left\{\frac{\rho_{0,j}}{\rho_{0,j+1}}\right\}_{j=0}^{n-1}, \left(\frac{\rho_{k,0}}{\rho_{0,0}}\right), \{\nu_j^k\}_{j=0}^n\right)$$
$$= \left(\frac{\rho_{0,n-1}}{\rho_{0,n}}\left(\frac{\rho_{0,n-2}}{\rho_{0,n-1}} \cdots \left(\frac{\rho_{0,0}}{\rho_{0,1}}\left(\frac{\rho_{k,0}}{\rho_{0,0}}\right)^{(\nu_0^k)}\right)^{(\nu_1^k)} \cdots\right)^{(\nu_{n-1}^k)}\right)^{(\nu_n^k)}$$

and $\vee$ is the set of numbers $\{\nu_j^k\}_{j=0}^n$, $k = 1, 2, \dots, p$, such that

$$\begin{cases} \nu_n^k = 1, \\ \nu_j^k \in \{0,1\} \qquad j = 0, \dots, n-1, . \\ \sum_{j=0}^n \nu_j^k = k \end{cases}$$

This formula for $n = 1, 2, 3, \dots,$ gives us

$$\frac{\rho_{0,1}}{\rho_{0,0}} = 1 + \left(\frac{\rho_{1,0}}{\rho_{0,0}}\right)',$$
$$\frac{\rho_{0,2}}{\rho_{0,1}} = 1 + \left(\frac{\rho_{0,0}}{\rho_{0,1}}\left(\frac{\rho_{2,0}}{\rho_{0,0}}\right)'\right)' + \left(\frac{\rho_{0,0}}{\rho_{0,1}}\frac{\rho_{1,0}}{\rho_{0,0}}\right)',$$
$$\frac{\rho_{0,3}}{\rho_{0,2}} = 1 + \left(\frac{\rho_{0,1}}{\rho_{0,2}}\left(\frac{\rho_{0,0}}{\rho_{0,1}}\left(\frac{\rho_{3,0}}{\rho_{0,0}}\right)'\right)'\right)'$$
$$+ \left(\frac{\rho_{0,1}}{\rho_{0,2}}\left(\frac{\rho_{0,0}}{\rho_{0,1}}\frac{\rho_{2,0}}{\rho_{0,0}}\right)'\right)'$$
$$+ \left(\frac{\rho_{0,1}}{\rho_{0,2}}\frac{\rho_{0,0}}{\rho_{0,1}}\left(\frac{\rho_{2,0}}{\rho_{0,0}}\right)'\right)' + \left(\frac{\rho_{0,1}}{\rho_{0,2}}\frac{\rho_{0,0}}{\rho_{0,1}}\frac{\rho_{1,0}}{\rho_{0,0}}\right)'.$$

Let us consider the system of functions

$$U_j(x) = L^{(m)}\left[\frac{x^j}{j!}\right], \quad j = 0, \dots, n-1,$$

and the corresponding system of functions, (see (9)),

$$V_0(x) = U_0(x)$$
$$V_j(x) = \frac{d}{dx}\left[\frac{1}{V_{j-1}}\frac{d}{dx}\left[\frac{1}{V_{j-2}} \cdots \frac{d}{dx}\left[\frac{1}{V_1}\frac{d}{dx}\left[\frac{U_j(x)}{V_0(x)}\right]\right]\right]\cdots\right],$$
$$j = 1, \dots, n-1.$$

For $j = 0, 1, 2, 3, \dots,$ as a consequence we get

$$\begin{aligned}
V_1 &= \left(\frac{L^{(m)}[x]}{\rho_{0,0}}\right)' = \left(\frac{x\rho_{0,0} + \rho_{1,0}}{\rho_{0,0}}\right)' \\
&= 1 + \left(\frac{\rho_{1,0}}{\rho_{0,0}}\right)' = \frac{\rho_{0,1}}{\rho_{0,0}}, \\
V_2 &= \left(\frac{1}{V_1}\left(\frac{L^{(m)}[x^2/2]}{V_0}\right)'\right)' \\
&= \left(\frac{\rho_{0,0}}{\rho_{0,1}}\left[\frac{(x^2/2)\rho_{0,0} + x\rho_{1,0} + \rho_{2,0}}{\rho_{0,0}}\right]'\right)' \\
&= \left(\frac{\rho_{0,0}}{\rho_{0,1}}\left(\frac{\rho_{2,0}}{\rho_{0,0}}\right)'\right)' + \left(\frac{\rho_{0,0}}{\rho_{0,1}}\frac{\rho_{1,0}}{\rho_{0,0}}\right)' \\
&\quad + \left(\frac{\rho_{0,0}}{\rho_{0,1}}x\left(1 + \left(\frac{\rho_{0,0}}{\rho_{0,0}}\right)'\right)\right)' = \frac{\rho_{0,2}}{\rho_{0,1}}.
\end{aligned}$$

Continuing this process we obtain that

$$V_j = \frac{\rho_{0,j}}{\rho_{0,j-1}}, \quad j = 1, 2, \dots, n-1.$$

Thus, we see that Theorem 3 establishes the formal correspondence between results of Theorems 1 and 2.

Example 1. Let

$$L^{(m)} = \sum_{k=0}^{m} P_k(x)\frac{d^k}{dx^k}, \tag{12}$$

where P_k is a polynomial such that $\deg P_k \le k$, $P_k \ne 0$, $k = 0, \dots, m-1$. The corresponding homogeneous differential equation for (12) is named after Euler, (see [**4**]). It is easy to check that, for all $n \in N$,

$$\rho_{0,n} = \text{const} \ne 0$$

which yields that all indices are normal.

2.2 *Normality for* $m = 1$. When $m = 1$, the conditions above naturally take a simplified form. In this case, our operator, (see (1)), reduces to

$$L^{(1)} = \frac{d}{dx} + \rho_0(x). \tag{13}$$

A (fundamental) solution of equation $L^{(1)}[u] = 0$ is

$$u(x) = \exp\left\{ -\int \rho_0(t)\, dt \right\} \tag{14}$$

and our system $\{Q_n\}$ of OPDO satisfies

$$\int_\Delta [Q_n(x)\rho_0(x) + Q_n'(x)]P(x)\, d\sigma(x), \quad \deg P \leq n-1,$$

or what is the same, (see (10)),

$$\int_\Delta \left[\frac{Q_n(x)}{u(x)}\right]' P(x)u(x)\, d\sigma(x) = 0, \quad \deg P \leq n-1. \tag{15}$$

The condition of normality for index n in Theorem 2 transforms into

$$u^{(n')}(x) \neq 0, \quad x \in \Delta, \quad n' = 1, 2, \ldots, n+1. \tag{16}$$

Example 2. For the system $\{Q_n\}$ of OPDO obtained from

$$0 = \int_0^{\pi/2} [Q_n(x)\sin x + Q_n'(x)\cos x]P(x)\, d\sigma(x), \quad \deg P \leq n-1,$$

it is possible to verify (16) for all n, (see [**6**]). Therefore, all indices are normal, $\deg Q_n = n$.

Let us give another condition for this case

Theorem 4. *Let $\rho_0(x)$ in* (13) *be infinitely differentiable in Δ and satisfy either*

$$\text{a)} \quad \frac{d^{n'}}{dx^{n'}}\rho_0(x) \leq 0, \quad n' = 0, \ldots, n, \tag{17}$$

or

$$(18) \qquad \text{b)} \quad (-1)^{n'} \frac{d^{n'}}{dx^{n'}} \rho_0(x) \geq 0, \quad n' = 0, \dots, n.$$

Then n is a normal index.

Proof. Let us first consider case a). We proceed by induction in order to prove that (16) takes place. For $n' = 1$, we have

$$u'(x) = -\rho_0(x), \quad u(x) > 0.$$

Assume that, for all $\nu = 2, \dots, n'$, $n' \leq n$,

$$u^{(\nu)} > 0.$$

Then, using Leibniz's formula, for n' one obtains

$$\begin{aligned} u^{(n')}(x) &= (-\rho_0 u)^{(n'-1)} \\ &= -\sum_{\nu=0}^{n'-1} \binom{n'-1}{\nu} \rho_0^{(\nu)} u^{(n'-1-\nu)} > 0. \end{aligned}$$

Case b) is treated analogously.

From this theorem we can derive several examples of classes of first degree linear operators $L^{(1)}$ for which all indices are normal (and $\deg Q_n = n$).

Example 3. The function $\rho_0(x) = x^\alpha$, $0 \leq \alpha \leq 1$, or

$$\rho_0(x) = \int_0^1 x^\alpha \, d\tau(\alpha)$$

satisfies condition (18) on any segment $\Delta = [a, b]$, $0 < a < b$.

Example 4. The function $\rho_0(x) = -e^{\alpha x}$, $\alpha \geq 0$, or more generally

$$\rho_0(x) = -\int_0^\infty e^{\alpha x} \, d\tau(\alpha)$$

satisfies (17) on any segment Δ.

The following example was stated in the introduction as a consequence of Theorem 4.

Example 5. Let

$$\rho_0(x) = \int_c^d \frac{d\tau(t)}{x-t}, \quad x \in [a,b], [a,b] \cap [c,d] = \varnothing,$$

then

$$\frac{d^n \rho_0(x)}{dx^n} = (-1)^n n! \int_c^d \frac{d\tau(t)}{(x-t)^{n+1}}.$$

It is easy to see that, if $\Delta = [a,b]$ is to the right of $[c,d]$, then (18) takes place, while if it is to the left, (17) is satisfied.

2.3 *Concluding remarks.* It would be desirable to obtain, for arbitrary fixed m, simple expressions in order that the conditions of Theorem 3 be fulfilled as we have for $m = 1$.

An important question for the further development of the theory is to study the localization of the zeros for the polynomials Q_n, $n \in N$. For example, if we know that the zeros are in a compact set of the complex plane, then advanced techniques of potential theory allow to describe the weak asymptotic behavior of $\{Q_n\}$, as $n \to \infty$. In such a way, due to the existing connection between SOP and OPDO indicated in the introduction, we could obtain similar results for general classes of Sobolev orthogonal polynomials.

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