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ON SAMPLING ASSOCIATED WITH SINGULAR STURM-LIOUVILLE EIGENVALUE PROBLEMS: THE LIMIT-CIRCLE CASE

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ABSTRACT. Sampling expansions are derived for solutions of second order singular Sturm-Liouville eigenvalue problems in the limit-circle case. In this setting special functions with continuous parameter satisfying such problems will be sampled in terms of these functions with a discrete parameter, sometimes orthogonal polynomials. As examples, the Legendre and Bessel functions are sampled. Sampling expansions of the associated integral transforms are also given. The analysis makes use of the approach derived by Fulton [**13**] and extends the range of examples studied by Butzer-Schöttler, Everitt and Zayed. A fully new example is the appearance of the Legendre function of the second kind in the analysis of sampling theory.

1. Introduction. Let $P_w(z)$, $w, z \in \mathbf{C}$, with $|z| < 1$ denote the Legendre function of the first kind, i.e. (see [**20, 23**])

$$P_w(z) = {}_2F_1\left(-w, w+1; 1; \frac{1-z}{2}\right), \tag{1.1}$$

where ${}_2F_1(a,b;c;\cdot)$ is the Gauss hypergeometric series. The following sampling expansion for $P_w(z)$ was derived in [**7**], (see also [**4**–**6**]):

$$\begin{aligned}\frac{P_{w-1/2}(x)}{w^2-1/4} &= \sum_{k=1}^{\infty} \frac{P_k(x)}{k(k+1)} \frac{(2k+1)\sin\pi(w-(k+(1/2)))}{\pi(w^2-(k+(1/2))^2)} \\ &\quad + \frac{\sin\pi(w+(1/2))}{\pi(w^2-(1/4))}\left\{\frac{1}{w^2-(1/4)} - \log\left(\frac{2}{1+x}\right) + 1\right\},\end{aligned} \tag{1.2}$$

where $w \in \mathbf{R} - \{1/2\}$, $x \in (-1, 1]$ and $P_k(\cdot)$ are the Legendre polynomials. The convergence of (1.2) is in the L^2-norm with respect to

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x for each $w \in \mathbf{R} - \{1/2\}$ and pointwise with respect to $w \in \mathbf{R} - \{1/2\}$ for each $x \in (-1, 1)$. Moreover, it is proved in [**6**, page 316] that the convergence is absolute and uniform with respect to w in any compact subset of $\mathbf{R}^+$ as well as with respect to x on the whole interval $[-1, 1]$. Expansion (1.2) can be simplified using the expansion of $\log(2/1 + x)$ in terms of the Legendre polynomials [**15**],

$$\log \frac{2}{1+x} = 1 + \sum_{k=1}^{\infty} (-1)^k P_k(x) \frac{2k+1}{k(k+1)}, \quad x \in (-1, 1]. \tag{1.3}$$

Substituting from (1.3) in (1.2), we obtain

$$P_{w-1/2}(x) = \sum_{k=0}^{\infty} P_k(x) \frac{(2k+1)\sin \pi(w - (k + (1/2)))}{\pi(w^2 - (k + (1/2))^2)}. \tag{1.4}$$

Noting that

$$\frac{2k+1}{w^2 - (k+1/2)^2} = \frac{1}{w - (k+1/2)} - \frac{1}{w + k + 1/2},$$

expansion (1.4) may be written in the following two-sided sampling expansion, (see [**6**, page 322]),

$$P_{w-1/2}(x) = \sum_{k=-\infty}^{\infty} P_{|k|}(x) \frac{\sin \pi(w - (k + (1/2)))}{\pi(w - (k + (1/2)))}.$$

The Legendre function $P_{w-1/2}(\cdot)$ satisfies the second order differential equation

$$-((1 - x^2)y')' + \frac{1}{4}y = w^2 y, \quad x \in (1, 1), w \in \mathbf{C}. \tag{1.5}$$

We notice that equation (1.5) has two singularities at ± 1. Both points are in the limit-circle case, which means that the eigenvalue problem associated with (1.5) should have two boundary conditions at ± 1, cf. [**16**]. There are several attempts to define these boundary conditions at the singular endpoints, among them is the boundary condition approach devised by Fulton [**12, 13**] and the Glazman-Krein-Naimark (GKN) conditions [**21**]. For historical comments concerning deriving boundary conditions at the limit-circle endpoints, see [**12, 13**]. Before we introduce the aim of this article, we mention some results concerning Legendre functions. In [**4**], the following continuous Legendre transform has been introduced

$$\hat{f}(\lambda) = \frac{1}{2}\int_{-1}^{1} f(x)P_\lambda(x)\,dx, \quad f \in L^2(-1,1). \tag{1.6}$$

Under the conditions $\sqrt{\lambda}\hat{f}(\lambda - 1/2)$ is a summable function, i.e., an $L^1(\mathbf{R})$-function in which f is continuous on $(-1,1)$, they derived the inversion formula, (cf. [**4**, page 57]),

$$f(x) = 4\int_0^\infty \hat{f}(\lambda - 1/2)P_{\lambda-1/2}(-x)\sin\pi\lambda\,d\lambda, \tag{1.7}$$

which has also been derived using a different technique in [**26**]. Other results concerning the Legendre transform include the sampling series derived in [**25**] and [**9**] by considering, respectively, the Legendre transforms

$$f^*(\lambda) = 2\sqrt{2}\pi\int_{-\pi/2}^{\pi/2} f(x)\sqrt{\cos x}\,P_{\sqrt{\lambda}-1/2}(\sin x)\,dx, \quad \lambda \in \mathbf{C}, \tag{1.8}$$

where $f \in L^2(-\pi/2,\pi/2)$ and

$$f^{**}(\lambda) = \int_{-1}^{1} f(x)P_{\sqrt{\lambda}-1/2}(x)\,dx, \quad \lambda \in \mathbf{C}, \tag{1.9}$$

f being an $L^2(-1,1)$-function. Then both transforms can be reconstructed via the following sampling representations

$$\begin{pmatrix} f^*(\lambda) \\ f^{**}(\lambda) \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} f^*((n+(1/2))^2) \\ f^{**}((n+(1/2))^2) \end{pmatrix} \frac{(2n+1)\sin\pi(\sqrt{\lambda}-(n+(1/2)))}{\pi(\lambda-(n+(1/2))^2)}. \tag{1.10}$$

In both cases, the sampled integral transforms are the same up to a change of variables. The sampling points are the same, namely, the eigenvalues of certain Legendre-type eigenvalue problems, which are again the same eigenvalue problems up to a change of variables, (see [**10**]). Moreover, expansions (1.10) were derived as examples of sampling series associated with second order singular problems; while Zayed [**25**] used the Titchmarsh form of the Legendre equation [**24**], Everitt et al. [**11**] used equation (1.5) together with the Glazman-Krein-Naimark boundary conditions.

Since the Legendre function satisfies a second order eigenvalue problem, it is an aim of the present work to derive a general sampling theorem for solutions of second-order singular eigenvalue problems with limit-circle endpoint(s). We use the theory developed by Fulton to define appropriate boundary conditions. This will not be just an extension of the Legendre case but so general as to cover all limit-circle cases. The Legendre function of the second kind

$$Q_w(z) = \frac{\sqrt{\pi}\Gamma(w+1)}{\Gamma(w+(3/2))(2z)^{w+1}}\,{}_2F_1\left(\frac{1}{2}w+1, \frac{1}{2}w+\frac{1}{2}; w+\frac{3}{2}; z^{-2}\right) \tag{1.11}$$

will appear, to the best of our knowledge, for the first time in sampling theory. We mention here that, although the Legendre function of the second kind is defined for $|z| > 1$, its domain of definition can be extended to include the interval $(-1, 1)$, cf. [**20**]. Due to the more general setting, the sampling points in the Legendre case will not necessarily be $\lambda_n = (n + (1/2))^2$, and the sampling functions $P_n(\cdot)$ in (1.4) are not necessarily polynomials but more general ones. The Legendre transforms will be extended to have more general kernels and sampling expansions will be derived accordingly. Expansions including the Bessel functions and the Hankel transform will also be given.

2. The sampling series. In this section we introduce the Sturm-Liouville problem using Fulton's approach for deriving boundary conditions at the limit-circle endpoint(s). Then we establish a sampling representation of the solutions of the differential equation of the problem.

Consider the differential equation

$$-y'' + q(x)y = \lambda y, \quad x \in I, \tag{2.1}$$

where $I \subseteq \mathbf{R}$ is an interval such that, when I is finite, $I = (a, b)$, $[a, b)$ or $(a, b]$, then $q(\cdot)$ will have singularities at the open endpoint(s) and is continuous otherwise. When I is infinite, $I = [a, \infty)$, (a, ∞),

$(-\infty, b]$, $(-\infty, b)$ or $(-\infty, \infty)$, $a, b \in \mathbf{R}$, then $q(\cdot)$ is assumed to be continuous throughout I. We assume that the limit-circle case holds at any singularity. Thus, the discreteness of the eigenvalues is guaranteed, [**24**]. Sufficient conditions for the limit-circle situation can be found in Titchmarsh's [**24**]. We start our investigations with the case $I = [a, \infty)$, $a \in \mathbf{R}$. Hence we have the condition

$$y(a)\cos\alpha + y'(a)\sin\alpha = 0, \quad \alpha \in [0, \pi). \tag{2.2}$$

Since the limit-circle case holds at ∞, i.e., all solutions of (2.1) are $L^2(a, \infty)$-solutions, there should be another boundary condition at ∞ to formulate the eigenvalue problem. With this aim, we outline the theory developed by Fulton [**12, 13**] to define the other boundary condition. Let u, v be two linearly independent solutions of (2.1) when $\lambda = 0$, such that their Wronskian satisfies $W_x(u, v) = u(x)v'(x) - u'(x)v(x) \equiv 1$ on $[a, \infty)$. Equation (2.1) can be written as a first order system or in a vector-valued form as

$$\frac{d}{dx}\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(\lambda - q) & 0 \end{pmatrix}\begin{pmatrix} y \\ y' \end{pmatrix}. \tag{2.3}$$

If we use the transformation

$$Z = \begin{pmatrix} u & v \\ u' & v' \end{pmatrix}^{-1}\begin{pmatrix} y \\ y' \end{pmatrix}, \quad Z = SY, \tag{2.4}$$

then Z satisfies

$$\frac{dZ}{dx} = \lambda B Z, \quad B(x) = \begin{pmatrix} u(x)v(x) & v^2(x) \\ -u^2(x) & -u(x)v(x) \end{pmatrix}. \tag{2.5}$$

Since the limit-circle case holds at ∞, then $u, v \in L^2(a, \infty)$. Consequently, $B(x) \in L^1(a, \infty)$. Following [**17**, page 54], the solutions of (2.5) have limits at ∞. Thus solutions of (2.5) can be determined according to the initial conditions

$$\lim_{z\to\infty} Z_\lambda(x) := \begin{pmatrix} \sin\gamma \\ -\cos\gamma \end{pmatrix}, \quad \gamma \in [0, \pi), \lambda \in \mathbf{C}. \tag{2.6}$$

Denoting the solution determined by (2.6) by $Z_{\infty,\lambda}$, we have

$$(Z_{\infty,\lambda})_1(\infty)\cos\gamma + (Z_{\infty,\lambda})_2(\infty)\sin\gamma = 0, \tag{2.7}$$

where the subscripts 1 and 2 denote the first and second components of the vector $Z_{\infty,\lambda}$, respectively. Using the transformation (2.4), we obtain the boundary condition which y satisfies at ∞, namely,

$$(Sy)_1(\infty)\cos\gamma + (Sy)_2(\infty)\sin\gamma = 0, \tag{2.8}$$

where

$$(Sy)_1(\infty) = \lim_{x\to\infty} W_x(y,v), \quad (Sy)_2(\infty) = -\lim_{x\to\infty} W_x(y,u). \tag{2.9}$$

Thus [**16**] we have put the Sturm-Liouville problem (2.1)–(2.2) and (2.8) into an operator theoretic frame.

Let $\phi_\lambda(\cdot)$ and $\psi_\lambda(\cdot)$ be the two solutions of (2.1) determined by the initial conditions

$$\begin{pmatrix} \phi_\lambda(a) & (S\psi_\lambda)_1(\infty) \\ \phi'_\lambda(a) & (S\psi_\lambda)_2(\infty) \end{pmatrix} = \begin{pmatrix} \sin\alpha & \sin\gamma \\ -\cos\alpha & -\cos\gamma \end{pmatrix}. \tag{2.10}$$

Let also

$$\omega_{\alpha,\gamma}(\lambda) := W_x(\phi_\lambda, \psi_\lambda), \tag{2.11}$$

which is independent of x, [**13**]. The functions $\phi_\lambda(x)$ and $\psi_\lambda(x)$ are entire functions of λ for $x \in [a,\infty)$. Therefore, $\omega_{\alpha,\gamma}(\lambda)$ is also an entire function of λ [**13**]. The eigenvalues of the problem (2.1)–(2.2) and (2.8) are the zeros of $\omega_{\alpha,\gamma}(\lambda)$, consisting of a sequence of real numbers, $\{\lambda_n\}_{n=0}^{\infty}$, with no finite limit points. All eigenvalues are simple from the algebraic and geometric points of view. Moreover, according to [**1–3**],

$$\lambda_n \sim n^2 \quad \text{as } n \to \infty. \tag{2.12}$$

The sequence of eigenfunctions corresponding to $\{\lambda_n\}_{n=0}^{\infty}$ is either $\{\phi_{\lambda_n}(\cdot)\}_{n=0}^{\infty}$ or $\{\psi_{\lambda_n}(\cdot)\}_{n=0}^{\infty}$. Each of these is an orthogonal basis of $L^2(a,\infty)$. Moreover, there are nonzero real numbers, k_n, such that

$$\psi_{\lambda_n}(x) = k_n \phi_{\lambda_n}(x). \tag{2.13}$$

The main result of this section is the following:

Theorem 2.1. *For all $\lambda \in \mathbf{C}$ the functions $\phi_\lambda(\cdot)$ and $\psi_\lambda(\cdot)$ admit the sampling expansions*

$$\phi_\lambda(x) = \sum_{n=0}^{\infty} \phi_{\lambda_n}(x) \frac{\omega_{\alpha,\gamma}(\lambda)}{(\lambda - \lambda_n)\omega'_{\alpha,\gamma}(\lambda_n)}, \tag{2.14}$$

$$\psi_\lambda(x) = \sum_{n=0}^{\infty} \psi_{\lambda_n}(x) \frac{\omega_{\alpha,\gamma}(\lambda)}{(\lambda - \lambda_n)\omega'_{\alpha,\gamma}(\lambda_n)}, \quad x \in [a, \infty). \tag{2.15}$$

Both series converge in mean with respect to x for $\lambda \in \mathbf{C}$ and pointwise with respect to $\lambda \in \mathbf{C}$ for $x \in [a, \infty)$.

Proof. Let $\lambda, \mu \in \mathbf{C}$, $\lambda \neq \mu$ and $M > a$. By Green's formula, [**24**, page 1], since the functions $\phi_\lambda(\cdot)$ and $\psi_\lambda(\cdot)$ satisfy (2.1), we have

$$\begin{aligned}(\lambda - \mu) \int_a^M \phi_\lambda(x)\bar{\psi}_\mu(x)\,dx &= [\phi_\lambda(x)\bar{\psi}'_\mu(x) - \phi'_\lambda(x)\bar{\psi}_\mu(x)]_a^M \\ &= [\phi_\lambda(x)\bar{\psi}'_\mu(x) - \phi'(x)_\lambda\bar{\psi}_\mu(x)](M) \\ &\quad - [\phi_\lambda(x)\bar{\psi}'_\mu(x) - \phi'_\lambda(x)\bar{\psi}_\mu(x)](a).\end{aligned} \tag{2.16}$$

Letting $\mu \to \lambda_n$ in (2.16) for some n and noting from (2.10) that

$$\begin{aligned}[\phi_\lambda(x)\bar{\psi}'_{\lambda_n}(x) - \phi'_\lambda(x)\bar{\psi}_{\lambda_n}(x)](a) &= \phi_\lambda(a)\bar{\psi}'_{\lambda_n}(a) - \phi'_\lambda(a)\bar{\psi}_{\lambda_n}(a), \\ &= -\sin\alpha\bar{\psi}'_{\lambda_n}(a) - \cos\alpha\bar{\psi}_{\lambda_n}(a) = 0,\end{aligned}$$

since $\psi_{\lambda_n}(\cdot)$ is an eigenfunction, i.e., satisfies (2.2), we obtain

$$(\lambda - \lambda_n) \int_0^M \phi_\lambda(x)\bar{\psi}_{\lambda_n}(x)\,dx = W_M(\phi_\lambda, \bar{\psi}_{\lambda_n}). \tag{2.17}$$

Since $\phi_\lambda(\cdot)$, $\psi_{\lambda_n}(\cdot)$ lie in the domain of definition of the operator associated with (2.1)–(2.2) and (2.8), (see [**12**]), then we have [**13**, page 55]

$$\begin{aligned}W_M(\phi_\lambda, \bar{\psi}_{\lambda_n}) &= D_M(S\phi_\lambda, S\bar{\psi}_{\lambda_n}) \\ &= \begin{vmatrix} (S\phi_\lambda)_1(M) & (S\bar{\psi}_{\lambda_n})_1(M) \\ (S\phi_\lambda)_2(M) & (S\bar{\psi}_{\lambda_n})_2(M) \end{vmatrix}.\end{aligned} \tag{2.18}$$

Hence

$$(2.19)\qquad (\lambda-\lambda_n)\int_a^M \phi_\lambda(x)\bar{\psi}_{\lambda_n}(x)\,dx = (S\phi_\lambda)_1(M)(S\bar{\psi}_{\lambda_n})_2(M) - (S\phi_\lambda)_2(M)(S\bar{\psi}_{\lambda_n})_1(M).$$

Letting $M\to\infty$, we obtain

$$(2.20)\qquad (\lambda-\lambda_n)\int_a^\infty \phi_\lambda(x)\bar{\psi}_{\lambda_n}(x)\,dx = (S\phi_\lambda)_1(\infty)(S\bar{\psi}_{\lambda_n})_2(\infty) - (S\phi_\lambda)_2(\infty)(S\bar{\psi}_{\lambda_n})_1(\infty).$$

Again, we make use of (2.10) to deduce

$$(2.21)\qquad \int_a^\infty \phi_\lambda(x)\bar{\psi}_{\lambda_n}(x)\,dx = -\frac{\cos\gamma(S\phi_\lambda)_1(\infty)+\sin\gamma(S\phi_\lambda)_2(\infty)}{\lambda-\lambda_n}.$$

We now show that

$$(2.22)\qquad \int_a^\infty \phi_\lambda(x)\bar{\psi}_{\lambda_n}(x)\,dx = -\frac{\omega_{\alpha,\gamma}(\lambda)}{(\lambda-\lambda_n)}.$$

Indeed, using [**13**, page 55] and the fact that $\phi_\lambda(\cdot),\psi_\lambda(\cdot)$ belong to the domain of definition of the operator associated with the Sturm-Liouville problem, we have

$$(2.23)\qquad \begin{aligned} W_\infty(\phi_\lambda,\bar{\psi}_\lambda) &= D_\infty(S\phi_\lambda,S\bar{\psi}_\lambda)\\ &= (S\phi_\lambda)_1(\infty)(S\bar{\psi}_\lambda)_2(\infty)\\ &\quad - (S\phi_\lambda)_2(\infty)(S\bar{\psi}_\lambda)_1(\infty)\\ &= -\cos\gamma(S\phi_\lambda)_1(\infty)-\sin\gamma(S\phi_\lambda)_2(\infty), \end{aligned}$$

proving relation (2.22). Taking the limit in (2.22) when $\lambda\to\lambda_n$ and since λ_n is a simple zero of $\omega_{\alpha,\gamma}(\lambda)$, then

$$(2.24)\qquad \int_a^\infty \phi_{\lambda_n}(x)\bar{\psi}_{\lambda_n}(x)\,dx = -\omega'_{\alpha,\gamma}(\lambda_n).$$

Since $\phi_\lambda(\cdot)\in L^2(a,\infty)$ for all $\lambda\in\mathbf{C}$, and since $\{\phi_{\lambda_n}(\cdot)\}_{n=0}^\infty$ is an orthogonal basis of $L^2(a,\infty)$, then

$$(2.25)\qquad \phi_\lambda(x) = \sum_{n=0}^\infty \phi_{\lambda_n}(x)\frac{\langle\phi_\lambda,\phi_{\lambda_n}\rangle}{\|\phi_{\lambda_n}\|^2}.$$

Combining relations (2.13), (2.22), (2.24) and (2.25), one obtains (2.14) where the convergence is pointwise with respect to λ for fixed $x \in [a, \infty)$ and is in the L^2-norm with respect to x for any $\lambda \in \mathbf{C}$. Similarly, we can prove (2.15). $\square$

Remark 2.2. (a) The results derived above can also be deduced in case we have two singular endpoints, both being in the limit-circle case. All we do is to define a boundary condition at the left (singular) endpoint in a similar way, (cf. [**13**, page 60]).

(b) Extensions can also be carried out if the canonical Sturm-Liouville equation (2.1) is replaced by

$$(p(x)y')' + q(x)y = \lambda r(x)y, \quad x \in I, \tag{2.26}$$

with $r, 1/p, q \in L^1_{\text{loc}}(I)$ and $r(x) > 0$ for almost all $x \in I$.

3. The Legendre case. In this section we derive a sampling expansion associated with the Legendre equation. Using the terminology of Section 2 above, we consider the eigenvalue problem

$$-((1-x^2)y')' + \frac{1}{4}y' = \lambda y, \quad -1 < x < 1, \tag{3.1}$$

$$(Sy)_1^-(-1)\cos\alpha + (Sy)_2^-(-1)\sin\alpha = 0, \tag{3.2}$$

$$(Sy)_1^+(1)\cos\gamma + (Sy)_2^+(1)\sin\gamma = 0, \tag{3.3}$$

where $\alpha, \gamma \in [0, \pi)$ and

$$\begin{pmatrix} (Sy)_1^{\mp}(\mp 1) \\ (Sy)_2^{\mp}(\mp 1) \end{pmatrix} := \lim_{x \to \mp 1} \begin{pmatrix} (1-x^2)W_x(y, v) \\ (1-x^2)W_x(y, u) \end{pmatrix}, \tag{3.4}$$

$u(x) = \tanh^{-1} x/2$, $v(x) = -1$, $-1 < x < 1$, are the solutions of (3.1), when $\lambda = 0$ for which $(1-x^2)W_x(u, v) \equiv 1$ on $(-1, 1)$. Let

$$\lambda = \left(\mu + \frac{1}{2}\right)^2, \quad s := \sqrt{\lambda}, \tag{3.5}$$

where the square root is defined with a branch on the negative real λ-axis. We take the Frobenius (logarithmic) solutions at ± 1. Those at

$+1$ are, (see [**14**, **18**–**19**]),

(3.6)
$$P(x,y)=P_\mu(x)=\sum_{k=0}^{\infty}c(s,k)\left(\frac{1-x}{2}\right)^k,$$

(3.7)
$$R(x,\lambda)=P(x,\lambda)\ln\left(\frac{1-x}{2}\right)+\sum_{k=1}^{\infty}c(s,k)\beta(s,k)\left(\frac{1-x}{2}\right)^k,$$

where

(3.8)
$$\begin{aligned}c(s,k):&=\frac{1}{(k!)^2}\prod_{i=1}^{k}\left(-s+i-\frac{1}{2}\right)\left(s+i-\frac{1}{2}\right)\\&=\frac{1}{(k!)^2}\prod_{i=1}^{k}\left(\left(i-\frac{1}{2}\right)^2-\lambda\right)\end{aligned}$$

and

(3.9)
$$\begin{aligned}\beta(s,k)&=\sum_{i=1}^{k}\frac{1}{(-s+i-(1/2))}+\frac{1}{(s+i-(1/2))}-2\sum_{i=1}^{k}\frac{1}{i}\\&=\sum_{i=1}^{k}\frac{2i-1}{(i-(1/2))^2-\lambda}-2\sum_{i=1}^{k}\frac{1}{i}.\end{aligned}$$

Here, $P_\mu(\cdot)$ is the Legendre function of the first kind introduced in Section 1. The other two Frobenius solutions at -1 are obtained from $P(x,\lambda)$, $R(x,\lambda)$ by replacing x by $-x$. Frobenius solutions at 1 and -1 are related by [**18, 19**],

(3.10)
$$P(x,\lambda)=A(s)P(-x,\lambda)+B(s)R(-x,\lambda),$$

(3.11)
$$R(x,\lambda)=\left(\frac{1-A^2(s)}{B(s)}\right)P(-x,\lambda)-A(s)R(-x,\lambda),$$

where

$$\begin{aligned} A(s) :&= -\frac{\cos s\pi}{\pi}\left\{\psi\left(-s+\frac{1}{2}\right)+\psi\left(s+\frac{1}{2}\right)+2C\right\} \\ &= -\sin s\pi - \frac{2}{\pi}\left(\psi\left(s+\frac{1}{2}\right)+C\right)\cos s\pi \\ &= -\sin s\pi - \frac{2}{\pi}\left(\pi\left(-s+\frac{1}{2}\right)+C\right)\cos s\pi, \end{aligned} \tag{3.12}$$

$$B(s) = -\frac{\cos s\pi}{\pi}, \quad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \tag{3.13}$$

and C is the Euler constant. The functions $P(\pm x, \lambda)$, $R(\pm x, \lambda)$, $A(s)$ and $B(s)$ are all entire in λ for $x \in [-1, 1)$, [**14**]. As the Legendre function of the first kind is included above, the Legendre function of the second kind, $Q_\mu(\cdot)$ is included since [**14**, page 218],

$$R(x,\lambda) = -2(\psi(\mu+1)+C)P_\mu(x) - 2Q_\mu(x). \tag{3.14}$$

Let $\phi_\lambda(\cdot)$ and $\psi_\lambda(\cdot)$ be the solutions of (3.1) for which

$$(S\phi_\lambda)_1^-(-1) = -\sin\alpha, \quad (S\phi_\lambda)_2^-(-1) = \cos\alpha, \tag{3.15}$$

$$(S\psi_\lambda)_1^+(1) = \sin\gamma, \quad (S\psi_\lambda)_2^+(1) = -\cos\gamma, \tag{3.16}$$

for all $\lambda \in \mathbf{C}$. Then, with the aid of [**14**, pages 218–219],

$$\begin{pmatrix} (SP(-x,\lambda))_1^-(-1) & (SP(-x,\lambda))_2^-(-1) \\ (SR(-x,\lambda))_1^-(-1) & (SR(-x,\lambda))_2^-(-1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \tag{3.17}$$

$$\begin{pmatrix} (SP(x,\lambda))_1^+(1) & (SP(x,\lambda))_2^+(1) \\ (SR(x,\lambda))_1^+(1) & (SR(x,\lambda))_2^+(1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \tag{3.18}$$

it is concluded that

$$\phi_\lambda(x) = \cos\alpha P(-x,\lambda) + \frac{1}{2}\sin\alpha R(-x,\lambda), \tag{3.19}$$

$$\psi_\lambda(x) = -\cos\lambda P(x,\lambda) + \frac{1}{2}\sin\gamma R(x,\lambda). \tag{3.20}$$

Hence [**14**, page 219],

$$\begin{aligned}\omega_{\alpha,\gamma}(\lambda) = & -2(\cos\alpha\cos\gamma)B(s) - (\cos\alpha\sin\gamma)A(s) \\ & + (\sin\alpha\cos\gamma)A(s) \\ & - \frac{1}{2}(\sin\alpha\sin\gamma)\left(\frac{1-A^2(s)}{B(s)}\right).\end{aligned} \tag{3.21}$$

Thus, the eigenvalues of problems (3.1)–(3.3), $\{\lambda_n\}_{n=0}^{\infty}$, are the zeros of $\omega_{\alpha,\gamma}(\lambda)$ and the corresponding eigenfunctions are either $\{\phi_{\lambda_n}(\cdot)\}_{n=0}^{\infty}$ or $\{\psi_{\lambda_n}(\cdot)\}_{n=0}^{\infty}$. As a consequence of Theorem 2.1 above, we now have the following sampling expansion of $\phi_\lambda(\cdot)$ and $\psi_\lambda(\cdot)$.

Theorem 3.1. *Let $\phi_\lambda(\cdot)$, $\psi_\lambda(\cdot)$ and $\omega_{\alpha,\gamma}(\lambda)$ be given by* (3.19), (3.20) *and* (3.21) *respectively. Then the following expansions*

$$\phi_\lambda(x) = \sum_{n=0}^{\infty} \phi_{\lambda_n}(x)\frac{\omega_{\alpha,\gamma}(\lambda)}{(\lambda-\lambda_n)\omega'_{\alpha,\gamma}(\lambda_n)} \tag{3.22}$$

$$\psi_\lambda(x) = \sum_{n=0}^{\infty} \psi_{\lambda_n}(x)\frac{\omega_{\alpha,\gamma}(\lambda)}{(\lambda-\lambda_n)\omega'_{\alpha,\gamma}(\lambda_n)} \tag{3.23}$$

hold for $\lambda \in \mathbf{C}$ and $x \in (-1,1)$. The convergence is pointwise with respect to $\lambda \in \mathbf{C}$ for $x \in (-1,1)$ and is in the L^2-norm with respect to x for $\lambda \in \mathbf{C}$.

Remark 3.2. The sampled functions $\phi_\lambda(\cdot)$ and $\psi_\lambda(\cdot)$ can be written as linear combinations of the Legendre functions of the first and second kind. Indeed, from (3.14) and the fact that $P(\cdot,\lambda)$ is the Legendre function of the first kind and using (3.19)–(3.20), the functions $\phi_\lambda(\cdot)$ and $\psi_\lambda(\cdot)$ can be explicitly written in the form

$$\begin{aligned}\phi_\lambda(x) = & \left[\cos\alpha - \sin\alpha\left(\psi(\sqrt{\lambda}+\frac{1}{2}\right)+C\right)\right]P_{\sqrt{\lambda}-(1/2)}(-x) \\ & - \sin\alpha Q_{\sqrt{\lambda}-(1/2)}(-x),\end{aligned} \tag{3.24}$$

and

$$\begin{aligned}\psi_\lambda(x) = & -\left[-\cos\gamma - \sin\gamma\left(\psi(\sqrt{\lambda}+\frac{1}{2}\right)+C\right)\right]P_{\sqrt{\lambda}-(1/2)}(x) \\ & - \sin\gamma Q_{\sqrt{\lambda}-(1/2)}(x),\end{aligned} \tag{3.25}$$

where $\psi(z)$ is given in (3.13) and C is the Euler constant. According to Theorem 3.1 above, we sampled $\phi_\lambda(\cdot)$ and $\psi_\lambda(\cdot)$ from their values at the eigenvalues. The eigenfunctions $\phi_{\lambda_n}(\cdot)$ or $\psi_{\lambda_n}(\cdot)$ are not necessarily polynomials as in the previous situations mentioned in Section 1. However, we have the possibility to derive sampling expansions for Legendre functions of the first kind or that of the second kind. As for the first kind, we have three choices. First, take $\alpha = \gamma = 0$, Theorem 3.2 below, and in this case, $\phi_\lambda(\cdot)$ and $\psi_\lambda(\cdot)$ are similar and the expansion will be in terms of the Legendre polynomials. Second, take $\alpha = 0$, $\gamma \neq 0$. In this case, $\phi_\lambda(x) = P_{\sqrt{\lambda}-(1/2)}(-x)$, but the eigenfunctions are no more polynomials. Finally, we can take $\alpha \neq 0$ and $\gamma = 0$, leading to a similar case as for the second one. As for the Legendre function of the second kind, it is clearly seen that it is not possible to isolate $Q_\mu(\pm x)$ from a direct choice of α and γ, but we can sample $Q_\mu(x)$ as a combination of two different expansions, one for $\phi_\lambda(x)$ and another for $P_{\sqrt{\lambda}-(1/2)}(-x)$.

The following theorem illustrates the claims of the previous remark.

Theorem 3.3. *For $\lambda \in \mathbf{C}$, we have*

$$P_{\sqrt{\lambda}-(1/2)}(x) = \sum_{n=0}^{\infty} P_n(x) \frac{(2n+1)\sin\pi(\sqrt{\lambda}-(n+(1/2)))}{\pi(\lambda-(n+(1/2))^2)}. \tag{3.26}$$

The convergence is uniform with respect to x on compact subsets of $(-1,1)$ for all $\lambda \in \mathbf{C}$ and is uniform with respect to λ on compact subsets of $\mathbf{C}$ for all $x \in (-1,1)$. Moreover, the convergence is uniform on $\mathbf{R}$ with respect to λ for all $x \in (-1,1)$.

Proof. We consider Theorem 3.1 with $\alpha = \gamma = 0$. Hence

$$\phi_\lambda(x) = P_{\sqrt{\lambda}-(1/2)}(-x), \quad \psi_\lambda(x) = -P_{\sqrt{\lambda}-(1/2)}(x), \tag{3.27}$$

$$\omega(\lambda) = \omega_{0,0}(\lambda) = \frac{2}{\pi}\cos\sqrt{\lambda}\pi. \tag{3.28}$$

The eigenvalues, the zeros of $\omega(\lambda)$, are $\lambda_n = (n+(1/2))^2$, $n = 0, 1, \dots$. We also have

$$\omega'(\lambda_n) = \omega'\left(\left(n+\frac{1}{2}\right)^2\right) = \frac{(-1)^{n+1}}{(n+(1/2))}, \tag{3.29}$$

and the corresponding eigenfunctions are

(3.30) $$\phi_{\lambda_n}(x) = P_n(-x) = (-1)^n P_n(x), \quad \psi_{\lambda_n}(x) = P_n(x),$$

where $P_n(x)$ are the Legendre polynomials. Therefore, $k_n = (-1)^n$. Applying Theorem 3.1, we obtain

(3.31)
$$P_{\sqrt{\lambda}-(1/2)}(-x) = \sum_{n=0}^{\infty} (-1)^n P_n(x) \frac{(2n+1)\sin\pi(\sqrt{\lambda}-(n+(1/2)))}{\pi(\lambda-(n+(1/2))^2)},$$

(3.32)
$$P_{\sqrt{\lambda}-(1/2)}(x) = \sum_{n=0}^{\infty} P_n(x) \frac{(2n+1)\sin\pi(\sqrt{\lambda}-(n+(1/2)))}{\pi(\lambda-(n+(1/2))^2)},$$

and the convergence properties are the same as in Theorem 3.1. As for the uniform convergence, we start with the uniform convergence with respect to x. We will use the first Stieltjes formula [**22**, page 197] for Legendre polynomials, which states

(3.33) $$|P_n(x)| \leq \sqrt{2}\frac{4}{\sqrt{\pi}} \frac{1}{\sqrt{n}\sqrt[4]{1-x^2}}, \quad -1 < x < 1,$$

$n = 0, 1, \dots$. Let $\lambda \in \mathbf{C}$ be fixed. Let $I \subset (-1,1)$ be compact. From (3.33) there is a positive constant C_I which depends neither on x nor on n such that

(3.34) $$|P_n(x)| \leq \frac{C_I}{\sqrt{n}}, \quad \text{for all } x \in I.$$

Hence,

(3.35)
$$\left| P_n(x) \frac{(2n+1)\sin\pi(\sqrt{\lambda}-(n+(1/2)))}{\pi(\lambda-(n+(1/2))^2)} \right| \leq \frac{C_{I,\lambda}(2n+1)}{\sqrt{n}|\lambda-(n+(1/2))^2|},$$

$x \in I$, $n \in \mathbf{N}_0$, where $C_{I,\lambda}$ is a positive constant, depending only on λ and I, which are fixed. Using the Weierstrass M-test for uniform convergence, relation (3.35) and

(3.36) $$\sum_{n=0}^{\infty} \frac{2n+1}{\sqrt{n}|\lambda-(n+(1/2))^2|} < \infty,$$

the uniform convergence of (3.31) and (3.32) follows on compact subsets of $(-1,1)$ since I is arbitrary.

We now prove the uniform convergence with respect to λ for all x on compact subsets of $\mathbf{C}$. Let x be fixed and $M \subset \mathbf{C}$ be compact. Using (3.33), there is an x-dependent positive constant D_x which does not depend on λ or n such that

$$|P_n(x)| \le \frac{D_x}{\sqrt{n}}. \tag{3.37}$$

To prove uniform convergence of (3.31) and (3.32) on M, it is sufficient to prove the uniform convergence of the series

$$\sum_{n=0}^{\infty} D_x \frac{(2n+1)\sin\pi(\sqrt{\lambda}-(n+(1/2)))}{\sqrt{n}\pi(\lambda-(n+(1/2))^2)}. \tag{3.38}$$

Since M is compact and the sequence $\{\lambda_n\}_{n=0}^{\infty}$ contains no finite limit points, then M contains at most a finite number of the λ_n's. Let $n_0 \in \mathbf{N}$ be such that $\lambda_n \notin M$ for all $n > n_0$. We rewrite (3.38) in the form

$$\begin{aligned}\sum_{n=0}^{n_0} D_x \frac{(2n+1)\sin\pi(\sqrt{\lambda}-(n+(1/2)))}{\sqrt{n}\pi(\lambda-(n+(1/2))^2)} \\ + \sum_{n=n_0+1}^{\infty} D_x \frac{(2n+1)\sin\pi(\sqrt{\lambda}-(n+(1/2)))}{\sqrt{n}\pi(\lambda-(n+(1/2))^2)}.\end{aligned} \tag{3.39}$$

The first part is bounded on M, since the zeros $(n+(1/2))^2$, $n = 0,\dots,n_0$ are simple zeros of the function $\sin\pi(\sqrt{\lambda}-(n+(1/2))^2)$. The general term of the second part is bounded by

$$D_{M,x}\frac{2n+1}{\sqrt{n}|\rho-(n+(1/2))^2|}, \tag{3.40}$$

$\lambda \in M$, where $D_{M,x}$ is a positive constant which depends only on M and x, which are fixed, and

$$\rho := \operatorname{dist}\left(\left(n_0+\frac{3}{2}\right)^2, M\right). \tag{3.41}$$

The bound (3.40) and the Weierstrass test prove the uniform convergence of (3.38) on M and, hence, the uniform convergence of (3.31)–(3.32) is established on compact subsets of $\mathbf{C}$. The uniform convergence on $\mathbf{R}$ can be derived similarly.

Remark 3.4. (a) Sampling Legendre functions can also be derived using Glazman-Krein-Naimark's boundary conditions, (cf. [**11, 21**]). The relationship between both GKN's boundary conditions and Fulton's conditions is not considered in this paper. However, it is interesting to see a separate publication on this topic.

(b) We can use the canonical form of the Legendre equation, (see [**24**]),

$$-y'' - \frac{1}{4}\sec^2 xy = \lambda y, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}. \tag{3.42}$$

In this case, if we take the boundary conditions to be

$$\lim_{x\to-(\pi/2)} y(x) = 0, \quad \lim_{x\to(\pi/2)} y(x) = 0, \tag{3.43}$$

then the solutions $\phi_\lambda(x)$ and $\psi_\lambda(x)$ are

$$\phi_\lambda(x) = 4\sqrt{\cos}x \int_0^{x+(\pi/2)} \frac{\cos\sqrt{\lambda}z}{(\cos z + \sin x)^{1/2}}\, dz, \tag{3.44}$$

$$\psi_\lambda(x) = 4\sqrt{\cos x} \int_0^{\pi/2-x} \frac{\cos\sqrt{\lambda}z}{(\cos z - \sin x)^{1/2}}\, dz. \tag{3.45}$$

Hence [**24**, page 77] $\psi_\lambda(x) = \phi_\lambda(-x)$ and the Wronskian is

$$\omega(\lambda) := W_x(\phi_\lambda, \psi_\lambda) = 8\pi\cos\sqrt{\lambda}\pi. \tag{3.46}$$

Thus the eigenvalues are $\lambda_n = (n+(1/2))^2$, $n = 0, 1, \dots$, and the corresponding eigenfunctions are either $\phi_{\lambda_n}(x) = (-1)^n 2\sqrt{2}\pi\sqrt{\cos x}P_n(\sin x)$ or $\psi_{\lambda_n}(x) = 2\sqrt{2}\pi\sqrt{\cos x}P_n(\sin x)$, $P_n(\cdot)$ are the Legendre polynomials. Finally, one can derive sampling expansions for $\phi_\lambda(\cdot)$ and $\psi_\lambda(\cdot)$,

$$\phi_\lambda(-x) = \psi_\lambda(x) = 2\sqrt{2}\sqrt{\cos x}P_{\sqrt{\lambda}-(1/2)}(x). \tag{3.47}$$

Relation (3.47) can be derived from (3.44)–(3.45) using integral representations of the Legendre functions [**23**, pages 202–208]. Everitt [**9**] has established the analysis of Legendre problems with the canonical equation (3.42).

(c) Expansion (3.26) has been extended to the Jacobi case by Butzer and Schöttler [**5**].

4. Sampling the Legendre transform. In this section we give extensions of the sampling results of [**11, 25**], equation (1.10) above. The sampled transform in this case has either $\phi_\lambda(\cdot)$ or $\psi_\lambda(\cdot)$ as its kernel.

Theorem 4.1. *Let $\phi_\lambda(\cdot)$ and $\psi_\lambda(\cdot)$ be the functions introduced above in* (3.24) *and* (3.25), *respectively. Let $g(\cdot) \in L^2(-1,1)$ and*

$$\begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \end{pmatrix} = \int_{-1}^{1} g(x) \begin{pmatrix} \phi_\lambda(x) \\ \psi_\lambda(x) \end{pmatrix} dx. \tag{4.1}$$

Then $f_1(\lambda), f_2(\lambda)$ are entire functions of λ and admit the sampling representations

$$\begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} f_1(\lambda_n) \\ f_2(\lambda_n) \end{pmatrix} \frac{\omega_{\alpha,\gamma}(\lambda)}{(\lambda-\lambda_n)\omega'_{\alpha,\gamma}(\lambda_n)}, \tag{4.2}$$

where $\omega_{\alpha,\gamma}(\lambda)$ is the entire function introduced in (3.21) *and $\{\lambda_n\}_{n=0}^{\infty}$ are its simple zeros. The sampling series* (4.2) *converge absolutely and uniformly on compact subsets of the complex plane.*

Proof. We prove the theorem for $f_1(\lambda)$, the case of $f_2(\lambda)$ being similar. Since $\{\phi_{\lambda_n}(\cdot)\}_{n=0}^{\infty}$ is a complete orthogonal set of $L^2(-1,1)$, by applying Parseval's equality to the integral transform $f_1(\lambda)$, we obtain

$$f_1(\lambda) = \sum_{n=0}^{\infty} \frac{\langle g(\cdot), \phi_{\lambda_n}(\cdot)\rangle \langle \phi_\lambda(\cdot), \phi_{\lambda_n}(\cdot)\rangle}{\|\phi_{\lambda_n}(\cdot)\|^2}. \tag{4.3}$$

But $\langle g(\cdot), \phi_{\lambda_n}(\cdot)\rangle = f_1(\lambda_n)$, $n = 0, 1, \ldots$. Moreover, from the calculations of the proof of Theorem 3.1 above,

$$\frac{\langle \phi_\lambda(\cdot), \phi_{\lambda_n}(\cdot)\rangle}{\|\phi_{\lambda_n}(\cdot)\|^2} = \frac{\omega_{\alpha,\gamma}(\lambda)}{(\lambda-\lambda_n)\omega'_{\alpha,\gamma}(\lambda_n)}. \tag{4.4}$$

Combining (4.3) and (4.4), one obtains (4.2), and the convergence is pointwise for $\lambda \in \mathbf{C}$. The proof of the absolute convergence on $\mathbf{C}$ is established as follows: Let $\lambda \in \mathbf{C}$. From the above calculations and by using the Cauchy-Schwarz inequality, one deduces

(4.5)
$$\begin{aligned}\sum_{n=0}^{\infty}\left|f_1(\lambda_n)\frac{\omega_{\alpha,\gamma}(\lambda)}{(\lambda-\lambda_n)\omega'_{\alpha,g}(\lambda_n)}\right| &= \sum_{n=0}^{\infty}\left|\frac{\langle g(\cdot),\phi_{\lambda_n}(\cdot)\rangle\langle\phi_\lambda(\cdot),\phi_{\lambda_n}(\cdot)\rangle}{\|\phi_{\lambda_n}(\cdot)\|^2}\right|\\ &\le \left(\sum_{n=0}^{\infty}\left|\frac{\langle g(\cdot),\phi_{\lambda_n}(\cdot)\rangle}{\|\phi_{\lambda_n}(\cdot)\|}\right|^2\right)^{1/2}\\ &\quad\times\left(\sum_{n=0}^{\infty}\left|\frac{\phi_\lambda(\cdot),\phi_{\lambda_n}(\cdot)\rangle}{\|\phi_{\lambda_n}(\cdot)\|}\right|^2\right)^{1/2}.\end{aligned}$$

As for the proof of the uniform convergence on compact subsets of $\mathbf{C}$, let $M \subset \mathbf{C}$ be compact, let $N > 0$. Define $\sigma_N(\lambda)$ to be

$$\sigma_N(\lambda) := \left|f_1(\lambda) - \sum_{n=0}^{N-1} f_1(\lambda_n)\frac{\omega_{\alpha,\gamma}(\lambda)}{(\lambda-\lambda_n)\omega'_{\alpha,\gamma}(\lambda_n)}\right|, \quad \lambda \in M. \tag{4.6}$$

To prove that the expansion of $f_1(\lambda)$ converges uniformly on M, it is sufficient to show that $\sigma_N(\lambda)$ approaches zero uniformly on M; in other words, $\sigma_N(\lambda) \to 0$ as $N \to \infty$ without depending on $\lambda \in M$. Indeed, again using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sigma_N(\lambda) &\le \sum_{n=N}^{\infty}\left|\frac{\langle g,\phi_{\lambda_n}\rangle\langle\phi_\lambda,\phi_{\lambda_n}(\cdot)\rangle}{\|\phi_{\lambda_n}(\cdot)\|^2}\right|\\ &\le \left(\sum_{n=N}^{\infty}\left|\frac{\langle g,\phi_{\lambda_n}(\cdot)\rangle}{\|\phi_{\lambda_n}(\cdot)\|}\right|^2\right)^{1/2}\left(\sum_{n=N}^{\infty}\left|\frac{\langle\phi_\lambda,\phi_{\lambda_n}(\cdot)\rangle}{\|\phi_{\lambda_n}(\cdot)\|}\right|^2\right)^{1/2}\\ &\le \left(\sum_{n=N}^{\infty}\frac{\langle g,\phi_{\lambda_n}(\cdot)\rangle}{\|\phi_{\lambda_n}(\cdot)\|}\right|^2\Bigg)^{1/2}\left(\int_{-1}^{1}|\phi_\lambda(x)|^2\,dx\right)^{1/2}.\end{aligned} \tag{4.7}$$

Following [**8, 10, 11**] there is a positive constant C_M, $\lambda \in M$, which is independent of λ, for which $\|\phi_\lambda(\cdot)\|^2 < C_M$. Hence,

$$\sigma_N(\lambda) \le C_M\left(\sum_{n=N}^{\infty}\left|\frac{\langle g,\phi_{\lambda_n}(\cdot)\rangle}{\|\phi_{\lambda_n}(\cdot)\|}\right|^2\right)^{1/2}, \quad \lambda \in M. \tag{4.8}$$

The righthand side inequality (4.8) goes to zero as N goes to ∞ without depending on λ, proving the uniform convergence as claimed. By the uniform convergence property, $f_1(\lambda)$ is analytic on compact subsets of $\mathbf{C}$, i.e., is entire. $\square$

Remark 4.2. The special case (1.10) arises if one takes $\alpha = \gamma = 0$. In case one chooses $\alpha = 0$, $\gamma \neq 0$ or $\alpha \neq 0$, $\gamma = 0$, one may sample the Legendre transform (1.9) but in this case the sampling points will be the zeros of the corresponding transcendental functions.

5. Sampling the Bessel functions. In this section we give less detailed sampling expansions of the Bessel functions and the corresponding Hankel transforms. Since the methods are almost similar, no proofs will be given in this section. In the following investigations, the positions of ϕ_λ and ψ_λ will be reversed, adapted to the standard literature. Thus, $\phi_\lambda(\cdot)$ and $\psi_\lambda(\cdot)$ will be related to the initial conditions at right (and left) endpoints, respectively.

Consider the eigenvalue problem

$$-y'' + \frac{\nu^2 - (1/4)}{x^2} y = \lambda y, \quad 0 < x \leq c < \infty, \tag{5.1}$$

where $0 < \nu < 1$, $\nu \neq 1/2$. The solutions [**12**]

$$u(x) = \frac{1}{\sqrt{2\nu}} x^{(1/2)-\nu}, \quad v(x) = \frac{1}{\sqrt{2\nu}} x^{(1/2)+\nu} \tag{5.2}$$

are both $L^2(0,c)$-solutions satisfying $W(u,v) \equiv 1$ on $(0,c]$, (see [**12**, page 83]). Thus equation (5.1) is in the limit circle case. For simplicity, we take the boundary conditions

$$(Sy)_1(0)\cos\alpha + (Sy)_2(0)\sin\alpha = 0, \tag{5.3}$$

$$y(c) = 0. \tag{5.4}$$

Using Titchmarsh's notations [**24**, pages 81–84], (see also [**12**]), the solutions $\phi_\lambda(\cdot)$ and $\theta_\lambda(\cdot)$, which satisfy

$$\phi_\lambda(c) = 0, \quad \phi'_\lambda(c) = -1 \tag{5.5}$$

$$\theta_\lambda(c) = 1, \quad \theta'_\lambda(c) = 0, \quad \lambda \in \mathbf{C} \tag{5.6}$$

are

(5.7)
$$\phi_\lambda(x) = -\frac{\pi\sqrt{cx}}{2\sin\nu\pi}[J_\nu(sx)J_{-\nu}(sc) - J_{-\nu}(sx)J_\nu(sc)],$$
(5.8)
$$\theta_\lambda(x) = \frac{s\sqrt{cx}}{2\sin\nu\pi}[J_\nu(sx)J'_{-\nu}(sc) - J_{-\nu}(sx)J'_\nu(sc)] + \frac{\phi_\lambda(x)}{2c},$$

where $s = \sqrt{\lambda}$ with a branch taken along the negative real axis. Again, using [**12**, page 85], it can be concluded that
(5.9)
$$\omega_\alpha(\lambda) = 2c\begin{cases} s^{-\nu}J_\nu(sc)2^{2\nu}(\Gamma(1+\nu)/\Gamma(1-\nu))\cot\alpha - s^\nu J_{-\nu}(sc) & \alpha \neq 0, \\ s^{-\nu}2^\nu\Gamma(1+\nu)J_\nu(sc) & \alpha = 0. \end{cases}$$

Relation (5.9) follows directly from [**12**, page 56].

The eigenvalues of problems (5.1), (5.3)–(5.4) are the zeros of $\omega_\alpha(\lambda)$. Let us denote them by $\{\lambda_n\}_{n=0}^\infty$. The function $\phi_\lambda(\cdot)$ satisfies the second boundary condition. Now we need to find the function $\psi_\lambda(\cdot)$, which satisfies the first boundary condition. The next lemma is devoted to this aim.

Lemma 5.1. *The function $\psi_\lambda(\cdot)$, which satisfies*

$$(S\psi_\lambda)_1(0) = \sin\alpha, \quad (S\psi_\lambda)_2(0) = -\cos\alpha, \ \lambda \in \mathbf{C} \tag{5.10}$$

is given by

$$\begin{aligned}\psi_\lambda(x) = [&\sin\alpha(S\theta_\lambda)_2(0) + \cos\alpha(S\theta_\lambda)_1(0)]\phi_\lambda(x) \\ &- [\cos\alpha(S\phi_\lambda)_1(0) + \sin\alpha(S\phi_\lambda)_2(0)]\theta_\lambda(x),\end{aligned} \tag{5.11}$$

where $(S\phi_\lambda)_i(0)$, $(S\theta_\lambda)_i(0)$, $i = 1, 2$, are given below.

Proof. From [**12**, page 85], we have

(5.12)
$$\begin{aligned}
(S\phi_\lambda)_1(0) &= \frac{\pi\sqrt{c\nu}}{\sqrt{2}\sin\nu\pi}\left(\frac{s}{2}\right)^{-\nu}\frac{1}{\Gamma(1-\nu)}J_\nu(cs),\\
(S\phi_\lambda)_2(0) &= -\frac{\pi\sqrt{c\nu}}{\sqrt{2}\sin\nu\pi}\left(\frac{s}{2}\right)^{\nu}\frac{1}{\Gamma(1+\nu)}J_{-\nu}(cs),\\
(S\theta_\lambda)_1(0) &= \frac{\pi\sqrt{\nu c}}{\sqrt{2}\sin\nu\pi}\left(\frac{s}{2}\right)^{-\nu}\frac{1}{\Gamma(1-\nu)}\left[sJ_\nu'(cs)+\frac{1}{2c}J_\nu(cs)\right],\\
(S\theta_\lambda)_2(0) &= -\frac{\pi\sqrt{\nu c}}{\sqrt{2}\sin\nu\pi}\left(\frac{s}{2}\right)^{\nu}\frac{1}{(1+\nu)}\left[sJ_{-\nu}'(cs)+\frac{1}{2c}J_{-\nu}(cs)\right].
\end{aligned}$$

Set

$$\psi_\lambda(x) = A\phi_\lambda(x) + B\theta_\lambda(x), \tag{5.13}$$

where A, B are constants which depend on λ only and need to be determined. From (5.10), we get the following system of linear equations in the unknowns A, B:

$$(S\phi_\lambda)_1(0)A + (S\theta_\lambda)_1(0)B = \sin\alpha, \tag{5.14}$$

$$(S\phi_\lambda)_2(0)A + (S\theta_\lambda)_2(0)B = -\cos\alpha. \tag{5.15}$$

This system has a unique nontrivial solution at every $\lambda \in \mathbf{C}$ since, cf. [**13**],

(5.16)
$$\begin{aligned}
\mathcal{D} = \begin{vmatrix} (S\phi_\lambda)_1(0) & (S\theta_\lambda)_1(0)\\ (S\phi_\lambda)_2(0) & (S\theta_\lambda)_2(0)\end{vmatrix} &= D_0(\phi_\lambda,\theta_\lambda) = W_0(\phi_\lambda,\theta_\lambda)\\
= W_c(\phi_\lambda,\theta_\lambda) = \begin{vmatrix} 0 & 1\\ -1 & 0\end{vmatrix} &= 1 \neq 0 \quad \text{for all } \lambda\in\mathbf{C}.
\end{aligned}$$

Solving this system, one obtains (5.11). □

Now we have the following two sampling theorems. The first concerns sampling representations of $\phi_\lambda(\cdot)$ and $\psi_\lambda(\cdot)$, while the second deals with sampling series for the associated integral transforms of Hankel type.

Theorem 5.2. *The function $\phi_\lambda(\cdot)$ and $\psi_\lambda(\cdot)$ defined in* (5.7) *and* (5.11) *have the sampling representations*

$$\begin{pmatrix}\phi_\lambda(x)\\ \psi_\lambda(x)\end{pmatrix} = \sum_{n=0}^{\infty}\begin{pmatrix}\phi_{\lambda_n}(x)\\ \psi_{\lambda_n}(x)\end{pmatrix}\frac{\omega_\alpha(\lambda)}{(\lambda-\lambda_n)\omega_\alpha'(\lambda_n)}. \tag{5.17}$$

Series (5.17) *converges pointwise with respect to* λ *for* $x \in (0, c]$ *and in the* $L^2(0, c)$*-norm with respect to* x *for* $\lambda \in \mathbf{C}$.

Theorem 5.3. *Let* $g(\cdot) \in L^2(0, c)$ *and*

$$\begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \end{pmatrix} = \int_0^c g(x) \begin{pmatrix} \phi_\lambda(x) \\ \psi_\lambda(x) \end{pmatrix} dx. \tag{5.18}$$

Then $f_1(\lambda)$ *and* $f_2(\lambda)$ *are entire functions of* λ *and admit the sampling formulae*

$$\begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} f_1(\lambda_n) \\ f_2(\lambda_n) \end{pmatrix} \frac{\omega_\alpha(\lambda)}{(\lambda - \lambda_n)\omega'_\alpha(\lambda_n)}. \tag{5.19}$$

Series (5.19) *converges absolutely and uniformly on compact subsets of the complex plane.*

Remark 5.4. The sampling theorem associated with Bessel's equation in [**25**] is a special case of the above one, when $\alpha = \pi/4$.

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