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A MULTI-INDEX BOREL-DZRBASHJAN TRANSFORM

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ABSTRACT. An integral transform involving a Fox's H-function is introduced. This integral transform is closely related to a multi-index analogue of the classical Mittag-Leffler function. Along with the basic operational and mapping properties of this transform, the new results presented here include complex and real inversion formulas and a convolution theorem.

1. Introduction. The role of the *Laplace transform*:

$$\mathcal{L}\{f(z);s\} = \int_0^\infty \exp(-sz) f(z)\,dz \tag{1}$$

in the operational calculus, and its use in various problems of applied analysis, engineering and other fields are well-known. The success of the Laplace transform motivates the search for other more general transforms of similar type. As an integral transform of resembling type, one can mention the *Borel-Dzrbashjan transform*, studied initially by Dzrbashjan [**5**], and later by Dimovski and Kiryakova [**3**]:

$$\mathcal{B}_{\rho,\mu}\{f(z);s\} = \rho s^{\mu\rho-1} \int_0^\infty \exp(-s^\rho z^\rho) z^{\mu\rho-1} f(z)\,dz, \qquad \rho>0, \quad \mu>0. \tag{2}$$

Another transform of the same type that is related to the Bessel differential operator is the well-known *Meijer transform*:

$$\mathcal{K}_\nu\{f(z);s\} = \int_0^\infty \sqrt{sz}K_\nu(sz) f(z)\,dz, \tag{3}$$

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where $K_\nu(z)$ is the Bessel function of the third kind.

In 1958, Obrechkoff [**11**] introduced a far reaching generalization of the Laplace and Meijer transforms, namely,

$$\mathcal{O}\{f(z);s\} = \beta \int_0^\infty K[(sz)^\beta] z^{\beta(\nu_m+1)-1} f(z)\,dz, \tag{4}$$

where $m \geq 1$ is an integer, $\beta > 0$, $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_m$ are real parameters, and the kernel-function $K(z)$ is expressed in the form

$$K(z) = \int_0^\infty \cdots \int_0^\infty \left[\prod_{k=1}^{m-1} u_k^{\nu_k - \nu_m - 1}\right] \times \exp\left(- u_1 - \cdots - u_{m-1} - \frac{z}{u_1 \cdots u_{m-1}}\right) du_1 \ldots du_{m-1}.$$

The transform (4) is called the *Obrechkoff transform* of order m. In [**1, 2, 4**] and [**8**, Chapter 3], Dimovski and Kiryakova studied this transform for the purposes of operational calculi for Bessel-type differential operators of arbitrary integer order $m > 1$. In particular, in [**4, 8**], they discovered that the kernel $K(z)$ is a Meijer G-function.

These integral transforms, besides being analogues to the Laplace transform, give examples for the so-called *convolution type transforms* [**7, 17**], since for all of them and their numerous particular cases, especially those following from (4), convolution operations and respective convolution properties have been found.

There is a class of integral transforms that is associated with a generalized hypergeometric function known as the *H-function*, whose definition we repeat here for the sake of completeness:

$$H_{p,q}^{m,n}\left[\sigma \middle| \begin{matrix} (a_j, A_j)_1^p \\ (b_k, B_k)_1^q \end{matrix}\right] = \frac{1}{2\pi i} \int_{\mathcal{C}'} \frac{\prod_{k=1}^m \Gamma(b_k - sB_k) \prod_{j=1}^n \Gamma(1 - a_j + sA_j)}{\prod_{k=m+1}^q \Gamma(1 - b_k + sB_k) \prod_{j=n+1}^p \Gamma(a_j - sA_j)} \sigma^s\,ds, \tag{5}$$

where $\mathcal{C}'$ is a suitable contour in $\mathbf{C}$, the orders (m,n,p,q) are integers with $0 \leq m \leq q$, $0 \leq n \leq p$ and the parameters $a_j \in \mathbf{R}$, $A_j > 0$,

$j = 1, \dots, p$, $b_k \in \mathbf{R}$, $B_k > 0$, $k = 1, \dots, q$, are such that $A_j(b_k + l) \neq B_k(a_j - l' - 1)$, $l, l' = 0, 1, 2, \dots$. For various types of contours and conditions for existence and analyticity of function (5), as well as asymptotic expansions as $\sigma \to 0$ and $\sigma \to \infty$, one can see [**8**, Appendix], [**10, 12, 14**].

For $A_1 = \cdots = A_p = B_1 = \cdots = B_q = 1$, the H-function turns into the simpler *Meijer's G-function* [**6**, Chapter 5], [**8, 12**]:

$$G_{p,q}^{m,n}\left[\sigma \left| \begin{matrix} (a_j)_1^p \\ (b_k)_1^q \end{matrix} \right.\right] = \frac{1}{2\pi i} \int_{\mathcal{C}'} \frac{\prod_{k=1}^m \Gamma(b_k - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{k=m+1}^q \Gamma(1 - b_k + s) \prod_{j=n+1}^p \Gamma(a_j - s)} \sigma^s ds.$$

The G- and H-*transforms* are defined as

$$\mathcal{G}\{f(z); s\} = \int_0^\infty G_{p,q}^{m,n}\left[sz \left| \begin{matrix} (a_j)_1^p \\ (b_k)_1^q \end{matrix} \right.\right] f(z)\, dz,$$

and

$$\mathcal{H}\{f(z); s\} = \int_0^\infty H_{p,q}^{m,n}\left[sz \left| \begin{matrix} (a_j, A_j)_1^p \\ (b_k, B_k)_1^q \end{matrix} \right.\right] f(z)\, dz, \tag{6}$$

respectively.

In this paper we consider a special H-transform of the form (6). This transform turns out to share many properties similar to those of the Laplace transform. Moreover, the inverse transform and the operational calculus, which is based on the transform, are related to the recently introduced *multi-index Mittag-Leffler function*. We derive some basic operational properties, complex and real inversion formulas, as well as a convolution theorem.

2. Multi-index Borel-Dzrbashjan transform.

We introduce in this section a multi-index Borel-Dzrbashjan transform and discuss some of its mapping and operational properties.

Definition 1. Let $m \geq 1$ be an integer and $\rho_i, \mu_i \in \mathbf{R}$ with $\rho_i > 0$, $i = 1, \dots, m$. Define the H-transform $\mathcal{B}_{(\rho_i),(\mu_i)}\{f(z); s\}$ by:

$$\begin{aligned} \mathcal{B}(s) = (\mathcal{B}f)(s) &= \mathcal{B}_{(\rho_i),(\mu_i)}\{f(z); s\} \\ &= \int_0^\infty H_{0,m}^{m,0}\left[sz \left| \begin{matrix} -- \\ (\mu_i - \frac{1}{\rho}, \frac{1}{\rho})_1^m \end{matrix} \right.\right] f(z)\, dz. \end{aligned} \tag{7}$$

We shall call the transform (7) a multi-index Borel-Dzrbashjan transform.

If $m = 1$ in (7), we obtain the Borel-Dzrbashjan transform (2), which justifies the adjective *multi-index*. If, additionally, $\mu = \rho = 1$, then the *Laplace integral transform* (1) follows as a special case, and for this reason we sometimes say the transform (7) is of *Laplace-type*.

In [**4**] and [**8**, Chapter 3], the *Obrechkoff integral transform* (4) has been represented as a G-transform of Laplace type, namely,

$$\mathcal{O}\{f(z);s\} = \beta s^{-\beta(\gamma_m+1)+1}\int_0^\infty G^{m,0}_{0,m}[\cdots]f(z)\,dz. \tag{8}$$

Due to the relation [**8**, Appendix], [**10, 14**],

$$G^{m,0}_{0,m}\left[z^\beta\middle|\begin{matrix}--\\(\nu_i+1-\frac{1}{\beta})_1^m\end{matrix}\right] = \frac{1}{\beta}H^{m,0}_{0,m}\left[z\middle|\begin{matrix}--\\(\nu_i+1-\frac{1}{\beta},\frac{1}{\beta})_1^m\end{matrix}\right], \quad \beta > 0,$$

the Obrechkoff transform is also a multi-index Borel-Dzrbashjan transform of form (7) with $\mu_i = \nu_i + 1$, $\rho_i = (1/\beta)$, $i = 1,\dots,m$; namely, $\mathcal{O}\{f(z);s\} = s^{-\beta\mu_m+1}\mathcal{B}_{(\rho_i),(\mu_i)}\{f(z);s\}$. Its properties, inversion theorems, and convolution discovered in [**4**] and [**8**, Chapter 3], can follow from our present presentation.

Since $K_\nu(z) = (1/2)G^{2,0}_{0,2}\left[(z^2/4)\middle|\begin{matrix}--\\\nu/2,-\nu/2\end{matrix}\right]$, [**6**], the *Meijer transform* (3) comes out as a special case of the Obrechkoff transform (4), (8), when $m = \beta = 2$, $\nu_i = \pm\nu/2$, namely: $\mathcal{K}_\nu\{f(z);s\} = 2^{\nu-2}s^{-\nu+1/2}\mathcal{O}\{f(z);s/2\}$. Thus, (3) can be seen also as a special case of the multi-index Borel-Dzrbashjan transform (7).

Throughout this paper we use the notation $H^{m,0}_{0,m}(sz)$ for $H^{m,0}_{0,m}\left[sz\middle|\begin{matrix}--\\(\mu_i-(1/\rho_i),(1/\rho_i))_1^m\end{matrix}\right]$ and reserve the letters α,μ and ρ to mean the following:

$$\begin{gathered}\alpha = \min_{1\le i\le m}\{\mu_i\rho_i\} - 1, \quad \mu = \mu_1+\cdots+\mu_m,\\ \frac{1}{\rho} = \frac{1}{\rho_1}+\cdots+\frac{1}{\rho_m}, \quad \sigma = \left(\frac{\rho_1}{\rho}\right)^{\rho/\rho_1}\cdots\left(\frac{\rho_m}{\rho}\right)^{\rho/\rho_m}.\end{gathered} \tag{9}$$

We consider (7) for the space Ξ_c of functions $f(z)$ such that $f(z)z^{\alpha}e^{-cz^{\rho}} \in L_1(\mathbf{R}^+)$ for some real c. The complex variable s varies in the complex domain

$$\mathcal{D}^c = \left\{ s : \Re(s^{\rho}) > \frac{c}{\sigma}, |\arg s| < \frac{\pi}{2\rho} \right\}.$$

The known asymptotics [**10, 14**]:

$$\begin{aligned} &H^{m,0}_{0,m}(z) = O(|z|^{\alpha}), \quad \text{as } z \to 0, \\ &H^{m,0}_{0,m}(z) \sim \exp(-\sigma z^{\rho}) z^{\rho(\mu - \frac{1}{\rho} - \frac{m-1}{2})}, \\ &\qquad \text{as } z \to \infty, \quad |\arg(z)| < \frac{\pi}{2\rho}, \end{aligned}$$

make it clear that the integral (7) is absolutely convergent for $f \in \Xi_c$ and $s \in \mathcal{D}^c$, and its value tends to zero as $s \to \infty$ in $\mathcal{D}^c$. Moreover, since

$$\begin{aligned} \frac{d}{ds} H^{m,0}_{0,m}(sz) &= \frac{\rho_m \mu_m - 1}{s} H^{m,0}_{0,m}(sz) \\ &\quad - \frac{\rho_m}{s} H^{m,0}_{0,m}\left[sz \middle| \begin{matrix} \text{---} \\ (\mu_i - \frac{1}{\rho_i}, \frac{1}{\rho_i})_1^{m-1}, \quad (\mu_m + 1 - \frac{1}{\rho_m}, \frac{1}{\rho_m}) \end{matrix} \right], \end{aligned}$$

the integral $\displaystyle\int_0^{\infty} f(z) \frac{d}{ds} H^{m,0}_{0,m}(sz)\, dz$ is also absolutely convergent for $s \in \mathcal{D}^c$. Hence, $\mathcal{B}(s)$ is an analytic function in the region $\mathcal{D}^c$, and in this region, $\mathcal{B}(s) \to 0$ as $s \to \infty$.

One can easily use definition (7) to evaluate the images of some functions. For example, using the formula for integrals of products of two different H-functions [**8, 10, 14**],

$$\begin{aligned} &\int_0^{\infty} z^{\beta-1} H^{s,t}_{u,v}\left[\eta z \middle| \begin{matrix} (c_i, C_i)_1^u \\ (d_l, D_l)_1^v \end{matrix} \right] H^{l,n}_{p,q}\left[\omega z^r \middle| \begin{matrix} (a_j, A_j)_1^p \\ (b_k, B_k)_1^q \end{matrix} \right] dz \\ &= \eta^{-\beta} H^{l+t,n+s}_{p+v,q+u}\left[\frac{\omega}{\eta^r} \middle| \begin{matrix} (a_j, A_j)_1^n, (1 - d_l - \beta D_l, r D_l)_1^v, (a_j, A_j)_{n+1}^p \\ (b_k, B_k)_1^l, (1 - c_i - \beta C_i, r C_i)_1^u, (b_k, B_k)_{l+1}^q \end{matrix} \right], \end{aligned}$$

we get

$$\begin{aligned} &\mathcal{B}_{(\rho_i),(\mu_i)}\left\{ H^{l,n}_{p,q}\left[\omega z^r \middle| \begin{matrix} (a_j, A_j)_1^p \\ (b_k, B_k)_1^q \end{matrix} \right], s \right\} \\ &= s^{-\beta} H^{l,m+n}_{p+m,q}\left[\frac{\omega}{s^r} \middle| \begin{matrix} (a_j, A_j)_1^n, \quad (1 - \mu_i + \frac{1}{\rho_i} - \frac{\beta}{\rho_i}, \frac{r}{\rho_i})_1^m, (a_j, A_j)_{n+1}^p \\ (b_k, B_k)_1^q \end{matrix} \right]. \end{aligned}$$

Also, from [**10, 12**], we find

$$(10) \qquad \mathcal{B}_{(\rho_i),(\mu_i)}\{z^k\} = s^{-(k+1)} \prod_{i=1}^{m} \Gamma\left(\mu_i + \frac{k}{\rho_i}\right), \quad \operatorname{Re} k > -\alpha - 1.$$

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function of order ρ and type σ, that is,

$$|f(z)| \le C \exp[(\sigma + \varepsilon)|z|^{\rho}], \quad \text{for any } \varepsilon > 0.$$

Suppose further that $\mu_i > 0$, $i = 1, 2, \dots, m$. Then $\alpha > -1$, and formula (10) holds for any nonnegative integer k and we arrive at

$$(11) \quad f(z) = \sum_{k=0}^{\infty} a_k z^k \overset{\mathcal{B}_{(\rho_i),(\mu_i)}}{\longmapsto} (\mathcal{B}f)(s) = \sum_{k=0}^{\infty} \frac{a_k \prod_{i=1}^{m} \Gamma(\mu_i + k/\rho_i)}{s^{k+1}}.$$

We now show that the transform (7) is an isomorphism on some subspaces of $L^2(\mathbf{R}^+)$. To this end, let (c, γ) be a pair of real numbers such that: either $c > 0$ and γ is arbitrary or $c = 0$ and $\gamma > 0$. These two conditions can be combined by means of the sign-symbol, as follows:

$$2\operatorname{sign} c + \operatorname{sign} \gamma \ge 0.$$

Let $\mathcal{M}_{c,\gamma}^{-1}(L_2)$ denote the subset of $L_2(\mathbf{R}^+)$ consisting of all functions f such that

$$f(z) = \lim_{N\to\infty} \frac{1}{2\pi i} \int_{1/2-iN}^{1/2+iN} f^*(s) z^{-s}\, ds, \quad z > 0,$$

where $f^*(s)s^{\gamma} \exp(\pi c|s|) \in L_2(1/2-i\infty, 1/2+i\infty)$, and the convergence is understood in the $L_2(\mathbf{R}^+)$ norm. Thus, $f^*(s)$ is the *Mellin transform* [**15**] of f:

$$(12) \qquad f^*(s) = \int_0^{\infty} z^{s-1} f(z)\, dz.$$

The set $\mathcal{M}_{c,\gamma}^{-1}(L_2)$, equipped with the norm

$$\|f(z)\|_{\mathcal{M}_{c,\gamma}^{-1}(L_2)} = \|f^*(s)\|_{L_2((1/2-i\infty,1/2+i\infty);|s|^{2\gamma}\exp(2\pi c|s|))},$$

is a Banach space. The space $\mathcal{M}^{-1}_{c,\gamma}(L_2)$ was introduced in [**16**] and it was shown in [**17**] that $f \in \mathcal{M}^{-1}_{c,\gamma}(L_2)$ if and only if

$$\|z^{\gamma} D^{\gamma} f(z)\|_{L_2(\mathbf{R}^+)} < \infty, \quad \text{if } c = 0,$$
$$\sum_{k=0}^{\infty} \frac{(2\pi c)^{2k}}{(2k)!} \left\| \left(z \frac{d}{dz} \right)^k z^{\gamma} D^{\gamma} f(z) \right\|^2_{L_2(\mathbf{R}^+)} < \infty, \quad \text{if } c > 0,$$

where D^{ρ} is the Riemann-Liouville operator of fractional integration if $\rho < 0$, and the Riemann-Liouville fractional differentiation if $\rho \geq 0$, [**13**]:

$$D^{\rho} y(z) = D^{\rho}_z y(z) = \frac{d^n}{dz^n} \int_0^z \frac{(z-\xi)^{n-\rho-1}}{\Gamma(n-\rho)} y(\xi)\, d\xi, \tag{13}$$
$$\text{with integer } n : n-1 \leq \rho < n.$$

The chain of the subspaces $\mathcal{M}^{-1}_{c,\gamma}(L_2)$ is well-ordered, that is,

$$\mathcal{M}^{-1}_{c,\gamma}(L_2) \subset \mathcal{M}^{-1}_{c',\gamma'}(L_2) \subset \mathcal{M}^{-1}_{0,0}(L_2) = L_2(\mathbf{R}^+),$$
$$\text{if } 2\text{sign}\,(c-c') + \text{sign}\,(\gamma - \gamma') \geq 0.$$

It is proved in [**16**] that the H-transform, defined as in (6), is well-defined in $L_2(\mathbf{R}^+)$ if

$$2\text{sign}\, c^* + \text{sign} \left(\gamma^* - \frac{1}{2} \right) > 0, \tag{14}$$

and, in this case, is an isomorphism from $\mathcal{M}^{-1}_{c,\gamma}(L_2)$ onto $\mathcal{M}^{-1}_{c+c^*,\gamma+\gamma^*}(L_2)$ where

$$c^* = \frac{1}{2} \left(\sum_{i=1}^{n} A_i - \sum_{i=n+1}^{p} A_i + \sum_{i=1}^{m} B_i - \sum_{i=m+1}^{q} B_i \right),$$
$$\gamma^* = \frac{1}{2} \left(\sum_{i=1}^{p} A_i - \sum_{i=1}^{q} B_i \right) + \frac{q-p}{2} + \Re \left(\sum_{i=1}^{p} a_i - \sum_{i=1}^{q} b_i \right).$$

For the Laplace-type H-transform (7), we have

$$c^* = \frac{1}{2\rho}, \gamma^* = \frac{1}{2\rho} + \frac{m}{2} - \mu,$$

and thus (14) is satisfied. Here μ stands for $\sum_1^m \mu_i$ but, if one considers arbitrary complex parameters μ_i, then $\mu := \sum_1^m \Re\mu_i$. Thus we obtain the following mapping property:

Theorem 1. *The transform $B_{(\rho_i),(\mu_i)}$, defined by* (7), *is an isomorphism from $\mathcal{M}^{-1}_{c,\gamma}(L_2)$ onto $\mathcal{M}^{-1}_{c+(1/2\rho),\gamma+(1/2\rho)+(m/2)-\mu}(L_2)$.*

Next we discuss some operational properties of the multi-index Borel-Dzrbashjan transform (7) that are analogues to the well-known Laplace transform's rules:

$$\mathcal{L}\left\{\int_0^z f(\varsigma)\,d\varsigma; s\right\} = \frac{1}{s}\mathcal{L}\{f(z); s\}, \quad \text{the “integral law,”}$$
$$\mathcal{L}\left\{\frac{d}{dz}f(z); s\right\} = s\mathcal{L}\{f(z); s\} - f(0), \quad \text{the “differential law.”}$$

It turns out that transform (7) is similarly related to a pair of “integration” and “differentiation” operators introduced in [**9**]:

Definition 2. Let $\mu_i \geq 0,\ i = 1, \dots, m$, and $f(z) = \sum_{k=0}^\infty a_k z^k$ be an analytic function in a disc $\mathcal{D}_r = \{z : |z| < r\}$. Define the operators $D_{(\rho_i),(\mu_i)}$ and $L_{(\rho_i),(\mu_i)}$ by:

$$D_{(\rho_i),(\mu_i)}f(z) = \sum_{k=1}^\infty a_k \frac{\Gamma(\mu_1 + \frac{k}{\rho_1})\cdots\Gamma(\mu_m + \frac{k}{\rho_m})}{\Gamma(\mu_1 + \frac{k-1}{\rho_1})\cdots\Gamma(\mu_m + \frac{k-1}{\rho_m})} z^{k-1},$$
$$L_{(\rho_i),(\mu_i)}f(z) = \sum_{k=0}^\infty a_k \frac{\Gamma(\mu_1 + \frac{k}{\rho_1})\cdots\Gamma(\mu_m + \frac{k}{\rho_m})}{\Gamma(\mu_1 + \frac{k+1}{\rho_1})\cdots\Gamma(\mu_m + \frac{k+1}{\rho_m})} z^{k+1}.$$

If $m = 1$, we get the so-called *Dzrbashjan-Gelfond-Leontiev operators of differentiation and integration* studied in [**3**] and [**8**, Chapter 2]. Thus we call the operators $D_{(\rho_i),(\mu_i)}$ and $L_{(\rho_i),(\mu_i)}$ the *multi-index Dzrbashjan-Gelfond-Leontiev differentiation and integration*, respectively.

The operator $D_{(\rho_i),(\mu_i)}$ can be considered as a fractional analogue of the hyper-Bessel differential operator. In fact, we have

$$D_{(\rho_i),(\mu_i)}f(z) = z^{-1}\prod_{i=1}^m \left(z^{1+(1-\mu_i)\rho_i} D^{1/\rho_i}_{z^{\rho_i}} z^{(\mu_i-1)\rho_i}\right) f(z). \tag{15}$$

To see (15), apply the formula

$$D_w^\beta w^\mu = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\beta)} w^{\mu-\beta}$$

with $w = z^{\rho_i}$ and obtain

$$\begin{aligned}\left(z^{1+(1-\mu_i)\rho_i} D_{z^{\rho_i}}^{1/\rho_i} z^{(\mu_i-1)\rho_i}\right) z^k &= w^{1-\mu_i+(1/\rho_i)} D_w^{1/\rho_i} w^{\mu_i-1+(k/\rho_i)} \\ &= w^{1-\mu_i+(1/\rho_i)} \frac{\Gamma(\mu_i+(k/\rho_i))}{\Gamma(\mu_i+(k-1)/\rho_i)} \\ &\quad \times w^{\mu_i-1+(k-1)/\rho_i} \\ &= \frac{\Gamma(\mu_i+(k/\rho_i))}{\Gamma(\mu_i+(k-1)/\rho_i)} z^k.\end{aligned}$$

Therefore, if $f(z) = \sum_{k=0}^\infty a_k z^k$, we have

$$\begin{aligned}&z^{-1} \prod_{i=1}^m \left(z^{1+(1-\mu_i)\rho_i} D_{z^{\rho_i}}^{1/\rho_i} z^{(\mu_i-1)\rho_i}\right) f(z) \\ &\qquad = z^{-1} \sum_{k=0}^\infty a_k \prod_{i=1}^m \left(z^{1+(1-\mu_i)\rho_i} D_{z^{\rho_i}}^{1/\rho_i} z^{(\mu_i-1)\rho_i}\right) z^k \\ &\qquad = z^{-1} \sum_{k=0}^\infty a_k \prod_{i=1}^m \frac{\Gamma(\mu_i+(k/\rho_i))}{\Gamma(\mu_i+(k-1/\rho_i))} z^k \\ &\qquad = D_{(\rho_i),(\mu_i)} f(z).\end{aligned}$$

Operator (15) encompasses many operators appearing in the literature. For example, the so-called *hyper-Bessel differential oeprator* in the form

$$\begin{aligned}B &= z^{\alpha_0} \frac{d}{dz} z^{\alpha_1} \frac{d}{dz} z^{\alpha_2} \cdots \frac{d}{dz} z^{\alpha_m} = z^{-\beta} \prod_{i=1}^m \left(z^{-\beta\nu_i+1} \frac{d}{dz} z^{\beta\nu_i}\right) \\ &= z^{-\beta} \beta^m \prod_{i=1}^m \left(\frac{z}{\beta} \frac{d}{dz} + \nu_i\right)\end{aligned}$$

is easily seen to be a special case of (15). Also, the operators

$$B_{\nu,n} = D z^{-\nu+1/n} (z^{-\nu+1/n} D)^{n-1} z^{\nu+1+2/n}, \quad n > 1,$$

and

$$B_m = DzD\dots zD,$$

considered by Krätzel (1963–1967) and by Ditkin, Prudnikov (1963) and Botashev (1965), respectively, are special cases of (15).

The next theorem shows that transform (7) "*algebraizes*" the operators $D_{(\rho_i),(\mu_i)}$ and $L_{(\rho_i),(\mu_i)}$, i.e., reduces them to multiplications by fixed rational functions, a property similar to that of the Laplace transform with respect to the ordinary integration and differentiation.

Theorem 2. *Let $\mu_i > 0$, $i = 1, \dots, m$. If $f(z)$ is an entire function of order ρ and type σ, then*:

$$\mathcal{B}_{(\rho_i),(\mu_i)}\Big\{L_{(\rho_i),(\mu_i)}f(z);s\Big\} = \frac{1}{s}\mathcal{B}_{(\rho_i),(\mu_i)}\{f(z);s\}, \tag{16}$$

and

$$\mathcal{B}_{(\rho_i),(\mu_i)}\{D_{(\rho_i),(\mu_i)}f(z);s\} = s\mathcal{B}_{(\rho_i),(\mu_i)}\{f(z);s\} - f(0)\prod_{i=1}^{m}\Gamma(\mu_i). \tag{17}$$

Proof. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Definition 2 and formula (11) yield

$$\begin{aligned}
&\mathcal{B}_{(\rho_i),(\mu_i)}\{L_{(\rho_i),(\mu_i)}f(z);s\}\\
&= \mathcal{B}_{(\rho_i),(\mu_i)}\Bigg\{\sum_{k=0}^{\infty} a_k \frac{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})}{\Gamma(\mu_1+\frac{k+1}{\rho_1})\cdots\Gamma(\mu_m+\frac{k+1}{\rho_m})} z^{k+1};s\Bigg\}\\
&= \sum_{k=0}^{\infty} \frac{a_k}{s^{k+2}} \frac{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})}{\Gamma(\mu_1+\frac{k+1}{\rho_1})\cdots\Gamma(\mu_m+\frac{k+1}{\rho_m})} \prod_{i=1}^{m}\Gamma\bigg(\mu_i+\frac{k+1}{\rho_i}\bigg)\\
&= \frac{1}{s}\sum_{k=0}^{\infty} a_k \frac{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})}{s^{k+1}}\\
&= \frac{1}{s}\sum_{k=0}^{\infty} a_k \mathcal{B}_{(\rho_i),(\mu_i)}\{z^k;s\} = \frac{1}{s}\mathcal{B}_{(\rho_i),(\mu_i)}\Bigg\{\sum_{k=0}^{\infty} a_k z^k;s\Bigg\}\\
&= \frac{1}{s}\mathcal{B}_{(\rho_i),(\mu_i)}\{f(z);s\},
\end{aligned}$$

which is (16).

To prove formula (17), notice that $f(0) = a_0$. Thus,

$$\begin{aligned}
&\mathcal{B}_{(\rho_i),(\mu_i)}\{D_{(\rho_i),(\mu_i)}f(z);s\} \\
&= \mathcal{B}_{(\rho_i),(\mu_i)}\bigg\{\sum_{k=1}^{\infty} a_k \frac{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})}{\Gamma(\mu_1+\frac{k-1}{\rho_1}\cdots\Gamma(\mu_m+\frac{k-1}{\rho_m})} z^{k-1};s\bigg\} \\
&= \sum_{k=1}^{\infty} \frac{a_k}{s^k} \frac{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})}{\Gamma(\mu_1+\frac{k-1}{\rho_1})\cdots\Gamma(\mu_m+k-1\rho_m)} \prod_{i=1}^{m}\Gamma\bigg(\mu_i+\frac{k-1}{\rho_i}\bigg) \\
&= \sum_{k=0}^{\infty} a_k \frac{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})}{s^k} - a_0\prod_{i=1}^{m}\Gamma(\mu_i) \\
&= s\sum_{k=1}^{\infty} a_k \frac{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})}{s^{k+1}} - f(0)\prod_{i=1}^{m}\Gamma(\mu_i) \\
&= s\sum_{k=0}^{\infty} a_k \mathcal{B}_{(\rho_i),(\mu_i)}\{z^k;s\} - f(0)\prod_{i=1}^{m}\Gamma(\mu_i) \\
&= s\mathcal{B}_{(\rho_i),(\mu_i)}\bigg\{\sum_{k=0}^{\infty} a_k z^k;s\bigg\} - f(0)\prod_{i=1}^{m}\Gamma(\mu_i) \\
&= s\mathcal{B}_{(\rho_i),(\mu_i)}\{f(z);s\} - f(0)\prod_{i=1}^{m}\Gamma(\mu_i),
\end{aligned}$$

which is (17). □

3. Inversion formulas. The formula for the inverse transform of the Borel-Dzrbashjan transform (2) involves the Mittag-Leffler function [**5**],

$$E_{(1/\rho),\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu+(k/\rho))}, \quad \rho > 0, \mu > 0. \tag{18}$$

To obtain a formula for the inverse transform of the multi-index Borel-Dzrbashjan transform (7), we need the multi-index Mittag-Leffler function introduced by the second author in [**9**]:

Definition 3. Let ρ_i and μ_i, $i = 1, \dots, m$, be as in Definition 1. The multi-index Mittag-Leffler function is

$$E_{(1/\rho_i),(\mu_i)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + k/\rho_1) \cdots \Gamma(\mu_m + k/\rho_m)}. \tag{19}$$

The reader is referred to [**9**] for the basic properties of the multi-index Mittag-Leffler function, its expression as a Wright's generalized hypergeometric function as well as Fox's H-function, and its representation by a Mellin-Barnes type contour integral. It is also proved there that the multi-index Mittag-Leffler function is an entire function of order ρ and type σ where ρ and σ are as in (9).

It turns out that the multi-index Mittag-Leffler function is closely related to the multi-index Borel-Dzrbashjan transform (7). Our starting point is an asymptotic formula that we state in the following lemma.

Lemma 1. *The following asymptotic formula for multi-index Mittag-Leffler functions* (19) *holds*:

$$|E_{(1/\rho_i),(\mu_i)}(z)| \leq C|z|^{\rho((1/2)+\mu-(m/2))} \exp(\sigma|z|^{\rho}), \quad |z| \to \infty, \tag{20}$$

with ρ, μ *and* σ *as in* (9).

Proof. The Stirling formula for the gamma function yields

$$\begin{aligned} \prod_{j=1}^{m} \Gamma\left(\mu_j + \frac{k}{\rho_j}\right) &\sim \prod_{j=1}^{m} \left(\frac{k}{\rho_j}\right)^{(k/\rho_j)+\mu_j-(1/2)} \exp\left(-\frac{k}{\rho_j}\right) \\ &\sim \left(\frac{k}{\rho}\right)^{(k/\rho)+\mu-(m/2)} \exp\left(-\frac{k}{\rho}\right) \prod_{j=1}^{m} \left(\frac{\rho}{\rho_j}\right)^{k/\rho_j} \\ &\sim \Gamma\left(\frac{1}{2} + \mu - \frac{m}{2} + \frac{k}{\rho}\right) \prod_{j=1}^{m} \left(\frac{\rho}{\rho_j}\right)^{k/\rho_j}. \end{aligned}$$

By definition (19), we find

$$\begin{aligned}|E_{(1/\rho_i),(\mu_i)}(z)| &\le \sum_{k=0}^{\infty} \frac{|z|^k}{\prod_{j=1}^{m} \Gamma(\mu_j + (k/\rho_j))} \\ &\le C \sum_{k=0}^{\infty} \frac{1}{\Gamma((1/2) + \mu - (m/2) + (k/\rho))} \\ &\quad \times \left[|z| \prod_{j=1}^{m} (\rho/\rho_j)^{1/\rho_j} \right]^k \\ &= E_{(1/\rho),(1/2)+\mu-(m/2)} \left(|z| \prod_{j=1}^{m} (\rho_j/\rho)^{1/\rho_j} \right),\end{aligned}$$

where $E_{(1/\rho),\mu}(z)$ is the Mittag-Leffler function (18). Then using the asymptotic formula

$$E_{1/\rho,\mu}(z) \sim \rho z^{\rho(1-\mu)} \exp(z^\rho), \quad z > 0, z \to \infty,$$

derived by Dzrbashjan [**5**], we obtain

$$\begin{aligned}&E_{(1/\rho),(1/2)+\mu-(m/2)} \left(|z| \prod_{j=1}^{m} (\rho_j/\rho)^{1/\rho_j} \right) \\ &\qquad \le C|z|^{\rho((1/2)-\mu+(m/2))} \exp \left(|z|^\rho \prod_{j=1}^{m} (\rho_j/\rho)^{\rho/\rho_j} \right) \\ &\qquad = C|z|^{\rho((1/2)-\mu+(m/2))} \exp(\sigma |z|^\rho).\end{aligned}$$

Thus

$$|E_{(1/\rho_i),(\mu_i)}(z)| \le C|z|^{\rho((1/2)-\mu+(m/2))} \exp(\sigma|z|^\rho),$$

which is (20). □

The asymptotic estimate (20) ensures that the multi-index Mittag-Leffler function belongs to Ξ_c, provided that $\mu_i > 0$, $i = 1, 2, \dots, m$, and $c > \sigma$. Using (11) for the power series (19), we obtain

$$\text{(21)} \quad \mathcal{B}_{(\rho_i),(\mu_i)} \Big\{ E_{(1/\rho_i),(\mu_i)}(\lambda z); s \Big\} = \frac{1}{s - \lambda}, \qquad |s| > |\lambda|, \lambda \ne 0, \mu_i > 0, \quad i = 1, \dots, m.$$

To obtain an inversion formula $\mathcal{B}^{-1}_{(\rho_i),(\mu_i)}$ for the multi-index Borel-Dzrbashjan transform (7) of a function $f \in \Xi_c$ in a complex contour integral form, we begin with the Cauchy integral formula for the image $\mathcal{B}(s) = \mathcal{B}_{(\rho_i),(\mu_i)}\{f(z); s\}$, an analytic function in the complex domain $\mathcal{D}^c$, namely,

$$\mathcal{B}(\lambda) = \frac{1}{2\pi i}\int_{\partial\mathcal{D}^c} \frac{\mathcal{B}(s)}{s-\lambda}\, ds, \quad \text{for } \lambda \in \mathcal{D}^c.$$

Here $\partial\mathcal{D}^c$ is the boundary of the domain $\mathcal{D}^c$, starting at $e^{-i\pi/2\rho}\infty$ and ending at $e^{i\pi/2\rho}\infty$. Applying the operator $\mathcal{B}^{-1}_{(\rho_i),(\mu_i)}$ to both sides, we get

$$f(z) = \frac{1}{2\pi i}\int_{\partial\mathcal{D}^c} \mathcal{B}(s)\mathcal{B}^{-1}_{(\rho_i),(\mu_i)}\left\{\frac{1}{s-\lambda}\right\} ds.$$

Formula (21) implies that

$$\mathcal{B}^{-1}_{(\rho_i),(\mu_i)}\left\{\frac{1}{s-\lambda}; z\right\} = E_{(1/\rho_i),(\mu_i)}(sz),$$

and therefore,

$$f(z) = \frac{1}{2\pi i}\int_{\partial\mathcal{D}^c} E_{(1/\rho_i),(\mu_i)}(sz)\mathcal{B}(s)\, ds. \tag{22}$$

A rigorous proof of formula (22) is lengthy and would be a subject of another paper.

Another complex inversion formula that effectively uses the *Mellin transform* techniques is the content of the following theorem.

Theorem 3 (Complex inversion formula). *Let $f(z) \in L_1(\mathbf{R}^+; z^{c-1})$ with $c < \alpha+1$ be of bounded variation at z. Then the following inversion formula holds*:

$$\frac{1}{2}(f(z+0) + f(z-0)) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\mathcal{B}^*(1-q)z^{-q}}{\prod_{i=1}^{m}\Gamma(\mu_i - q/\rho_i)}\, dq. \tag{23}$$

Here the integral is understood as a Cauchy principal value, and $\mathcal{B}^*(1-q)$ is the Mellin transform (12) *of $\mathcal{B}(s)$.*

Proof. From the asymptotics of $H^{m,0}_{0,m}[sz|^{--}_{(\mu_i-\frac{1}{\rho_i},\frac{1}{\rho_i})_1^m}]$ and the condition on f, it is clear that

$$\int_0^\infty \int_0^\infty \left| s^{-q} H^{m,0}_{0,m}\left[sz \middle|^{--}_{(\mu_i - \frac{1}{\rho_i}, \frac{1}{\rho_i})_1^m} \right] f(z) \right| dz\, ds < \infty.$$

Therefore, one can apply the Fubini theorem to obtain

$$\begin{aligned} \mathcal{B}^*(1-q) &= \int_0^\infty s^{-1}\mathcal{B}(s)\, ds \\ &= \int_0^\infty s^{-q}\, ds \int_0^\infty H^{m,0}_{0,m}\left[sz \middle|^{--}_{(\mu_i - \frac{1}{\rho_i}, \frac{1}{\rho_i})_1^m} \right] f(z)\, dz \\ &= \int_0^\infty f(z)\, dz \int_0^\infty s^{-q} H^{m,0}_{0,m}\left[sz \middle|^{--}_{(\mu_i - \frac{1}{\rho_i}, \frac{1}{\rho_i})_1^m} \right] ds. \end{aligned}$$

The inner integral is the multi-index Borel-Dzrbashjan transform of s^{-q}, and by (10), has the value $z^{q-1}\prod_{i=1}^m \Gamma(\mu_i - (q/\rho_i))$, and therefore

$$\begin{aligned} \mathcal{B}^*(1-q) &= \prod_{i=1}^m \Gamma\left(\mu_i - \frac{q}{\rho_i}\right) \int_0^\infty z^{q-1} f(z)\, dz \\ &= \prod_{i=1}^m \Gamma\left(\mu_i - \frac{q}{\rho_i}\right) f^*(q). \end{aligned} \tag{24}$$

The inversion theorem for the Mellin transform [**15**] now yields

$$\begin{aligned} (f(z+0) + f(z-0))/2 &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(q) z^{-q}\, dq \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\mathcal{B}^*(1-q) z^{-q}}{\prod_{i=1}^m (\mu_i - (q/\rho_i))}\, dq, \end{aligned}$$

where the integral is understood as a Cauchy principal value. □

One can also find a real inversion formula, *analogous to the Post-Widder real inversion formula* for the Laplace transform, and to other existing real inversion formulas for the Meijer and Obrechkoff transforms [**4**]. The technique from Hirshman and Widder [**7**] is applied:

Theorem 4 (Real inversion formula). *Let f be of bounded variation at z and $f(z)$, $zf'(z) \in L_2(\mathbf{R}^+)$. Then*
(25)
$$f(z) = \lim_{n\to\infty} \frac{n^{((1/\rho)-\mu)}}{(n!)^m} \prod_{i=1}^{m}\prod_{j=0}^{n}\left(j+\mu_i+\frac{z}{\rho_i}\frac{d}{dz}\right)\left\{\frac{1}{z}(\mathcal{B}f)\left(\frac{n^{1/\rho}}{z}\right)\right\}.$$

Proof. Use the formula

$$\frac{1}{\Gamma(s)} = \lim_{n\to\infty} s(1+s)\left(1+\frac{s}{2}\right)\cdots\left(1+\frac{s}{n}\right)n^{-s},$$

and the property that

$$|\Gamma(s)s(1+s)\left(1+\frac{s}{2}\right)\cdots\left(1+\frac{s}{n}\right)n^{-s}| = \left|\frac{\Gamma(n+s+1)}{\Gamma(n+1)}n^{-s}\right|$$

is uniformly bounded with respect to n and $s=(1/2)+it$ [**6**]. From (24) we have

$$\mathcal{B}^*(s) = \prod_{i=1}^{m}\Gamma\left(\mu_i - \frac{1-s}{\rho_i}\right)f^*(1-s),$$

and, therefore,

$$\begin{aligned} f^*(s) &= \frac{\mathcal{B}^*(1-s)}{\prod_{i=1}^{m}\Gamma(\mu_i-(s/\rho_i))} \\ &= \mathcal{B}^*(1-s)\lim_{n\to\infty}\prod_{i=1}^{m}\prod_{j=0}^{n}\left\{\left(j+\mu_i-\frac{s}{\rho_i}\right)n^{-\mu_i+(s/\rho_i)}\right\}(n!)^{-m}. \end{aligned}$$

Since $f(z)$, $zf'(z)\in L_2(\mathbf{R}^+)$, then $f^*(s)$,

$$sf^*(s)\in L_2((1/2)-i\infty,(1/2)+i\infty),$$

and therefore $f^*(s)\in L_1((1/2)-i\infty,(1/2)+i\infty)$. Then the Lebesgue dominance convergent theorem and the inversion theorem for the Mellin transform [**15**] can be applied to get

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}-i\infty} f^*(s)z^{-s}\,ds \\
&= \lim_{n\to\infty}\frac{1}{n^{\sum_i^m \mu_i}(n!)^m}\frac{1}{2\pi i} \\
&\quad\times\prod_{i=1}^{m}\prod_{j=0}^{n}\bigg[\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}-i\infty}\bigg(j+\mu_i-\frac{s}{\rho_i}\bigg)\mathcal{B}^*(1-s)\bigg(\frac{z}{n^{1/\rho}}\bigg)^{-s}\,dz\bigg] \\
&= \lim_{n\to\infty}\frac{1}{(n!)^m n^{\mu}}\prod_{j=0}^{n}\prod_{i=1}^{m}\bigg(j+\mu_i+\frac{z}{\rho_i}\frac{d}{dz}\bigg) \\
&\quad\times\frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}-i\infty}\mathcal{B}^*(1-s)\bigg(\frac{z}{n^{1/\rho}}\bigg)^{-s}\,ds.
\end{aligned}
$$

Hence we arrive at

$$
f(z) = \lim_{n\to\infty}\frac{n^{(1/\rho)-\mu}}{(n!)^m}\prod_{j=0}^{n}\prod_{i=1}^{m}\bigg(j+\mu_i+\frac{z}{\rho_i}\frac{d}{dz}\bigg)\bigg(\frac{1}{z}\mathcal{B}\bigg(\frac{n^{1/\rho}}{z}\bigg)\bigg),
$$

which is (25). □

If $m = 1$ in Theorem 4, the resulting real inversion formula for the Borel-Dzrbashjan transform (2) seems to be new.

4. Convolution property. In this section we find a suitable operation that serves as a convolution of our multi-index Borel-Dzrbashjan transform.

Let $\mu_i > 0$, $i = 1, 2, \dots, m$. Define the operation $*$ by

$$
(f * g)(z) = I_{(\rho_i),m}^{(2\mu_i-1),(-\mu_i)}(f\circ g)(z),
$$

where $\circ$ denotes the operation

$$
\begin{aligned}
(f\circ g)(z) &= \int_0^1\cdots\int_0^1\prod_{i=1}^{m}[t_i(1-t_i)]^{\mu_i-1} f\bigg(z\prod_{i=1}^{m}t_i^{1/\rho_i}\bigg) \\
&\quad\times g\bigg(z\prod_{i=1}^{m}(1-t_i)^{1/\rho_i}\bigg)\,dt_1\dots dt_m,
\end{aligned}
\tag{26}
$$

and the following denotation for the generalized operators of fractional integro-differentiation is used [**8**]:

$$I^{(\gamma_i),(\delta_i)}_{(\beta_i),m} f(z) = \begin{cases} \int_0^1 H^{m,0}_{m,m}\left[\sigma\Big|\begin{matrix}(\gamma_i+\delta_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i})_1^m\\(\gamma_i+1-\frac{1}{\beta_i},\frac{1}{\beta_i})_i^m\end{matrix}\right] f(z\sigma)d\sigma, \text{ if } \sum_{i=1}^m \delta_i > 0,\\ f(z), \quad \text{if } \delta_1=\delta_2=\cdots=\delta_m=0,\\ \left[\prod_{i=1}^m\prod_{j=1}^{\eta_i}(\frac{1}{\beta_i}z\frac{d}{dz}+\mu_i+j)\right] I^{(\gamma_i+\delta_i),(\eta_i-\delta_i)}_{(\beta_i),m} f(z) \text{ if } \sum_{i=1}^m \delta_i < 0,\\ \qquad\qquad \text{with integers } \eta_i : \eta_i-1\le\delta_i<\eta_i. \end{cases}$$

Theorem 5. *The operator $*$ is a convolution of the multi-index Borel-Dzrbashjan transform in $L_1(\mathbf{R}^+, z^{c-1})$, $c < (\alpha+1)/2$, namely,*

$$\mathcal{B}_{(\rho_i),(\mu_i)}\{(f*g)(z);s\} = s\mathcal{B}_{(\rho_i),(\mu_i)}\{f(z);s\}\cdot\mathcal{B}_{(\rho_i),(\mu_i)}\{g(z);s\}. \tag{27}$$

Proof. Let p and q be complex numbers with $\operatorname{Re} p$, $\operatorname{Re} q < \alpha+1$. Then

$$z^{-p} * z^{-q} = z^{-p-q}\prod_{i=1}^m \frac{\Gamma(\mu_i-p/\rho_i)\Gamma(\mu_i-q/\rho_i)}{\Gamma(\mu_i-(p+q)/\rho_i)}, \tag{28}$$

which follows by evaluating $z^{-p}\circ z^{-q}$, as repeated beta-integrals, arising from (26) and then $I^{(2\mu_i-1),(-\mu_i)}_{(\rho_i),m}\{z^{-p-q}\}$. Then, by formula (10), one can easily verify (27) for any two power functions z^{-p} and z^{-q}, i.e.,

$$\mathcal{B}_{(\rho_i),(\mu_i)}\{z^{-p}*z^{-q};s\} = s\mathcal{B}_{(\rho_i),(\mu_i)}\{z^{-p};s\}\cdot\mathcal{B}_{(\rho_i),(\mu_i)}\{z^{-q};s\}.$$

To prove (27) in the case of arbitrary functions $f(z), g(z)\in L_1(\mathbf{R}^+, z^{c-1})$, $c<(\alpha+1)/2$, we use the complex inversion formula (23):

$$\begin{aligned} f(z)*g(z) &= \left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(\mathcal{B}f)^*(1-p)z^{-p}}{\prod_{i=1}^m\Gamma(\mu_i-p/\rho_i)}\,d\rho\right] \\ &\quad * \left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(\mathcal{B}g)^*(1-q)z^{-q}}{\prod_{i=1}^m\Gamma(\mu_i-q/\rho_i)}\,dq\right] \\ &= \frac{1}{(2\pi i)^2}\int_{c-i\infty}^{c+i\infty}\int_{c-i\infty}^{c+i\infty}\frac{(z^{-p}*z^{-q})(\mathcal{B}f)^*(1-p)(\mathcal{B}g)^*(1-q)}{\prod_{i=1}^m\Gamma(\mu_i-p/\rho_i)\Gamma(\mu_i-q/\rho_i)}\,dq\,dp, \end{aligned}$$

where $(\mathcal{B}f)^*(p)$ is the Mellin transform of $\mathcal{B}_{(\rho_i),(\mu_i)}\{f;s\}$.

Because $\mu_i > 0$, $i = 1,\dots,m$, we have $\alpha + 1 > 0$ and hence $\operatorname{Re} p = \operatorname{Re} q = c < (\alpha+1)/2 < \alpha + 1$. Thus formula (28) is applicable and yields

$$f(z) * g(z) = \frac{1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} \int_{c-i\infty}^{c+i\infty} \frac{z^{-(p+q)}}{\prod_{i=1}^m \Gamma(\mu_i - (p+q)/\rho_i)} \times (\mathcal{B}f)^*(1-p)(\mathcal{B}g)^*(1-q)\, dq\, dp.$$

Making the substitution $p = \sigma - q$ in the p-integral so that σ runs over the contour $(2c - i\infty, 2c + i\infty)$, we obtain

$$(fz) * g(z) = \frac{1}{(2\pi i)^2} \int_{2c-i\infty}^{2c+i\infty} \frac{z^{-\sigma}}{\prod_{i=1}^m \Gamma(\mu_i - \sigma/\rho_i)} \times \int_{c-i\infty}^{c+i\infty} (\mathcal{B}f)^*(1-\sigma+q)(\mathcal{B}g)^*(1-q)\, dq\, d\sigma. \tag{29}$$

Applying the Parseval formula for the Mellin transform [**15**], we get

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (\mathcal{B}f)^*(1-\sigma+q)(\mathcal{B}g)^*(1-q)\, dq = \int_0^\infty s^{1-\sigma} (\mathcal{B}f)(s)(\mathcal{B}g)(s)\, ds = (s(Bf)(s)(Bg)(s))^*(1-\sigma). \tag{30}$$

Substituting (30) in (29), we find

$$f(z) * g(z) = \frac{1}{2\pi i} \int_{2c-i\infty}^{2c+i\infty} \frac{(s(Bf)(s)(Bg)(s))^*(1-\delta) z^{-\sigma}}{\prod_{i=1}^m \Gamma(\mu_i - \sigma/\rho_i)}\, d\sigma, \tag{31}$$

with $2c < \alpha + 1$. Formula (23) tells us that the right-hand side of (31) is the multi-index Borel-Dzrbashjan inverse of $s(Bf)(s)(Bg)(s)$. Applying $\mathcal{B}_{(\rho_i),(\mu_i)}$ to (31) yields (27). □

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