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FLUCTUATION OF SECTIONAL CURVATURE FOR CLOSED HYPERSURFACES

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ABSTRACT. Liebmann proved in 1899 that the only closed surfaces in Euclidean three-space that have constant Gauss curvature are round spheres. Thus, if a closed surface in three-space is not a topological sphere, its Gauss curvature must fluctuate. We consider quantitative formulations of this fact, also in higher dimensions.

0. Introduction. Consider a smooth closed manifold M of dimension n which has an immersion $f : M \to (\mathbf{R}^{n+1}, \mathrm{can})$ as a hypersurface in Euclidean space. The immersion pulls back the canonical Riemannian metric on $\mathbf{R}^{n+1}$ to a Riemannian metric on M, called the induced metric, which we denote by $f^*\mathrm{can}$. If M is not diffeomorphic to S^n, the sectional curvature of $f^*\mathrm{can}$ must fluctuate. For if the sectional curvature is constant, it must be positive. Then the shape operator is everywhere definite, so the hypersurface is diffeomorphic to S^n by a theorem of Hadamard.

We seek a lower bound for the amount of fluctuation of sectional curvature, dependent on M, but independent of the particular immersion f as far as possible. For any closed Riemannian manifold, the set of values of the sectional curvature forms a closed bounded interval. The task at hand is to give a lower bound for the length $l(\mathrm{sec})$ of this interval for the Riemannian metrics $f^*\mathrm{can}$. Because of scaling, it is clear that such a bound cannot depend on M alone, but must have some dependence on the immersion f. It turns out that it is possible to give a lower bound depending only on the topology of M and its volume with respect to $f^*\mathrm{can}$.

1. Fluctuation of sectional curvature. Let F be some fixed field, and $\beta_j(M;F) = \dim H_j(M;F)$ the Betti numbers of M with respect to the field F and $\beta(M;F)$ their sum. Then $l(\mathrm{sec})$ can be estimated from below by $\mathrm{vol}\,(M)$ and $\beta(M;F)$.

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Proposition 1.1. *Suppose that M is a smooth closed manifold of dimension n which is not diffeomorphic to S^n. For any smooth immersion $f : M \to (\mathbf{R}^{n+1}, \mathrm{can})$ of M, the length $l(\sec)$ of the range of the sectional curvature of the induced metric f^*can satisfies*

$$l(\sec) > \left(\frac{\mathrm{vol}\,(S^n)}{\mathrm{vol}\,(M)} \cdot \frac{\beta}{2} \right)^{2/n},$$

where β is the sum of the Betti numbers of M over some field and $\mathrm{vol}\,(M)$ *is the volume of M with respect to the induced metric.*

Proof. By shrinking a large round sphere until it touches $f(M)$, we see that there is a point where the sectional curvatures are positive. But if all sectional curvatures are positive everywhere, the shape operator is positive definite. Then M is diffeomorphic to S^n by Hadamard's theorem but this is excluded by assumption. Hence sec must take the value zero.

The Gauss-Kronecker curvature G is the determinant of the shape operator, so $G = \lambda_1\lambda_2 \cdots \lambda_n$ where the λ_j are the principal curvatures. If n is even, write

$$G = (\lambda_1\lambda_1) \cdots (\lambda_{n-1}\lambda_n) = K_{\sigma_{12}} K_{\sigma_{23}} \cdots K_{\sigma_{n-1n}}$$

and if n is odd, write

$$G^2 = \lambda_1\lambda_2\lambda_1\lambda_3\lambda_2\lambda_3 \cdots \lambda_{n-1}\lambda_n = K_{\sigma_{12}} K_{\sigma_{13}} K_{\sigma_{23}} \cdots K_{\sigma_{n-1,n}},$$

where σ_{ij} is a section spanned by principal directions for λ_i and λ_j and K_σ is the sectional curvature for σ. Whether n is even or odd, the assumption $|\sec| \le c$ yields $|G| \le c^{n/2}$.

The Chern-Lashof inequality

$$\int_M |G|\, d\,\mathrm{vol} \ge \frac{\mathrm{vol}\,(S^n)}{2}\beta$$

from [**1**] now yields the desired result, since if $|G| \le c^{n/2}$, we get $c^{n/2}\mathrm{vol}\,(M) \ge \mathrm{vol}\,(S^n)\beta/2$ while sec takes the value zero. □

If M is an orientable surface of genus $p \ge 1$ immersed in $\mathbf{R}^3$, Proposition 1.1 yields the inequality $\mathrm{area}\,(M) l(K(M)) > 4\pi(p+1)$.

This bound has the correct order of magnitude for $p \to \infty$, as one sees by building a ladder-like surface consisting of $p-1$ identical pieces each with one rung and two U-shaped endpieces. The surface has genus p, the maximum and minimum of K on the surface is independent of p for $p \geq 2$, and the area grows linearly with p.

Using the Gauss-Bonnet theorem, this inequality can be improved.

Proposition 1.2. *Let M be a smooth, closed surface of genus $p \geq 1$ isometrically immersed in $\mathbf{R}^3$. Then*

$$\operatorname{area}(M) l(K(M)) > 4\pi(p + 2\sqrt{p} + 1)$$

if M is orientable and

$$\operatorname{area}(M) l(K(M)) > 2\pi(p + 2\sqrt{p} + 2)$$

if M is non-orientable.

Proof. Assume that M is orientable; the non-orientable case involves only inessential changes.

We use the Gauss-Bonnet theorem

$$\int_M K\, dA = 2\pi\chi(M) = 4\pi(1-p)$$

and the inequality

$$\int_{K>0} K\, dA \geq 4\pi$$

which is related to the surjectivity of the Gauss map. Splitting the left-hand side of the Gauss-Bonnet theorem into contributions from the sets where $K > 0$ and $K \leq 0$ and using the inequality, we obtain

$$\int_{K\leq 0} K\, dA \leq -4\pi p.$$

Denote the area of the set where $K > 0$ by A_+; then the area of the set where $K \leq 0$ is $\operatorname{area}(M) - A_+$. The interval $K(M)$ contains a point to the right of $4\pi/A_+$ since the average of K on that set is at least as large.

In the same way it contains a point to the left of $-4\pi p/(\text{area}\,(M)-A_+)$. Hence,

$$l(K(M)) > 4\pi\left(\frac{1}{A_+} + \frac{p}{\text{area}\,(M) - A_+}\right),$$

and choosing the value of A_+ that makes the right-hand side as small as possible yields the desired inequality. □

After imposing further topological restrictions, Heinz Hopf's Gauss-Bonnet theorem for hypersurfaces can be used to improve the inequality in Proposition 1.1 along the same lines. Then the Chern-Lashof inequality can be replaced by an appeal to the surjectivity of the Gauss map as above.

For a torus T in $\mathbf{R}^3$, Proposition 1.2 yields an appreciable improvement over Proposition 1.1. This suggests the problem of finding the sharp lower bound for $\text{area}\,(T)l(K(T))$ where T is an arbitrary smooth torus isometrically immersed in $\mathbf{R}^3$. Proposition 1.2 yields the lower bound 16π for this quantity. For a torus immersed as a tube of constant circular cross-section, it is easy to determine the sharp bound for $\text{area}\,(T)l(K(T))$ by explicit calculation. The infimum is approached, but not attained, by an anchor ring where the smaller radius tends to zero while the larger stays fixed. Perhaps the resulting estimate $\text{area}\,(T)l(K(T)) > 8\pi^2$ holds in general. This estimate can also be verified for knotted tori. If T is knotted, an inequality of Langevin and Rosenberg [**2**] yields the estimate $\text{area}\,(T)l(K(T)) > 32\pi > 8\pi^2$ as in the proof of Proposition 1.2.

REFERENCES

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