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NORMAL HYPERBOLICITY FOR FLOWS AND NUMERICAL METHODS

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ABSTRACT. In this paper we prove that normally hyperbolic invariant manifolds persist between flows and numerical methods in both directions. This means that normal hyperbolicity of flows is preserved under numerical methods and that normally hyperbolicity for numerical methods is inherited by flows.

1. Definitions and statement of theorems. Let M be a smooth complete Riemannian manifold with a distance d arising from the Riemannian metric and $\mathrm{Diff}\,(M)$ be the set of diffeomorphisms on M with the strong topology and distance d_{C^1}. A *flow* is a map $\varphi : \mathbf{R}\times M \to M$ that satisfies the group property: $\varphi^s(\varphi^t(x)) = \varphi^{s+t}(x)$.

Definition 1. For $p \geq 1$, let φ be a C^{p+1} flow on M. A C^{p+1} function $N : \mathbf{R} \times M \to M$ is called a *numerical method of order* p for φ^t if there are positive constants K and h_0 such that $d(\varphi^h(x), N^h(x)) \leq Kh^{p+1}$, for all $h \in [0, h_0]$ and $x \in M$. Here h stands for a stepsize of N. We denote the i-th iterate of $N^h(x)$ by $(N^h)^i(x)$.

Numerical methods arise from computer simulation and numerical approximation. For instance, both explicit and implicit Runge-Kutta methods satisfy the above conditions (see [**1**]).

It is well known that the time-h map of the flow and the numerical method of stepsize h are C^1 close polynomially in terms of h.

Lemma 1 [**6**]. *Let N be a numerical method of order p for a C^{p+1} flow φ on a compact manifold M. Then there is a constant K_1 such that $d_{C^1}(\varphi^h, N^h) \leq K_1 h^p$ for all sufficiently small h. Moreover, given $T > 0$, there is a constant K_2 such that $d_{C^1}(\varphi^T, (N^{T/n})^n) \leq K_2 n^{1-p}$ for all large positive integers n.*

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We formulate normal hyperbolicity by making an adaptation of the cone-field argument used in [**17**].

Definition 2. We say that Λ is a *compact invariant manifold* of a diffeomorphism f on M if it is a C^1 compact boundaryless submanifold of M satisfying $f(\Lambda) = \Lambda$. We say that Λ is *normally hyperbolic* for f if there exists a splitting $TM \mid \Lambda = E^s \oplus T\Lambda \oplus E^u$ and constants $\alpha > 0$, $K > 0$ and $0 < \mu < 1 < \lambda$ such that, for all $x \in \Lambda$ and $n \in \mathbf{N}$,

$$\begin{gathered}\dim(E^s_x),\ \dim(T_x\Lambda),\ \text{and } \dim(E^u_x) \text{ are constants,}\\ Df_x^{-1}(C^s_x) \subset C^s_{f^{-1}(x)}, \qquad Df_x(C^u_x) \subset C^u_{f(x)},\\ |Df_x^n v| \le K\mu^n |v|, \quad \text{for all } v \in C^s_x,\\ |Df_x^{-n} v| \le K\lambda^{-n}|v|, \quad \text{for all } v \in C^u_x,\end{gathered}$$

where

$$\begin{aligned}C^s_x &= \{(v^s, v^{cu}) \in E^s_x \times (T_x\Lambda \times E^u_x) : |v^{uc}| \le \alpha |v^s|\},\\ C^u_x &= \{(v^{sc}, v^u) \in (E^s_x \times T_x\Lambda) \times E^u_x : |v^{sc}| \le \alpha |v^u|\}.\end{aligned}$$

Let φ^t be a flow on M and Λ_φ be a compact invariant manifold of the time-T map φ^T of the flow. We say that φ^t is *normally hyperbolic* on Λ_φ if φ^T is normally hyperbolic on Λ_φ.

Let N be a numerical method for a flow on M satisfying that N^h has a compact invariant manifold Λ_h for all sufficiently small $h > 0$. We say that N is *normally hyperbolic* on $\{\Lambda_h\}$ if each individual N^h is normally hyperbolic on Λ_h with respect to the constants α, K, μ and λ independent of h. We say that $\{\Lambda_h\}$ is *isolated* if there exists a neighborhood U, independent of h, of Λ_h such that $\Lambda_h \subset \operatorname{int}(U)$ and $\Lambda_h = \cap_{n=-\infty}^{\infty} (N^h)^n(U)$.

The following proposition derives general conditions for a closed submanifold to be normally hyperbolic for numerical methods.

Proposition 1. *Let Λ be a normally hyperbolic invariant manifold for a diffeomorphism f on M. Then, for each $x \in \Lambda$, there exist unique subspaces $\mathbf{E}^s_x \subset C^s_x$ and $\mathbf{E}^u_x \subset C^u_x$ such that the splitting $T_xM \mid \Lambda = \mathbf{E}^s_x \oplus T_x\Lambda \oplus \mathbf{E}^u_x$ is Df-invariant and varies continuously* *with x, and for all $n \geq 0$,*

$$\|Df^n \mid \mathbf{E}^s_x\| \leq K\mu^n, \qquad \|Df^{-n} \mid \mathbf{E}^u_x\| \leq K\lambda^{-n},$$
$$\|Df^n \mid \mathbf{E}^s_x\| \cdot \|Df^{-n} \mid T_{f^n(x)}\Lambda\| \leq K\mu^n,$$
$$\|Df^{-n} \mid \mathbf{E}^u_x\| \cdot \|Df^n \mid T_{f^n(x)}\Lambda\| \leq K\lambda^{-n}.$$

For a proof of the proposition, refer to [**17**] and [**18**].

The following theorem states normally hyperbolic invariant manifold of flows persists under numerical methods.

Theorem 1. *Let $p \geq 2$, φ^t be a C^{p+1} flow on M, and N be a numerical method of order p for φ^t. If φ^t has a C^1 compact normally hyperbolic invariant manifold Λ_φ, then for all sufficiently small $h > 0$, there exists a normally hyperbolic invariant manifold Λ_h for N^h.*

The persistence of hyperbolic periodic orbits is shown early in [**3**], and later in [**2**] and [**5**] for one-step methods, in [**4**] for multistep methods, and in [**19**] for general numerical methods. The case of *stable* normally hyperbolic tori is investigated in [**14**]. Our theorem is a generalized version of the above results.

For many local properties of flows persisted by numerical methods, refer to the extensive volume of Stuart and Humphries [**22**] and the long bibliography therein. For structural stability results of flows under numerical methods, see our previous works [**10**], [**11**], [**12**] and [**13**].

Next, we consider the converse of Theorem 1, that is, normal hyperbolicity of a numerical method is inherited by the flow.

Theorem 2. *Let $p \geq 2$, φ^t be a C^{p+1} flow on M, and let N be a numerical method of order p for φ^t. If N^h has a C^1 compact invariant manifold Λ_h for all sufficiently small $h > 0$, N is normally hyperbolic on $\{\Lambda_h\}$, and $\{\Lambda_h\}$ is isolated with respect to a common neighborhood U. Then there is a normally hyperbolic invariant manifold Λ_φ for φ^t.*

In [**7**], Hagen assumes that a numerical method has a smooth compact normally hyperbolic invariant manifold Λ *independent of stepsize h*

and showed the existence of a compact invariant manifold for the flow. In our result, we allow the invariant manifolds for the numerical method varies with the stepsize, which is more practical for numerical computations.

2. Proof of Theorem 1. The proof presented here relies on the proof of the stable manifold theorem. We make modifications from the method developed by Conley as given in McGehee [**15**] and Moser [**16**] for the two-dimensional case, the approach given in Hirsch and Pugh [**8**] for the high dimensional case, and the proof given in Hirsch, Pugh and Shub [**9**] for normal hyperbolicity. See also [**21**].

Because Λ_φ is normally hyperbolic, the tangent bundle of M along Λ_φ splits as the sum of three bundles $TM \mid \Lambda_\varphi = \mathbf{E}^s \oplus T\Lambda_\varphi \oplus \mathbf{E}^u$. We want the normal bundle η of φ^t to be smooth. It is no loss of generality to make a convenient choice of η: let η^s and η^u be smooth subbundles of $TM \mid \Lambda_\varphi$ with approximating $\mathbf{E}^s$ and $\mathbf{E}^u$ so that $TM \mid \Lambda_\varphi = \eta^s \oplus T\Lambda_\varphi \oplus \eta^u$, and choose $\eta = \eta^s \oplus \eta^u$. Let $\eta^\delta(r) = \{v \in \eta^\delta : |v| \leq r\}$, for $\delta = s, u$, be the r disk bundles and $\eta(r) = \eta^s(r) \oplus \eta^u(r)$. Let $\pi^s : \eta \to \eta^s$ be a projection along η^u onto η^s and $\pi^u : \eta \to \eta^u$ be a projection along η^s onto η^u.

We want to view a tubular neighborhood of Λ_φ, as a bundle not over Λ_φ, but over some higher dimensional manifold. Let exp be the exponential map from tangent space to the manifold and set $X = \exp \eta^s(r_0)$, where $r_0 > 0$ is small enough so that $X(r)$ is a manifold (with boundary). It is clear that $TX \mid \Lambda_\varphi = \eta^s \oplus \eta^c$. For every point $x \in \Lambda$, affinely translate $\eta^u_x(r)$ from the origin to all points in η^s_x. Therefore we have an extension of $\eta^u(r)$ to X near Λ_φ, still denoting it η^u. Exponentiating the extension down to the manifold gives a tubular neighborhood $Y(r)$ of X in M. Let η^{sc} be a differentiable extension of TX to the neighborhood $Y(r)$ of Λ_φ.

For convenience, we change the Riemannian norm of M so that the time-T map φ^T of the flow is normally hyperbolic with respect to the constants $\alpha = K = 1$. For $x \in Y(r)$, let $C^u_x = \{(v^{sc}, v^u) \in \eta^{sc}_x \times \eta^u_x : |v^{sc}| \leq |v^u|\}$. If $r > 0$ is small enough, then for all $x \in Y(r)$, $D\varphi^T_x(C^u_x) \subset C^u_{f(x)}$.

For $x \in X$, let $D^u_0 = \exp(\eta^u_x)$. Then let D^u_0 be a disk in the tubular neighborhood $Y(r)$ satisfying: (i) D^u_0 has the same dimension as η^u_x,

(ii) the tangent space $T_x D_0^u$ is contained in the cone C_x^u, (iii) the boundary of D_0 is in the boundary of $Y(r)$, and (iv) D_0^u goes all the way across $Y(r)$. In local coordinates we could assume that D_0^u is the graph of a function form η_x^u into η_x^{sc}. Because of the invariance of the bundles under φ^T, $(\varphi^T)^n(D_0^u) \cap Y(r)$ is a disk with the same properties as above for all $n \geq 0$. And $D_n^u = (\varphi^T)^{-n}((\varphi^T)^n(D_0^u) \cap Y(r)) \subset D_0^u$ is a nested set of disks which converges to a single point. This point is the unique point in D_0^u which stays in $Y(r)$ for all forward iterates. Let

$$W_r^{s\varphi} = \bigcup_{x \in X} \bigcap_{n \geq 0} (\varphi^T)^{-n}((\varphi^T)^n(\exp(\eta_x^u)) \cap Y(r)).$$

Then the stable manifold $W_r^{s\varphi}$ consists of all points whose forward φ^T orbits never leave $Y(r)$. Applying this result to φ^{-T} produces the unstable manifold $W_r^{u\varphi}$. Thus $\Lambda_\varphi = W_r^{u\varphi}(\Lambda_\varphi) \cap W_r^{s\varphi}(\Lambda_\varphi)$.

By Lemma 1, we have $(N^{T/n})^n \to \varphi^T$ in the C^1 topology as $n \to \infty$. Take n sufficiently large so that $DN_x^{T/n}(C_x^u) \subset C^u_{N^{T/n}(x)}$. If D_0^{uN} is a disk of the same type as above, then $\cap_{n \geq 0}(N^{T/n})^{-n}((N^{T/n})^n(D_0^{uN}) \cap Y(r))$ is a single point. Thus the set

$$W_r^{sN} = \bigcup_{x \in X} \bigcap_{n \geq 0} (N^{T/n})^{-n}((N^{T/n})^n(\exp(\eta_x^u)) \cap Y(r))$$

consists of all points which stays in $Y(r)$ for all forward $N^{T/n}$-iterates. Similarly, we can get the unstable manifold W_r^{uN} when applying this result to $(N^{T/n})^{-n}$. The stable and unstable manifolds, W_r^{sN} and W_r^{uN}, are of C^1 and transverse to each other; therefore there exists $\Lambda_{T/n} = W_r^{uN} \cap W_r^{sN}$ which is $N^{T/n}$-invariant and is of C^1. Consider $T/n = h$, then Λ_h is N^h-invariant. By the C^1 persistence of $D\varphi^h$-invariant splitting $T_x M \mid \Lambda_\varphi = \mathbf{E}_x^s \oplus T_x \Lambda_h \oplus \mathbf{E}_x^u$, there exists a DN^h-invariant splitting $T_x M = \mathbf{E}_x^{sN} \oplus T_x \Lambda_h \oplus \mathbf{E}_x^{uN}$ of which N^h is normally hyperbolicity.

3. Proof of Theorem 2. We shall apply the abstract invariant manifold theorems in Theorem 4.1 of [**9**] and Theorem 3.1 of [**20**] to construct the local unstable and stable manifolds for points in Λ_h. Then consider the time-h map of the flow as a C^1 perturbation of N^h, and construct its local unstable and stable manifolds. The graph transform method is essential.

By Proposition 1, Λ_h has normally hyperbolic DN^h-invariant splitting $TM \mid \Lambda_h = \mathbf{E}^s \oplus T\Lambda_h \oplus \mathbf{E}^u$. For $r > 0$, let $X(r) = \exp \mathbf{E}^u(r)$. Fix $r_0 > 0$ so small that $X(r_0) \subset U$ is a compact manifold. Let η^u and η^s be smooth and trivial subbundles of $TM \mid X(r_0)$ with approximating $\mathbf{E}^u$ and $\mathbf{E}^s$ so that $TM \mid X(r_0) = \eta^u \oplus T\Lambda_h \oplus \eta^s$, and choose $\eta = \eta^u \oplus \eta^s$. Set $\eta(r) = \eta^u(r) \oplus \eta^s(r)$. For $\delta = u, s$, let $\eta^\delta(r) = \{v \in \eta^\delta : |v| \le r\}$ and $\pi^\delta : \eta \to \eta^\delta$ be a natural projection. If $\sigma : \eta^u(r) \to \eta$ is a section, then we can define the *slope* of σ at $v_x \in \eta^u(r)$ to be $\limsup_{v_y \to v_x}(|s(v_x) - s(v_y)|_s / d_u(v_x, v_y))$, where $\sigma(v_x) = (v_x, s(v_x)) \in \eta^u \times \eta^s$, $|\cdot|_s$ is the norm on η^s and d_u is the Finsler metric on η^u. Let $\Sigma(1, r) = \{$section $\sigma : \eta^u(r) \to \eta(r)$ such that *slope* $(\sigma) \le 1\}$ be a complete metric space with the C^0 sup norm.

For $r > 0$ small, $x \in X(r)$ and $v_x \in \eta(r)$, we define a bundle map F by

$$F(v_x) = \exp^{-1}_{N^h(x)} \circ N^h \circ \exp v_x.$$

Then F is a C^1 bundle map on $\eta(r)$. One can take $0 < r_1 \le r_0$ small so that, for $\sigma \in \Sigma(1, r_1)$ and $x \in X(r_1)$, $\pi^u \circ F \circ \sigma : \eta^u(r) \mid_{X(r_1)} \to \eta^u \mid_{X(r_1)}$ is invertible. We denote its inverse by $g : \eta^u(r)|_{X(r_1)} \to \eta^u$. We define a graph transform $F_\#$ of F over $X(r_1)$ by

$$F_\#(\sigma)_x = F \circ \sigma \circ g_x \quad \text{for } x \in X(r_1).$$

Then $F_\#$ is a contraction on $\Sigma(1, r_1)$ and has a unique fixed point $\sigma^{uN^h} \in \Sigma(1, r_1)$. For $0 < r \le r_1$, let $W_r^{uN^h}(x) = \exp \sigma_x^{uN^h}(\eta_x^u(r))$. Similarly, we get stable manifolds for N^h by $W_r^{sN^h}(x) = W_r^{uN^h}(x, (N^h)^{-1})$. Let $W_r^{\sigma N^h}(\Lambda_h) = \cup_{x \in \Lambda_h} W_r^{\sigma N^h}(x)$ for $\sigma = u, s$. Then $\Lambda_h = W_r^{uN^h}(\Lambda_h) \cap W_r^{sN^h}(\Lambda_h)$.

By Lemma 1, we have that φ^h is $O(h^p)$-close to N^h in the C^1 topology with $p \ge 2$. By Theorem 4.1 of [**9**], see also Theorem 2 of [**19**], one can take $h > 0$ sufficiently small so that the bundle map G_h defined by $G_h(v_x) = \exp^{-1}_{N^h(x)} \circ \varphi^h \circ \exp v_x$ has a well-defined graph transform $G_{h\#} = G_h \circ \sigma \circ g'$, where g' is a right inverse of $\pi^u \circ G_h \circ \sigma$. Moreover, $G_{h\#}$ is a contraction of $\Sigma(1, r)$, so has a fixed point $\sigma^{u\varphi}$. That is, we can construct for φ^h a manifold $W_r^{u\varphi} = \sigma^{u\varphi}(X(r))$. These results when applied to the inverse of φ^h produce $W_r^{s\varphi}$. Let $\Lambda_\varphi = W_r^{u\varphi} \cap W_r^{s\varphi}$. By backward and forward invariance of $W_r^{u\varphi}$ and $W_r^{s\varphi}$, Λ_φ is φ^h-invariant. Since Λ_h is normally hyperbolic and isolated, one can take $h > 0$ small enough so that φ^h is normally hyperbolic on Λ_φ and Λ_φ is *locally maximal* in the sense that there is a neighborhood V of Λ_φ such that any φ^h-invariant set contained entirely in V is a subset of Λ_φ. By uniqueness, we have that φ^t is normally hyperbolic on Λ_φ.

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