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ENUMERATIVE TRIANGLE GEOMETRY PART 1: THE PRIMARY SYSTEM, S

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1. Introduction. This article presents a procedure for counting Euclidean constructible objects in triangle geometry–points and lines in Part 1, together with circles in Part 2. If A, B, C are the vertex angles of the reference triangle, then each object is a function $F(A, B, C)$, both as a construction and in the form of homogeneous coordinates. The counting procedure depends on the fact that objects occur in sets formally of size 6:

$$\begin{gathered} F(A,B,C),\ F(B,C,A),\ F(C,A,B), \\ F(A,C,B),\ F(B,A,C),\ F(C,B,A). \end{gathered}$$

For homogeneous coordinates, we shall use trilinears. Basic lore on trilinears, presented in the references, is assumed.

2. Primary system, S. Let $S_0 := \{A, B, C\}$, where the object $A := 1 : 0 : 0$ may be interpreted as the point with trilinears $1 : 0 : 0$ or as the line with equation $u\alpha + v\beta + w\gamma = 0$ having coefficients $u : v : w = 1 : 0 : 0$, i.e., the line BC, which may be called the A-sideline just as the point having trilinears $1 : 0 : 0$ is the A-vertex.

Three operations, or *opera* (singular *opus*) will now be defined and eventually applied to objects in S_0 and in succeeding generations of objects. Let $U = u : v : w$ and $X = x : y : z$.

$$\text{(1)} \quad \text{Opus 1}: \quad U \cdot X := \begin{cases} U & \text{if } X = U \\ vz - wy : wx - uz : uy - vx & \text{if } X \neq U; \end{cases}$$

if U and X are interpreted as distinct points, then $U \cdot X$ is their line, and if U and X are interpreted as distinct lines, then $U \cdot X$ is their point of intersection.

$$\text{(2)} \quad \begin{aligned} \text{Opus 2}: \quad U \| X := v(ay - bx) &+ w(az - cx) : w(bz - cy) \\ &+ u(bx - ay) : u(cx - az) + v(cy - bz) \end{aligned}$$

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if $X \neq a : b : c$, where a, b, c are real variables satisfying

$$0 < a \leq b + c, \qquad 0 < b \leq c + a, \qquad 0 < c \leq a + b.$$

Geometrically, a, b, c may be interpreted as the sidelengths of ΔABC, and $U \| X$ as the line through point U parallel to line X.

$$\text{(3)} \qquad \text{Opus 3}: \quad U \perp X := vz' - wy' : wx' - uz' : uy' - vx'$$

if these three coordinates are not all zero, where

$$\begin{aligned} x' &:= 2abcx - cy(a^2 + b^2 - c^2) - bz(c^2 + a^2 - b^2), \\ y' &:= 2abcy - az(b^2 + c^2 - a^2) - cx(a^2 + b^2 - c^2), \\ z' &:= 2abcz - bx(c^2 + a^2 - b^2) - ay(b^2 + c^2 - a^2). \end{aligned}$$

To see that $U \perp X$ may be interpreted as the line through U perpendicular to X because (3) agrees with the standard formula (e.g., [**1**, Article 4625]), note that

$$\text{(4)} \qquad x' : y' : z' = x - yc_1 - zb_1 : y - za_1 - xc_1 : z - xb_1 - ya_1,$$

where

$$\begin{aligned} a_1 &= \cos A := (b^2 + c^2 - a^2)/(2bc), \\ b_1 &= \cos B := (c^2 + a^2 - b^2)/(2ca), \\ c_1 &= \cos C := (a^2 + b^2 - c^2)/(2ab). \end{aligned}$$

Opus 3 makes Opus 2 redundant, since $U \| X = U \perp (U \perp X)$. This causes no difficulty in the sequel. Regarding the three opera that have now been defined, it is easy to see that the coordinates of $U \perp X$, $U \cdot X$ and $U \| X$ are polynomials in a, b, c whenever those of U and X are polynomials in a, b, c.

Next we introduce a systematic procedure for generating objects using the three opera.

For $n \geq 1$, define

$$\text{(5)} \qquad S_n = \left\{ U \circ X : U \in S_{n-1}, X \in S_{n-1}, \circ \in \{\cdot, \|, \perp\} \right\}$$

and

$$\text{(6)} \qquad S = \bigcup_{n=0}^{\infty} S_n.$$

Each element of S is an *object* and may be interpreted as either a point or a line. The coordinates of every object are, or may be taken to be, polynomials in a, b, c. Since

$$\sin A : \sin B : \sin C = a : b : c$$

and

$$\cos A = (b^2 + c^2 - a^2)/(2bc),$$

each object is also a function of A, B, C, indeed, a polynomial function in

$$\sin A, \sin B, \sin C, \cos A, \cos B, \cos C.$$

However, we shall continue to represent these as a, b, c, a_1, b_1, c_1, respectively, as in the following display of all the objects in S_1.

TABLE 1. The 15 objects comprising S_1.

$A = 1 : 0 : 0$	$B = 0 : 1 : 0$	$C = 0 : 0 : 1$
$A \parallel A = 0 : b : c$	$B \parallel B = a : 0 : c$	$C \parallel C = a : b : 0$
$A \perp A = 0 : b_1 : -c_1$	$B \perp B = -a_1 : 0 : c_1$	$C \perp C = a_1 : -b_1 : 0$
$A \perp B = 0 : a_1 : 1$	$B \perp C = 1 : 0 : b_1$	$C \perp A = c_1 : 1 : 0$
$A \perp C = 0 : 1 : a_1$	$B \perp A = b_1 : 0 : 1$	$C \perp B = 1 : c_1 : 0$

Equations (5) and (6) show that every object in S has coordinates of the form

$$\hat{f}(A, B, C) : \hat{g}(A, B, C) : \hat{h}(A, B, C),$$

which we abbreviate as

$$f_{abc} : g_{bca} : h_{cab}. \tag{7}$$

If $n \geq 0$, then for each object in U in S_n, there are five others, for a total of six, corresponding to the six permutations of A, B, C (and a, b, c). The six objects are counted here separately, even if not distinct. Thus S_n is a union of classes of six formal objects. Each class we call a *hex*. We now observe something from Table 1 that, in Section 4, we will extend to all of S: the objects in each hex are given by the six forms

$$(8a) \qquad f_{abc} : g_{bca} : h_{cab},$$

$$(8b) \qquad h_{abc} : f_{bca} : g_{cab},$$

$$(8c) \qquad g_{abc} : h_{bca} : f_{cab},$$

and

$$(9a) \qquad f_{acb} : h_{bac} : g_{cba},$$

$$(9b) \qquad g_{acb} : f_{bac} : h_{cba},$$

$$(9c) \qquad h_{acb} : g_{bac} : f_{cba}.$$

Any one of these six fits the form (7) and determines the other five.

To see why g and h trade places when passing from (8a) to (9a), think of (8a) as a geometric point, P, so that $g_{bca}/h_{cab} = \beta/\gamma$, where β and γ are directed distances from P to lines CA and AB, respectively. Then the transformation from $F(A, B, C)$ in (8a) to $F(A, C, B)$ in (9a) carries lines CA and AB onto lines BA and AC, respectively, so that for (9a) we obtain $g_{cba}/h_{bac} = \gamma/\beta$.

It will be expedient to reduce notation even more: $f_{abc} : g_{bca} : h_{cab}$ will be written as $f_{ab} : g_{bc} : h_{ca}$, and similarly for forms (8b)–(9c). The six objects will be written as $U_{ab}, U_{bc}, U_{ca}; U_{ac}, U_{ba}, U_{cb}$, respectively. Let T_{ab} be the set of objects (8a)–(8c), and let T_{ac} be the set of objects (9a)–(9c). Then T_{ab} and T_{ac} are (a pair of) *bicentric triangles*, and if they are not identical, *proper bicentric triangles.* In any case, we write these triangles as matrices:

$$(10) \qquad T_{ab} = \begin{pmatrix} f_{ab} & g_{bc} & h_{ca} \\ h_{ab} & f_{bc} & g_{ca} \\ g_{ab} & h_{bc} & f_{ca} \end{pmatrix} \quad \text{and} \quad T_{ac} = \begin{pmatrix} f_{ac} & h_{ba} & g_{cb} \\ g_{ac} & f_{ba} & h_{cb} \\ h_{ac} & g_{ba} & f_{cb} \end{pmatrix}.$$

When speaking algebraically, "point" will be synonymous with "object," so that a point is by definition an ordered triple of trilinears, and a "triangle" is a set of three objects. On the other hand, when speaking geometrically, we shall use the term *triangle* only when the three objects are interpreted as geometric points, not lines. This distinction becomes necessary when formulating relations between triangles, such as perspective, homothety and similarity in terms of trilinears.

If the six objects $U_{ab}, U_{bc}, U_{ca}, U_{ac}, U_{ba}, U_{cb}$ are not distinct, then one of them equals another, and by permuting a, b, c, we see that U_{ab} equals one of the other five objects so that there are five cases to consider. In order to examine these, let $\sigma =$ area of ΔABC, and

$$\begin{aligned} k(a,b,c) &:= \frac{2\sigma}{af_{ab} + bg_{gc} + ch_{ca}}; \\ \hat{f}_{ab} &= k(a,b,c)f_{ab}, \\ \hat{g}_{ab} &= k(c,a,b)g_{ab}, \\ \hat{h}_{ab} &= k(b,c,a)h_{ab}. \end{aligned}$$

Write

$$\text{(11)} \qquad \hat{T}_{ab} = \begin{pmatrix} \hat{f}_{ab} & \hat{g}_{bc} & \hat{h}_{ca} \\ \hat{h}_{ab} & \hat{f}_{bc} & \hat{g}_{ca} \\ \hat{g}_{ab} & \hat{h}_{bc} & \hat{f}_{ca} \end{pmatrix} \quad \text{and} \quad \hat{T}_{ac} = \begin{pmatrix} \hat{f}_{ac} & \hat{h}_{ba} & \hat{g}_{cb} \\ \hat{g}_{ac} & \hat{f}_{ba} & \hat{h}_{cb} \\ \hat{h}_{ac} & \hat{g}_{ba} & \hat{f}_{cb} \end{pmatrix}.$$

Then, as triangles (not as matrices), $\hat{T}_{ab} = T_{ab}$ and $\hat{T}_{ac} = T_{ac}$; in (11) the vertices are given in actual trilinear distances.

Case 1. $U_{ab} = U_{ac}$. Here the first rows of the matrices in (11) are equal, so that the hex consists of three (not necessarily distinct) objects,

$$\text{(12)} \qquad \hat{f}_{ab} : \hat{g}_{bc} : \hat{g}_{cb}, \quad \hat{g}_{ac} : \hat{f}_{bc} : \hat{g}_{ca}, \quad \hat{g}_{ab} : \hat{g}_{ba} : \hat{f}_{ca},$$

which, taken in order, comprise the triangle

$$\text{(13)} \qquad \begin{pmatrix} \hat{f}_{ab} & \hat{g}_{bc} & \hat{g}_{cb} \\ \hat{g}_{ac} & \hat{f}_{bc} & \hat{g}_{ca} \\ \hat{g}_{ab} & \hat{g}_{ba} & \hat{f}_{ca} \end{pmatrix}.$$

In practice, the multiplier $k(a, b, c)$ is often symmetric in a, b, c, and when this happens, the nine common factors in (13) cancel, leaving

$$\text{(14)} \qquad \begin{pmatrix} f_a & g_{bc} & g_{cb} \\ g_{ac} & f_b & g_{ca} \\ g_{ab} & g_{ba} & f_c \end{pmatrix},$$

where the symbol f_a has been introduced in lieu of f_{ab} since $f_{ab} = f_{ac}$. We use the form (14) as the definition of *central triangle*. A central triangle is a *proper central triangle* if the three objects are noncollinear and *degenerate* otherwise. (If $k(a,b,c)$ is not symmetric, the triangle (13) is not that given by (14); nevertheless, (13) has the functional form of (14), so that we are safe in using (14) to define central triangle. This definition has been used previously in [**3**], [**4**], [**6**]. Case 1 as presented here is essentially van Lamoen's proof in [**5**] that bicentric triangles with equal first rows constitute a central triangle.)

Case 2. $U_{ab} = U_{ba}$. This is equivalent to $U_{bc} = U_{cb}$, which is equivalent to the three simultaneous equations $\hat{h}_{ab} = \hat{h}_{ac}$, $\hat{f}_{bc} = \hat{g}_{ba}$, $\hat{g}_{ca} = \hat{f}_{cb}$. Thus, the triangle obtained by replacing rows 1, 2, 3 of T_{ab} in order by rows 2, 3, 1, has the form (14) with top row $\hat{h}_{ab} : \hat{f}_{bc} : \hat{f}_{cb}$. The two bicentric triangles in this case are distinct, even though they have the same set of vertices.

An example illustrating Case 2 is the hex of $(A\|A) \cdot (B\|B)$, given by $U_{ab} = 1/a : 1/b : -1/c$, consisting of three points, which may be written as

$$1/a : 1/b : -1/c, \quad -1/a : 1/b : 1/c, \quad 1/a : -1/b : 1/c.$$

These are the vertices of the central triangle given by $f_a = -1/a$ and $g_{bc} = 1/b$.

Case 3. $U_{ab} = U_{cb}$. This is equivalent to $U_{ca} = U_{ba}$. In the manner of Case 2, the triangle obtained by replacing rows 1, 2, 3 of T_{ab} in order by rows 3, 1, 2, has the form (14) with top row $\hat{g}_{ab} : \hat{h}_{bc} : \hat{h}_{cb}$.

Case 4. $U_{ab} = U_{bc}$. By cyclically permuting a, b, c, we have $U_{bc} = U_{ca}$ and $U_{ca} = U_{ab}$, so that the hex consists of two (not necessarily distinct) objects, called in [**3**] (a pair of) *bicentric points*, given by

$$\hat{f}_{ab} : \hat{f}_{bc} : \hat{f}_{ca} \quad \text{and} \quad \hat{f}_{ac} : \hat{f}_{ba} : \hat{f}_{cb}, \tag{15}$$

i.e., $\hat{f}_{ab}, \hat{f}_{bc}, \hat{f}_{ca}$ are the actual trilinear distances of the three here-identical vertices (8a)–(8c).

Case 5. $U_{ab} = U_{ca}$. By cyclic permutation of a, b, c, we have $U_{bc} = U_{ab}$, which is Case 4.

Finally, if the two points in (15) are equal (or if we have a central triangle (13) of identical vertices), then $f_a : f_b : f_c$ is a *triangle center*, or simply a *center*. Centers thus defined comprise a proper subset of the set of centers defined in [**2**] and [**3**].

Examples selected from familiar objects of triangle geometry (if not in the primary system S, then in the Euclidean system ES to be developed in Part 2) include the following:

Centers. Incenter ($f_{ab} = 1$), centroid ($f_{ab} = bc$), circumcenter ($f_{ab} = \cos A$), orthocenter ($f_{ab} = \sec A$), symmedian point ($f_{ab} = a$), Fermat point ($f_{ab} = \csc(A + \pi/3)$), Clawson point ($f_{ab} = \tan A$).

Bicentric points. The 1st and 2nd Brocard points ($f_{ab} = ac^2$ [so that $f_{ac} = ab^2$]).

Central triangles. ΔABC, medial, orthic, tangential, pedal triangles and cevian triangles of centers.

A center $u : v : w$ can be interpreted as the line with trilinear equation $u\alpha + v\beta + w\gamma = 0$. Examples of *central lines* include the Euler line and Brocard axis.

A word about the development of the notion of bicentric triangles may be appropriate. The notions of central triangles and bicentric points were in use at the time the author suggested to Floor van Lamoen a definition of bicentric triangles in terms of only two functions f and g (instead of three functions f, g, h), and which includes bicentric points and central triangles as special cases. In a subsequent communication, van Lamoen gave the definition of bicentric triangle introduced in [**5**] and used here. Further developments are given in [**6**].

In summary, each generation S_n, hence S, contains four kinds of hexes: centers, bicentric points, central triangles and bicentric triangles. For central triangles, we wish to attach single-letter labels A, B, C to the vertices. If $U_{ab} = U_{ac}$, the A-vertex is U_{ab}, and write U_a (and similarly for B, C, U_b and U_c).

3. Concerning "interpretations" and "triangles." Opera 1–3 are defined in Section 2 by trilinear forms, and it is stated that $U \cdot X$ can be interpreted either as the line of points U and X or as the point of intersection of lines U and X. Opus 2, it is stated, can be interpreted as the line through point U parallel to line X. Regarding Opus 2, we have

$$U\|X = U \cdot (X \cdot X_6),$$

where $X_6 = a : b : c$; thus, $U\|X$ has two interpretations, one as a line, the other as a point.

Opus 3 is given by $U \cdot (x' : y' : z')$ with $x' : y' : z'$ as in (4) so that Opus 3, too, has two interpretations. (The object $x' : y' : z'$, interpreted as a point, is the intersection of the line at infinity with the line through U perpendicular to line X.)

Let us consider this duality of interpretation further. If $P = p : q : r$ is an object, then its possible interpretations are as the point with trilinears $p : q : r$ or the line with trilinear equation $p\alpha + q\beta + r\gamma = 0$. Each is readily obtainable from the other if $pqr \neq 0$: the line is the trilinear polar of the isogonal conjugate of the point, and the point is the isogonal conjugate of the trilinear pole of the line. (Trilinear poles and polars are discussed in [**4**].)

Practically speaking, when a geometric construction is to be made, we therefore have two routes from which to choose: (i) interpret A, B, C as 3 points or else 3 lines, whichever yields the desired construction directly, and carry out the steps or, (ii) interpret A, B, C oppositely from (i) so that the dual of the desired object is constructed, and then construct the required pole or polar of that dual. In many cases, these two routes are remarkably dissimilar.

Algebraically, there is no need for interpretations, and we may proceed with trilinears without ever mentioning geometric meanings. However, whenever we wish, we may call upon either of two geometric interpretations. This freedom of choice calls for a certain understanding about triangles. Central triangles and bicentric pairs of triangles have been unambiguously defined as certain *sets* of three objects, but we avoid calling a triangle an object. On the other hand, we may speak of a triangle or hex as being "in" S (or some other system), but here, "in" means containment as a subset, not as an object.

4. Least constructions and labels. Up to this point, any one of the objects in a hex could be labeled as the AB-object. We shall now give a procedure for identifying a specific object as the AB-object of its hex. To begin, in S_0, the AB-object is A, and in S_1, the AB-objects are $A, A\|A$, $A \perp A$ and $A \perp B$. The order in which these have just been listed suggests a systematic order in which, for each $n \geq 0$, the AB-objects (hence all objects) in S_{n+1} can be generated from those in S_n.

Define "precedes," indicated by the symbol "$\prec$," by writing $A \prec B \prec C$, and extend $\prec$ to the AB-objects in S_1 as follows:

$$A \prec A\|A \prec A \perp A \prec A \perp B.$$

Then fill in between with all the other objects according to the ordering of labels shown here:

$$AB,\ BC,\ CA,\ AC,\ BA,\ CB.$$

Thus the ordering of all the objects in S_1 is given by

(16)

$$\begin{aligned}&A \prec B \prec C \prec A\|A \prec B\|B \prec C\|C \prec A \perp A \prec B \perp B \prec C \perp C\\ &\prec A \perp B \prec B \perp C \prec C \perp A \prec A \perp C \prec B \perp A \prec C \perp B.\end{aligned}$$

Each of these 15 representations we shall call the *least construction* of the object in question. To order AB-objects, it suffices to define and order their least constructions. Suppose this has been done for arbitrary $n \geq 0$. Let $T_1, T_2, T_3, \ldots, T_k$ be the AB-objects in S_n arranged in increasing order, and let $U_1, U_2, U_3, \ldots, U_l$ be the least constructions in S_n arranged in increasing order. Then the first k AB-objects in S_{n+1} are, in order, $T_1, T_2, T_3, \ldots, T_k$. Next, arrange $T_i \cdot U_j$ for $i = 1, 2, 3, \ldots, k$ and $j = 1, 2, 3, \ldots, l$ in order according to the lexicographic ordering of the pairs (i, j), i.e.,

$$\begin{aligned}(1,1) < (1,2) < \cdots < (1,l) < (2,1) < (2,2) < \cdots < (2,l) < \cdots\\ < \cdots < (k,1) < (k,2) < \cdots < (k,l).\end{aligned}$$

Retain in order as a least construction each $T_i \cdot U_j$ which is not among the T_i and which lies not in the hex of any construction already retained. Then arrange all $T_i\|U_j$ in the same manner and retain as least constructions those that represent new hexes; finally, do the same with all $T_i \perp U_j$. (The objects in Tables 1 and 2, for example, are listed in the order just described.) Up to this point, only the AB-objects in

S_n have been ordered. To complete the ordering of all objects in S_n, begin by inserting between each neighboring pair $U_{ab} \prec U'_{ab}$ the other five objects in the hex of U_{ab}, getting

$$U_{ab} \prec U_{bc} \prec U_{ca} \prec U_{ac} \prec U_{ba} \prec U_{cb} \prec U'_{ab},$$

and then expel each which has already occurred. The ordering of objects in S_{n+1} is now finished. By induction, every object in the primary system S has a unique position in an ordered list.

Example 1. When generating S_1 from S_0, we find that $A \cdot B = C$ and also $A = B \cdot C$. Neither of these constructions alters the AB label assigned to A in S_0, and both are expelled in the procedure for ordering the objects in S_1.

Example 2. When generating S_2 from S_1, the hex containing $(A\|A) \cdot (B \perp B)$ is identical to that containing $(A \perp A) \cdot (B\|B)$. The AB-object for this hex is determined from the first of the two constructions, which is the least construction.

Henceforth, the AB-object of a hex will be called the *type* of the hex (and also the type of each object in the hex). For example, the type of the reference triangle ABC is $1:0:0$. Less transparently, corresponding to Example 2, the type of $-cb_1 : ac_1 : ab_1$ is $bc_1 : -ca_1 : ba_1$.

Example 3. The hex of $(A \cdot ((B \perp B)\|C)) \cdot (B \cdot ((C \perp C)\|A))$ consists of a pair of bicentric points:

$$abb_1 : bcc_1 : caa_1 \quad \text{and} \quad acc_1 : baa_1 : cbb_1.$$

Example 4. The reference triangle ABC, given by the identity matrix, is central. Permuting the vertices yields the pair

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

of proper bicentric triangles.

Example 5. The hex of $(A\|A)\cdot(B\|B)$ consists of six objects grouped here in the form of two triangles, T_{ab} and T_{ac}, the first having rows

$$(A\|A)\cdot(B\|B), \quad (B\|B)\cdot(C\|C), \quad (C\|C)\cdot(A\|A);$$

and the second,

$$(A\|A)\cdot(C\|C), \quad (B\|B)\cdot(A\|A), \quad (C\|C)\cdot(B\|B):$$

$$T_{ab} = \begin{pmatrix} 1/a & 1/b & -1/c \\ -1/a & 1/b & 1/c \\ 1/a & -1/b & 1/c \end{pmatrix} \quad \text{and} \quad T_{ac} = \begin{pmatrix} 1/a & -1/b & 1/c \\ 1/a & 1/b & -1/c \\ -1/a & 1/b & 1/c \end{pmatrix}.$$

This is a pair of proper bicentric triangles; neither should be mistaken for a central triangle, even though the set of vertices of each is that of the (central) anticomplementary triangle.

In general, if a central triangle has the form (14), then the relabeling of vertices, or rearrangement of rows, indicated by

$$\begin{pmatrix} g_{ab} & g_{bc} & f_{ca} \\ f_{ab} & g_{bc} & g_{ca} \\ g_{ab} & f_{bc} & g_{ca} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} g_{ab} & f_{bc} & g_{ca} \\ g_{ab} & g_{bc} & f_{ca} \\ f_{ab} & g_{bc} & g_{ca} \end{pmatrix}$$

yields proper bicentric triangles.

The distinction made in Examples 4 and 5 is significant inasmuch as theorems about bicentric triangles apply to pairs associated with a central triangle in this manner.

5. Generating S_2 from S_1. To generate S_2 from S_1, the following four reduction formulas for arbitrary objects X, Y, Z are helpful. Proofs of these are easy and omitted.

$$X\|(Y\|Z) = X\|Z \tag{17}$$

$$X\|(Y \perp Z) = X \perp Z \tag{18}$$

$$X \perp (Y\|Z) = X \perp Z \tag{19}$$

$$X \perp (Y \perp Z) = X\|Z. \tag{20}$$

An inventory of types in S_2 is now given: one center $b_1c_1 : c_1a_1 : a_1b_1$; no proper bicentric points; and triangles as shown in Tables 2 and 3.

TABLE 2. Central triangles in $S_2 - S_1$ (by type).

	Least construction	Type (i.e., U_a)
1	$A \cdot (A \parallel A)$	$0 : c : -b$
2	$A \cdot (A \perp A)$	$0 : c_1 : b_1$
3	$(A \parallel A) \cdot (B \parallel B)$	$bc : ca : -ab$
4	$(A \parallel A) \perp A$	$bb_1 - cc_1 : -c : b$
5	$(A \perp A) \cdot (B \perp A)$	$-1 : c_1 : b_1$
6	$(A \perp B) \cdot (B \perp A)$	$a_1 : b_1 : -a_1 b_1$
7	$(A \perp A) \perp A$	$b_1^2 + c_1^2 : c_1 : b_1$

At least three of the seven central triangles in Table 2 have occurred in the literature: #2, the orthic; #4, the anticomplementary and #7, the reflection of ΔABC about the circumcenter.

In summary, $S_2 - S_1$ consists of 24 hexes, 1 of size 1, 7 of size three, and 16 of size six, for a formal total of 118 objects. Counting these with the objects of Table 1, we find that S_2 has a total of 133 objects in 28 hexes. Ordered by $\prec$, the list begins as in (16) and ends in the six objects of type $(A \perp B) \perp C$:

$$\begin{aligned}(A \perp B) \perp C \prec (B \perp C) \perp A \prec (C \perp A) \perp B \prec (A \perp C) \perp B \\ \prec (B \perp A) \perp C \prec (C \perp B) \perp A.\end{aligned}$$

Certain types in S_2 may be of particular interest. For example, the bicentric triangles T_{ab} and T_{ac} of type $(A \perp B) \cdot (B \perp C)$ are mutually homothetic and similar to ΔABC. In fact, each can be obtained as a $90°$ rotation of ΔABC about its circumcenter, followed by a dilation; if, then, $\mathcal{L}(\Delta ABC)$ is any line, then the line $\mathcal{L}(T_{ab})$ is perpendicular to $\mathcal{L}(\Delta ABC)$; e.g., the Euler lines of T_{ab} and T_{ac} are each perpendicular to the Euler line of ΔABC. The conic passing through the vertices of T_{ab} and T_{ac} awaits investigation elsewhere.

TABLE 3. Proper bicentric triangles in $S_2 - S_1$ (by type).

	Least construction	Type (i.e., U_{ab})
1	$A \cdot (A \perp B)$	$0 : 1 : -a_1$
2	$(A \parallel A) \cdot (B \perp A)$	$b : cb_1 : -bb_1$
3	$(A \parallel A) \cdot (B \perp B)$	$bc_1 : -ca_1 : ba_1$
4	$(A \parallel A) \cdot (B \perp C)$	$bb_1 : c : -b$
5	$(A \parallel A) \parallel B$	$ab : -c^2 : bc$
6	$(A \parallel A) \perp B$	$ba_1 + c : cc_1 : -bc_1$
7	$(A \perp A) \cdot (B \perp C)$	$-b_1^2 : c_1 : b_1$
8	$(A \perp A) \parallel B$	$ab_1 : cc_1 : cb_1$
9	$(A \perp A) \perp B$	$c_1 - a_1b_1 : c_1^2 : b_1c_1$
10	$(A \perp B) \cdot (B \perp C)$	$a_1b_1 : 1 : -a_1$
11	$(A \perp B) \cdot (C \perp A)$	$1 : -c_1 : a_1c_1$
12	$(A \perp B) \parallel B$	$aa_1 : -c : aa_1$
13	$(A \perp B) \parallel C$	$a : b : -ba_1$
14	$(A \perp B) \perp A$	$c_1 - a_1b_1 : 1 : -a_1$
15	$A \perp B) \perp B$	$a_1^2 + 1 : c_1 : -c_1a_1$
16	$(A \perp B) \perp C$	$2a_1 : -b_1 : a_1b_1$

Regarding the ordering of the 133 objects comprising S_2, the objects listed in Table 2 do not all precede those in Table 3. With this in mind, along with the prospects of someday examining S_3 in detail, we have placed an ordered listing of the 133 objects in the Appendix.

6. Constructions in S and constructions not in S. Suppose X is a point in S_n. One may ask for the least number k such that the isogonal conjugate of X is in S_{n+k}. Of course, k may vary, depending on X, but if we regard X as an indeterminate point (i.e., $X = x : y : z$ where x, y, z are indeterminates or real variables), then an upper bound for the least k can be obtained by performing opera that always yield the isogonal conjugate of X. Moreover, instead of isogonal conjugate, one may ask for the least k for other constructions. In general, to each object $\mathcal{O}$ in S, we have the *generation-increase number* $\mathcal{G} = \mathcal{G}(\mathcal{O})$ defined as the least k such that $\mathcal{O} \in S_{n+k}$, where the objects to which the three opera can be applied all lie in S_n or are adjoined to S_n as indeterminate objects.

Upper bounds for generation-increase numbers for selected constructions will now be itemized; proofs are essentially given in [**2**]. Suppose U, V, W, X, L, L' are objects in S_n or adjoined to S_n as indeterminate objects; in each of these cases, we interpret U, V, W, X as points and L, L' as lines; relative to this interpretation of those six objects, each listed construction has a geometric interpretation, given in parentheses.

i. $\mathcal{G}$(pedal triangle of X) ≤ 2;
ii. $\mathcal{G}$(antipedal triangle of X) ≤ 3;
iii. $\mathcal{G}$(cevian triangle of X) ≤ 2;
iv. $\mathcal{G}$(anticevian triangle of X) ≤ 10;
v. $\mathcal{G}$(midpoint of U and V) ≤ 4;
vi. $\mathcal{G}$(reflection-in-U of V) ≤ 4;
vii. $\mathcal{G}$(reflection-in-U of L) ≤ 7;
viii. $\mathcal{G}$(reflection-in-L of U) ≤ 6;
ix. $\mathcal{G}$(reflection-in-L of L') ≤ 9;
x. $\mathcal{G}$(isogonal conjugate of X, if X is not on a sideline of ΔABC) ≤ 7;
xi. $\mathcal{G}$(isotomic conjugate of X, if X is not on a sideline of ΔABC) ≤ 8;
xii. $\mathcal{G}$(harmonic conjugate of W relative to U and V, if W is on line UV) ≤ 8;
xiii. $\mathcal{G}$(trilinear pole of L if L is not a sideline of ΔABC) ≤ 10;
xiv. $\mathcal{G}$(trilinear polar of U if $U \notin \{A, B, C\}$) ≤ 5;
xv. $\mathcal{G}$(U-Ceva conjugate of V, if neither U nor V is a sideline of ΔABC) ≤ 12;
xvi. $\mathcal{G}$(U-line conjugate of V, if neither U nor V is a sideline of ΔABC) ≤ 14.

Missing from this list are angle bisectors and hence also the incenter, I, which has the particularly simple polynomial trilinears $1 : 1 : 1$. To see that S does not contain the angle bisector of $\angle ABC$, note first that if points A, B, C each have Cartesian coordinates (x, y) where x and y are rational numbers, then every line in S has rational slope. However, if $A = (1, 1)$, $B = (0, 0)$, $C = (1, 0)$, then the bisector of $\angle ABC$ has irrational slope. Therefore this bisector is not in S. (I thank Matthew Cook for this simple argument.)

Since angle bisectors are not generally in S, one might wonder how isogonal conjugates are constructed using (only) opera 1–3; see [**2**].

The fact that $I \notin S$ suggests a future study of the system $S[I]$, defined inductively via opera 1–3 on the set $\{A, B, C, I\}$. One asks, for example, for the simplest polynomial $p(a, b, c)$, homogeneous in a, b, c and symmetric in b and c, for which the triangle center $p(a, b, c) : p(b, c, a) : p(c, a, b)$ is not in $S[I]$.

7. Toggles. Suppose T_{ab} and T_{ac} are bicentric triangles given by $T_{ab} = \{U_{ab}, U_{bc}, U_{ca}\}$ and $T_{ac} = \{U_{ac}, U_{ba}, U_{cb}\}$. Let

$$T^*_{ab} = \{U_{ab} \cdot U_{bc}, U_{bc} \cdot U_{ca}, U_{ca} \cdot U_{ab}\}$$

and

$$T^*_{ac} = \{U_{ac} \cdot U_{ba}, U_{ba} \cdot U_{cb}, U_{cb} \cdot U_{ac}\}.$$

Let $\mathcal{H} = T_{ab} \cup T_{ac}$ and $\mathcal{H}^* = T^*_{ab} \cup T^*_{ac}$. Then $T^*_{ab}, T^*_{ac}, \mathcal{H}^*$ are the *toggles* of $T_{ab}, T_{ac}, \mathcal{H}$, respectively. Note that $(T^*_{ab})^* = T_{ab}$, $(T^*_{ac})^* = T_{ac}$, and $(\mathcal{H}^*)^* = \mathcal{H}$. (In [**4**], T^*_{ab} is called the unary cofactor triangle of T_{ab}; it is proved that the two triangles are perspective, and their center of perspective defines the eigencenter of T_{ab}.)

Example 6. This is the first of two interpretations of item 8 in Table 3. Here $A' = ab_1 : cc_1 : cb_1$ is interpreted as the point of intersection of lines $0 : b_1 : -c_1$ and $c : 0 : -a$; the first line, $A \perp A$, is the A-altitude, and the other passes through vertex B and the symmedian point $a : b : c$. The hex of which A' is the type is represented in Figure 1 as a pair of bicentric triangles, $A'B'C'$ and $A''B''C''$, each triply perspective to the other and each triply perspective to ΔABC.

The centers of perspective will now be presented: the lines

$$A \cdot A' = A \cdot A'', \quad B \cdot B' = B \cdot B'', \quad C \cdot C' = C \cdot C''$$

concur in the orthocenter, $X_4 = \sec A : \sec B : \sec C$; the lines

$$A \cdot C'' = A \cdot B'', \quad B \cdot A' = B \cdot C'', \quad C \cdot B' = C \cdot A''$$

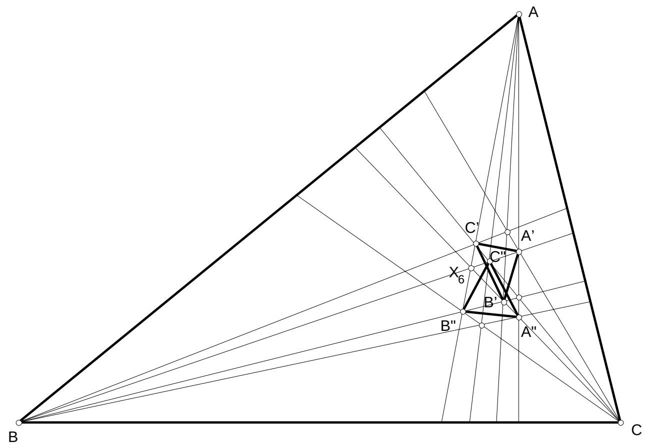

FIGURE 1. Bicentric triangles $A'B'C'$ and $A''B''C''$ triply perspective to each other and to triangle ABC. See Example 6.

concur in the symmedian point, $X_6 = a : b : c$; the lines

$$A \cdot B', \quad B \cdot C', \quad C \cdot A' \text{ concur in the point } P_{ab} := abb_1 : bcc_1 : caa_1;$$
$$A \cdot C'', \quad B \cdot A'', \quad C \cdot B'' \text{ concur in the point } P_{ac} := acc_1 : baa_1 : cbb_1.$$

These last two centers of perspective are bicentric points, and as is the case for all such pairs P and Q, the object $P \cdot Q$ is central. In the case at hand, the triangle center $P_{ab} \cdot P_{ac}$ is of interest as a "new" point on the Euler line. It is easy to confirm that it is in S_5 and that its trilinears are given by

$$P_{ab} \cdot P_{ac} = bc(bcb_1c_1 - a^2a_1^2) : ca(cac_1a_1 - b^2b_1^2) : ab(aba_1b_1 - c^2c_1^2),$$

or

$$l(A, B, C) : l(B, C, A) : l(C, A, B),$$

where $l(A, B, C) = (\csc A)(\sin 2B \sin 2C - \sin^2 2A)$. This point is named the *Bailey point*, honoring V.C. Bailey, Professor Emeritus of Mathematics, University of Evansville, in celebration of his ninety-fourth birthday.

On the first day of 1999, Peter Yff found an interesting way to write trilinears for the Bailey point. As a prelude, note that the form $\alpha =$

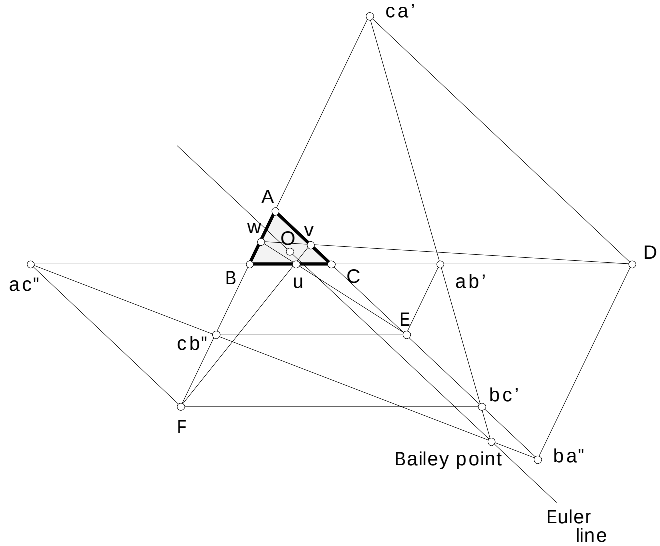

FIGURE 2. Bailey point: a new point on the Euler line of ΔABC.

$\cos A + t\cos B\cos C$ (for the first of three trilinears) expresses the best-known centers on the Euler line, since the values $t = 0, \infty, 1, 2, -1, -2$ give trilinears $\alpha : \beta : \gamma$ for the circumcenter, orthocenter, centroid, nine-point center, de Longchamps point, and Euler infinity point, respectively. Regarding t as a real variable gives a "thinline" that is a *proper* subset of the Euler line. Of course, every point on the Euler line is given by some t, but for points not on the aforementioned thinline, t is a nonconstant function of A, B, C. What Yff found was that, for the Bailey point,

$$t = \frac{\sin 2B \sin 2C + \sin 2C \sin 2A + \sin 2A \sin 2B}{-1 + \cos 2A \cos 2B \cos 2C}.$$

Example 7. This is the second interpretation of item 8 in Table 3. Here,

$$A' = ab_1 : cc_1 : cb_1 (= X_{ab} \text{ in Figure 3})$$

is interpreted as the line $ab_1\alpha + cc_1\beta + cb_1\gamma = 0$ through the point $A \perp A = 0 : b_1 : -c_1$ parallel to the line $\beta = 1$. The triangle T_{ab} with

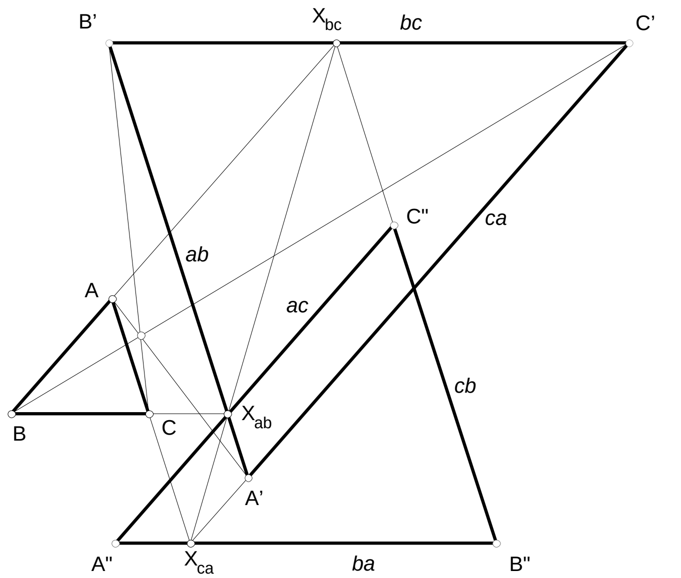

FIGURE 3. Bicentric triangles $A'B'C'$ and $A''B''C''$, triply perspective to each other and homothetic to triangle ABC. See Example 7.

sides

$$(A \perp A)\|B, \qquad (B \perp B)\|C, \qquad (C \perp C)\|A$$

is labeled $A'B'C'$ in Figure 3. Its vertices are the toggle of the sides, e.g.,

$$\begin{aligned} A' &= ((A \perp A)\|B) \cdot ((B \perp B)\|C) \\ &= cc_1(aa_1 - bb_1) : ab_1(cc_1 - aa_1) : ac_1(bb_1 - cc_1). \end{aligned}$$

The triangle T_{ac} with AC-side $(A \perp A)\|C$ is labeled $A''B''C''$ in Figure 3, and

$$A'' = bb_1(aa_1 - cc_1) : ab_1(cc_1 - bb_1) : ac_1(bb_1 - aa_1).$$

Notably, T_{ac} is triply perspective to T_{ab} and homothetic to ΔABC.

8. Purebreds. An object is a *purebred* if it has a construction in which neither B nor C appears, i.e., the only symbols that do occur are

$A, \cdot, \|, \perp$ and grouping symbols. The original generation S_0 contains one purebred, namely A; the first generation S_1 contains two more purebreds, namely $A\|A$ and $A \perp A$. The four purebreds in $S_2 - S_1$ are given by

$$A \cdot (A\|A) \prec A \cdot (A \perp A) \prec (A\|A) \perp A \prec (A \perp A) \perp A.$$

Table 4 gives a complete ordered list of the 15 purebreds in S_3.

Clearly, the hex generated by a purebred is a central triangle, which may be a triangle center, as in row 12 of Table 4. If the hex is not a single point, then it is a triangle with labeled vertices, and any two such triangles can be conveniently labeled as $A_1B_1C_1$ and $A_2B_2C_2$. One of the most common kinds of construction of "special points" in traditional triangle geometry is as the concurrence of lines A_1A_2, B_1B_2, C_1C_2. These lines always form a central triangle given by the toggles

$$\begin{aligned} A' &= (B_1 \cdot B_2) \cdot (C_1 \cdot C_2), \\ B' &= (C_1 \cdot C_2) \cdot (A_1 \cdot A_2), \\ C' &= (A_1 \cdot A_2) \cdot (B_1 \cdot B_2). \end{aligned}$$

We are particularly interested in pairs $A_1B_1C_1$ and $A_2B_2C_2$ for which $A' = B' = C'$. Examples illustrated by Table 4 are obtained by taking A_1 as in row 1 (i.e., $\Delta A_1B_1C_1 = \Delta ABC$):

for A_2 as in row 2, $A' = a : b : c =$ symmedian point;
for A_2 as in row 3, $A' = 1_1/a : 1/b_1 : 1/c_1 =$ orthocenter;
for A_2 as in row 4, $A' = 1/a : 1/b : 1/c =$ centroid;
for A_2 as in row 5, $A' = a_1 : b_1 : c_1 =$ circumcenter.

In general, if $A_2B_2C_2$ is given by (14), then it is easy to show that the lines AA_2, BB_2, CC_2 concur if and only if $g_{cb}g_{ac}g_{ba} = g_{bc}g_{ca}g_{ab}$, and that the point of concurrence, a triangle center we shall denote by X, is given according to one of three cases:

(i) if $g_{ab} = g_{ac}$, then $X = g_{ab} : g_{bc} : g_{ca}$;

(ii) if $g_{ab} = g_{ba}$ and $g_{ab} \neq g_{ac}$, then $X = 1 : 1 : 1$;

(iii) if $g_{ab} = g_{cb}$, $g_{ab} \neq g_{ac}$ and $g_{ab} \neq g_{ba}$, then $X = g_{ab}g_{ac} : g_{bc}g_{ba} : g_{ca}g_{cb}$.

TABLE 4. Purebreds in S_3.

	Least construction	Type (i.e., U_{ab})
1	A	$1 : 0 : 0$
2	$A \parallel A$	$0 : b : c$
3	$A \perp A$	$0 : b_1 : -c_1$
4	$A \cdot (A \parallel A)$	$0 : c : -b$
5	$A \cdot (A \perp A)$	$0 : c_1 : -b_1$
6	$(A \parallel A) \perp A$	$bb_1 - cc_1 : -c : b$
7	$(A \perp A) \perp A$	$b_1^2 + c_1^2 : c_1 : b_1$
8	$A \perp (A \cdot (A \parallel A))$	$0 : b + ca_1 : c + ba_1$
9	$A \perp (A \cdot (A \perp A))$	$0 : b_1 - c_1a_1 : a_1b_1 - c_1$
10	$A \perp ((A \parallel A) \perp A)$	$0 : b_1(cc_1 - bb_1) + b + ca_1 : c_1(bb_1 - cc_1)$ $+c + ba_1$
11	$A \perp ((A \perp A) \perp A)$	$0 : b_1(b_1^2 + c_1^2 + 1) - a_1c_1 : -c_1(b_1^2 + c_1^2$ $+1) + a_1b_1$
12	$(A \cdot (A \parallel A)) \perp A$	$a : b : c$ (symmedian point, or line at infinity)
13	$(A \cdot (A \perp A)) \perp A$	$0 : b_1 : c_1$
14	$((A \parallel A) \perp A) \perp A$	$a : b + b_1(bb_1 - cc_1) : c + c_1(cc_1 - bb_1)$
15	$((A \perp A) \perp A) \parallel A$	$-a : b(b_1^2 + c_1^2) : c(b_1^2 + c_1^2)$

Conversely, if one of these cases holds, then clearly $g_{cb}g_{ac}g_{ba} = g_{bc}g_{ca}g_{ab}$, and the three lines concur. An example illustrating case (iii) is given by $g_{ab} = c + a - b$.

The fact that purebreds help account for central triangles raises the question of other sufficient conditions for centrality. One other such condition is that U_{ab} be of the form $\mathcal{O}_{ab} \cdot \mathcal{O}_{ba}$, as in rows 3 and 6 of Table 2. However, these two conditions are not exhaustive, as illustrated by row 5 of Table 2. It is an open question whether an object whose *least* construction uses B or C can be a purebred.

In Part 2, we shall allow circles, in addition to the objects considered in the primary system.

Appendix

Ordering of objects in the primary system

1.	A	1	0	0
2.	B	0	1	0
3.	C	0	0	1

End of S_0; beginning of S_1:

4.	$A\|A$	0	b	c
5.	$B\|B$	a	0	c
6.	$C\|C$	a	b	0
7.	$A \perp A$	0	b_1	$-c_1$
8.	$B \perp B$	$-a_1$	0	c_1
9.	$C \perp C$	a_1	$-b_1$	0
10.	$A \perp B$	0	a_1	1
11.	$B \perp C$	1	0	b_1
12.	$C \perp A$	c_1	1	0
13.	$A \perp C$	0	1	a_1
14.	$B \perp A$	b_1	0	1
15.	$C \perp B$	1	c_1	0

End of S_1; beginning of S_2:

16.	$A \cdot (A\|A)$	0	c	$-b$
17.	$B \cdot (B\|B)$	$-c$	0	a
18.	$C \cdot (C\|C)$	b	$-a$	0
19.	$A \cdot (A \perp A)$	0	c_1	b_1
20.	$B \cdot (B \perp B)$	c_1	0	a_1
21.	$C \cdot (C \perp C)$	b_1	a_1	0
22.	$A \cdot (A \perp B)$	0	1	$-a_1$
23.	$B \cdot (B \perp C)$	$-b_1$	0	1
24.	$C \cdot (C \perp A)$	1	$-c_1$	0
25.	$A \cdot (A \perp C)$	0	$-a_1$	1
26.	$B \cdot (B \perp A)$	1	0	$-b_1$
27.	$C \cdot (C \perp B)$	$-c$	1	0

28.	$(A\|A)\cdot(B\|B)$	bc	ca	$-ab$
29.	$(B\|B)\cdot(C\|C)$	$-bc$	ca	ab
30.	$(C\|C)\cdot(A\|A)$	bc	$-ca$	ab
31.	$(A\|A)\cdot(B\perp A)$	b	cb_1	$-bb_1$
32.	$(B\|B)\cdot(C\perp B)$	$-cc_1$	c	ac_1
33.	$(C\|C)\cdot(A\perp C)$	ba_1	$-aa_1$	a
34.	$(A\|A)\cdot(C\perp A)$	c	$-cc_1$	bc_1
35.	$(B\|B)\cdot(A\perp B)$	ca_1	a	$-aa_1$
36.	$(C\|C)\cdot(B\perp C)$	$-bb_1$	ab_1	b
37.	$(A\|A)\cdot(B\perp B)$	bc_1	$-ca_1$	ba_1
38.	$(B\|B)\cdot(C\perp C)$	cb_1	ca_1	$-ab_1$
39.	$(C\|C)\cdot(A\perp A)$	$-bc_1$	ac_1	ab_1
40.	$(A\|A)\cdot(C\perp C)$	b_1c	ca_1	$-a_1b$
41.	$(B\|B)\cdot(A\perp A)$	$-b_1c$	c_1a	ab_1
42.	$(C\|C)\cdot(B\perp B)$	bc_1	$-c_1a$	a_1b
43.	$(A\|A)\cdot(B\perp C)$	bb_1	c	$-b$
44.	$(B\|B)\cdot(C\perp A)$	$-c$	cc_1	a
45.	$(C\|C)\cdot(A\perp B)$	b	$-a$	aa_1
46.	$(A\|A)\cdot(C\perp B)$	cc_1	$-c$	b
47.	$(B\|B)\cdot(A\perp C)$	c	aa_1	$-a$
48.	$(C\|C)\cdot(B\perp A)$	$-b$	a	bb_1
49.	$(A\|A)\|B$	ab	$-c^2$	bc
50.	$(B\|B)\|C$	ca	bc	$-a^2$
51.	$(C\|C)\|A$	$-b^2$	ab	ca
52.	$(A\|A)\|C$	ac	bc	$-b^2$
53.	$(B\|B)\|A$	$-c^2$	ba	ca
54.	$(C\|C)\|B$	ab	$-a^2$	cb
55.	$(A\|A)\perp A$	bb_1-cc_1	$-c$	b
56.	$(B\|B)\perp B$	c	cc_1-aa_1	$-a$
57.	$(C\|C)\perp C$	$-b$	a	aa_1-bb_1
58.	$(A\|A)\perp B$	ba_1+c	cc_1	$-bc_1$
59.	$(B\|B)\perp C$	$-ca_1$	cb_1+a	aa_1
60.	$(C\|C)\perp A$	bb_1	ab_1	ac_1+b
61.	$(A\|A)\perp C$	ca_1+b	$-cb_1$	bb_1
62.	$(B\|B)\perp A$	cc_1	ab_1+c	$-ac_1$
63.	$(C\|C)\perp B$	$-ba_1$	aa_1	bc_1+a

64.	$(A \perp A) \cdot (B \perp B)$	b_1c_1	c_1a_1	a_1b_1
65.	$(A \perp A) \cdot (B \perp A)$	-1	c_1	b_1
66.	$(B \perp B) \cdot (C \perp B)$	c_1	-1	a_1
67.	$(C \perp C) \cdot (A \perp C)$	b_1	a_1	-1
68.	$(A \perp B) \cdot (B \perp A)$	a_1	b_1	$-a_1b_1$
69.	$(B \perp C) \cdot (C \perp B)$	$-b_1c_1$	b_1	c_1
70.	$(C \perp A) \cdot (A \perp C)$	a_1	$-c_1a_1$	c_1
71.	$(A \perp A) \cdot (B \perp C)$	$-b_1^2$	c_1	b_1
72.	$(B \perp B) \cdot (C \perp A)$	c_1	$-c_1^2$	a_1
73.	$(C \perp C) \cdot (A \perp B)$	b_1	a_1	$-a_1^2$
74.	$(A \perp A) \cdot (C \perp B)$	$-c_1^2$	b_1	c_1
75.	$(B \perp B) \cdot (A \perp C)$	a_1	$-a_1^2$	c_1
76.	$(C \perp C) \cdot (B \perp A)$	a_1	b_1	$-b_1^2$
77.	$(A \perp A) \| B$	ab_1	cc_1	cb_1
78.	$(B \perp B) \| C$	ac_1	bc_1	aa_1
79.	$(C \perp C) \| A$	bb_1	ba_1	ca_1
80.	$(A \perp A) \| C$	ac_1	bc_1	bb_1
81.	$(B \perp B) \| A$	cc_1	ba_1	ca_1
82.	$(C \perp C) \| B$	ab_1	aa_1	cb_1
83.	$(A \perp A) \perp A$	$b_1^2+c_1^2$	c_1	b_1
84.	$(B \perp B) \perp B$	c_1	$c_1^2+a_1^2$	a_1
85.	$(C \perp C) \perp C$	b_1	a_1	$a_1^2+b_1^2$
86.	$(A \perp A) \perp B$	$c_1-a_1b_1$	c_1^2	b_1c_1
87.	$(B \perp B) \perp C$	c_1a_1	$a_1-b_1c_1$	a_1^2
88.	$(C \perp C) \perp A$	b_1^2	a_1b_1	$b_1-a_1c_1$
89.	$(A \perp A) \perp C$	$b_1-a_1c_1$	b_1c_1	b_1^2
90.	$(B \perp B) \perp A$	c_1^2	$c_1-b_1a_1$	c_1a_1
91.	$(C \perp C) \perp B$	a_1b_1	a_1^2	$a_1-c_1b_1$
92.	$(A \perp B) \cdot (B \perp C)$	a_1b_1	1	$-a_1$
93.	$(B \perp C) \cdot (C \perp A)$	$-b_1$	b_1c_1	1
94.	$(C \perp A) \cdot (A \perp B)$	1	$-c_1$	c_1a_1
95.	$(A \perp C) \cdot (C \perp B)$	a_1c_1	$-a_1$	1
96.	$(B \perp A) \cdot (A \perp C)$	1	b_1a_1	$-b_1$
97.	$(C \perp B) \cdot (B \perp A)$	$-c_1$	1	c_1b_1

98.	$(A \perp B) \cdot (C \perp A)$	1	$-c_1$	a_1c_1
99.	$(B \perp C) \cdot (A \perp B)$	b_1a_1	1	$-a_1$
100.	$(C \perp A) \cdot (B \perp C)$	$-b_1$	c_1b_1	1
101.	$(A \perp C) \cdot (B \perp A)$	1	a_1b_1	$-b_1$
102.	$(B \perp A) \cdot (C \perp B)$	$-c_1$	1	b_1c_1
103.	$(C \perp B) \cdot (A \perp C)$	c_1a_1	$-a_1$	1
104.	$(A \perp B) \| B$	aa_1	$-c$	a_1c
105.	$(B \perp C) \| C$	b_1a	bb_1	$-a$
106.	$(C \perp A) \| A$	$-b$	c_1b	cc_1
107.	$(A \perp C) \| C$	aa_1	a_1b	$-b$
108.	$(B \perp A) \| A$	$-c$	bb_1	b_1c
109.	$(C \perp B) \| B$	c_1a	$-a$	cc_1
110.	$(A \perp B) \| C$	a	b	$-ba_1$
111.	$(B \perp C) \| A$	$-cb_1$	b	c
112.	$(C \perp A) \| B$	a	$-ac_1$	c
113.	$(A \perp C) \| B$	a	$-ca_1$	c
114.	$(B \perp A) \| C$	a	b	$-ab_1$
115.	$(C \perp B) \| A$	$-bc_1$	b	c
116.	$(A \perp B) \perp A$	$c_1 - a_1b_1$	1	$-a_1$
117.	$(B \perp C) \perp B$	$-b_1$	$a_1 - b_1c_1$	1
118.	$(C \perp A) \perp C$	1	$-c_1$	$b_1 - c_1a_1$
119.	$(A \perp C) \perp A$	$b_1 - a_1c_1$	$-a_1$	1
120.	$(B \perp A) \perp B$	1	$c_1 - b_1a_1$	$-b_1$
121.	$(C \perp B) \perp C$	$-c_1$	1	$a_1 - c_1b_1$
122.	$(A \perp B) \perp B$	a_1^2+1	c_1	$-c_1a_1$
123.	$(B \perp C) \perp C$	$-a_1b_1$	b_1^2+1	a_1
124.	$(C \perp A) \perp A$	b_1	$-b_1c_1$	c_1^2+1
125.	$(A \perp C) \perp C$	a_1^2+1	$-b_1a_1$	b_1
126.	$(B \perp A) \perp A$	c_1	b_1^2+1	$-c_1b_1$
127.	$(C \perp B) \perp B$	$-a_1c_1$	a_1	c_1^2+1

128. $(A \perp B) \perp C$	$2a_1$	$-b_1$	a_1b_1
129. $(B \perp C) \perp A$	b_1c_1	$2b_1$	$-c_1$
130. $(C \perp A) \perp B$	$-a_1$	c_1a_1	$2c_1$
131. $(A \perp C) \perp B$	$2a_1$	c_1a_1	$-c_1$
132. $(B \perp A) \perp C$	$-a_1$	$2b_1$	a_1b_1
133. $(C \perp B) \perp A$	b_1c_1	$-b_1$	$2c_1$

End of S_2.

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