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VOLTERRA INCLUSIONS IN BANACH SPACES

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ABSTRACT. A solution characterization and a uniqueness for the Volterra inclusion (VI) are studied by means of the Laplace transform.

1. Introduction. The aim of this paper is to study the Volterra inclusion (or relation)

$$\text{(VI)} \qquad v(t) \in \mathcal{A} \int_0^t v(t-s)\, d\mu(s) + \mathcal{F}(t), \quad t \geq 0$$

by means of the Laplace transform.

Sums, compositions, or limits of closed operators are not necessarily closed but relatively closed (see Bäumer and Neubrander [**2**]). Since the inverse B^{-1} of an operator B is usually multivalued, the degenerate Volterra equation $Bv(t) = A\int_0^t v(t-s)d\mu(s) + f(t)$ leads to (VI) with a possibly nonclosed multivalued operator $\mathcal{A} := B^{-1}A = \{(x,y) : x \in D(A),\ y \in D(B), \text{ and } Ax = By\}$ and a possibly multivalued function $\mathcal{F} = B^{-1}f$. Another example which leads to the consideration of (VI) for multivalued operators and/or multivalued functions is a control problem of the following type. Let A be a single-valued linear operator on a Banach space X, and let $Z_t \subseteq X$, $t \geq 0$. The problem to be considered is to find all forcing terms $f : [0,\infty) \to X$ with $f(t) \in Z_t$ for which there exists a solution of the Volterra equation

$$u(t) = A \int_0^t u(t-s)\, d\mu(s) + f(t)$$

with a given property (P) (like $u(0) = x_0$ for some $x_0 \in X$). Define a multivalued function $\mathcal{F}$ by $\mathcal{F}(t) := Z_t$ for every $t \geq 0$. Then the control problem is equivalent to finding all solutions of the inclusion

$$u(t) \in A \int_0^t u(t-s)\, d\mu(s) + \mathcal{F}(t)$$

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with the property (P). If $Z_t = Z$ for all $t \geq 0$ for a linear subspace Z of X, then the operator $\mathcal{A}x := Ax + Z$ is multivalued and linear (see Section 2 for the definition of a multivalued linear operator) and the control problem is equivalent to finding all the solutions of the homogeneous Volterra inclusion

$$u(t) \in \mathcal{A} \int_0^t u(t-s)\, d\mu(s)$$

with the given property (P).

Section 1 includes one more example of an equation that leads to (VI) with a relatively closed operator $\mathcal{A}$.

The inclusion (VI) is a generalization of the multivalued Cauchy problem $(du/dt) \in \mathcal{A}u(t) + f(t)$, $0 < t < \infty$; $u(0) = x$ and the Volterra equation $v(t) = A\int_0^t v(t-s)\, d\mu(s) + f(t)$, $t \geq 0$ (see Knuckles and Neubrander [**7**] and Kim [**5**], [**6**]).

For (VI) we assume that $\mathcal{F}$ is a single- or multivalued function from $[0,\infty)$ to a Banach space X, and $\mu : [0,\infty) \to \mathbf{C}$ a normalized function of local bounded variation. $\mathcal{A} \subseteq X \times X$ is assumed to be a single- or multivalued, linear operator with domain $D(\mathcal{A})$. Further we assume that there exists an auxiliary Banach space $X_{\mathcal{A}}$ which is continuously embedded in X (i.e., $X_{\mathcal{A}} \hookrightarrow X$), for which $D(\mathcal{A})$ is a subspace of $X_{\mathcal{A}}$ and the graph of $\mathcal{A}$ is closed in $X_{\mathcal{A}} \times X$ (see Section 2). A function $v \in C([0,\infty); X)$ is said to be a solution of (VI) if

(i) the convolution $v * d\mu(t) := \int_0^t v(t-s)d\mu(s)$ of v and $d\mu$, $t \geq 0$, is locally Bochner integrable in X_A, and

(ii) $v * d\mu(t) \in D(\mathcal{A})$ for all $t \geq 0$ and (VI) holds.

If $\mathcal{F}$ is single-valued, if $\mathcal{A}$ is a single-valued closed linear operator with sufficiently nice spectrum and resolvent, and if (VI) models a well-posed evolutionary problem where t denotes the time variable, then properties of the equation (VI) are studied extensively (see Prüss [**9**]).

We characterize the solutions of (VI) in terms of the resolvent inclusion

$$0 \in (I - \widehat{d\mu}(\lambda)\mathcal{A})y(\lambda) - \widehat{\mathcal{F}}(\lambda), \quad \lambda > \omega,$$

which is sometimes easier to be solved than (VI) and obtain a uniqueness for (VI) in Section 3. The results generalize those results in [**6**].

The methods of integrated and convoluted solution operator families have been well applied to the study of the well-posed Volterra equations (VI) with both $\mathcal{A}$ and $\mathcal{F}$ single-valued (see [**5**], [**6**]). Those concepts of integrated and convoluted solution operator families for Volterra equations which assume the existence of the resolvent $(I-\widehat{d\mu}(\lambda)\mathcal{A})^{-1}$ of the operator $\mathcal{A}$ on an interval (ω,∞) extend automatically to multivalued closed linear operators $\mathcal{A}$. However, the assumption of the resolvent $(I-\widehat{d\mu}(\lambda)\mathcal{A})^{-1}$ of a multivalued operator $\mathcal{A}$, as a single-valued bounded linear operator in the definition of a convoluted (or integrated) solution operator family for the Volterra inclusion (VI), implies immediately that $\mathcal{A}$ is single-valued. Hence the methods of integrated and convoluted solution operator families don't seem to extend properly to multivalued closed linear operators.

2. Preliminaries. Let X and Y be Banach spaces throughout.

Let $\mathcal{A}\subseteq X\times Y$ be a multivalued operator with domain $D(\mathcal{A}):=\{x\in X:(x,y)\in\mathcal{A}$ for some $y\in X\}$ and range $\mathrm{Ran}\,(\mathcal{A}):=\{y\in X: y\in\mathcal{A}x,\ x\in D(\mathcal{A})\}$. $\mathcal{A}$ is called linear if $D(\mathcal{A})$ is a linear subspace of X and

$$\mu\mathcal{A}x\subseteq\mathcal{A}(\mu x)\quad\text{and}\quad\mathcal{A}x+\mathcal{A}y\subseteq\mathcal{A}(x+y)\tag{2.1}$$

for all x, $y\in D(\mathcal{A})$ and $\mu\in\mathbf{C}-\{0\}$. The relations in (2.1) are equivalent to

$$\mu\mathcal{A}x=\mathcal{A}(\mu x)\quad\text{and}\quad\mathcal{A}x+\mathcal{A}y=\mathcal{A}(x+y)\tag{2.2}$$

for all x, $y\in D(\mathcal{A})$ and $\mu\in\mathbf{C}-\{0\}$, respectively in order (see Cross [**3**], Favini and Yagi [**4**] or [**7**]).

A multi- or single-valued operator $\mathcal{A}\subseteq X\times Y$ is said to be relatively closed if there exist auxiliary Banach spaces $X_{\mathcal{A}}$ and $Y_{\mathcal{A}}$ such that

(i) $D(\mathcal{A})\subseteq X_{\mathcal{A}}\hookrightarrow X$ and $\mathrm{R}(\mathcal{A})\subseteq Y_{\mathcal{A}}\hookrightarrow Y$, and

(ii) $\mathcal{A}$ is closed in $X_{\mathcal{A}}\times Y_{\mathcal{A}}$, i.e., if $D(\mathcal{A})\ni x_n\to x$ in $X_{\mathcal{A}}$ and $\mathcal{A}x_n\ni y_n\to y$ in $Y_{\mathcal{A}}$, then $x\in D(\mathcal{A})$ and $y\in\mathcal{A}x$.

In this case, more specifically $\mathcal{A}$ is said to be $(X_{\mathcal{A}}\times Y_{\mathcal{A}})$-closed, and $(X_{\mathcal{A}}\hookrightarrow X)$-closed if $Y_{\mathcal{A}}=Y=X$.

Examples of relatively closed linear operators include sums, compositions, or limits of relatively closed linear operators. For relatively closed operators and the following, refer to [**2**] or [**7**].

We give an example of an equation that leads to an equation of (VI)-type with a relatively closed operator $\mathcal{A}$. Consider the integral equation

$$(2.3)\quad v(t) = A\int_0^t v(t-s)\,d\mu(s) + B\int_0^t v(t-s)\,d\mu(s) + f(t), \quad t \geq 0$$

where μ is a scalar valued function of bounded variation and f a Banach space $(X, \|\cdot\|)$-valued function on $[0,\infty)$. Suppose that A and B are single-valued closed linear operators on X with domain D. The operator $\mathcal{A} := A + B : D \to X$ is not necessarily closed but $([D] \hookrightarrow X)$-closed where $[D]$ is the space D equipped with the norm $\|x\|_{[D]} = \|x\| + \|Ax\|$ for every $x \in D$ as usual (see [**7**]). Thus, (2.3) can be handled as a (VI)-type: $v(t) = \mathcal{A}\int_0^t v(t-s)\,d\mu(s) + f(t)$, $t \geq 0$, where $\mathcal{A}$ is a single valued $([D] \hookrightarrow X)$-closed linear operator.

Proposition 2.1. *Suppose that $\mathcal{A} \subseteq X \times X$ is linear and $(X_{\mathcal{A}} \hookrightarrow X)$-closed. If $u : [0,\infty) \to X_{\mathcal{A}}$ is Bochner integrable, $u(t) \in D(\mathcal{A})$ for all $t \in [0,\infty)$, and $\mathcal{A}u(\cdot) \ni y(\cdot) : [0,\infty) \to X$ is Bochner integrable, then $\int_0^\infty u(s)\,ds \in D(\mathcal{A})$ and $\int_0^\infty y(s)\,ds \in \mathcal{A}\int_0^\infty u(s)\,ds$.*

In the following we list some basic notation and Laplace transform results for Section 3 that could be skipped. One can refer to [**1**], [**5**], or [**8**] for details. By $BV_\varepsilon([0,\infty);\mathbf{C})$ for $\varepsilon \geq 0$ we denote the space consisting of $\mathbf{C}$-valued functions f of local bounded variation on $[0,\infty)$ satisfying that $f(0) = 0$ and that for some constant $M \geq 0$, $\mathrm{var}_{[0,t]}(f) \leq Me^{\varepsilon t}$ for all $t \geq 0$. By $\mathbf{C}_\omega$ we denote the set $\{\lambda \in \mathbf{C} : \mathrm{Re}\,\lambda > \omega\}$ and by $\mathbf{N}_0$ the set $\{0\} \cup \mathbf{N}$. The exponential growth bound of a function $f \in L^1_{loc}([0,\infty);X)$ is defined as

$$\omega(f) := \inf\{\omega \in \mathbf{R} : \sup_{t \geq \tau} \|e^{-\omega t} f(t)\| < \infty \text{ for some } \tau \geq 0\}.$$

A function f is said to be exponentially bounded if $\omega(f) < \infty$. The n-th normalized antiderivative of $f \in L^1_{loc}([0,\infty);X)$ is the function defined by $f^{[n]}(t) := \int_0^t ((t-s)^{n-1}/(n-1)!) f(s)\,ds$ for $t \geq 0$. For $f \in L^1_{\mathrm{loc}}([0,\infty);X)$, we define

$$(2.4)\qquad \mathrm{abs}\,(f) := \inf\{\mathrm{Re}\,\lambda : \widehat{f}(\lambda) := \lim_{T\to\infty}\int_0^T e^{-\lambda t} f(t)\,dt \text{ exists}\}$$

and f is said to be Laplace transformable if $\mathrm{abs}(f) < \infty$. Clearly, $\mathrm{abs}(f) \le \omega(f)$. If $\mathrm{abs}(f) < \infty$ and if ω is a constant such that $\omega \ge \{\mathrm{abs}(f),\ 0\}$, then $\omega(f^{[1]}) \le \omega$ and

$$\lambda\widehat{f^{[1]}}(\lambda) = \widehat{f}(\lambda) \tag{2.5}$$

for all $\lambda \in \mathbf{C}_\omega$. The notation $\mathrm{abs}_X(f)$ sometimes replaces $\mathrm{abs}(f)$ when it is necessary to specify the space X in which the integral in (2.4) converges. The elementary fact will be used implicitly in Theorem 3.1 that if X_1 is continuously embedded in X and $\mathrm{abs}_{X_1}(f) < \infty$, then $\mathrm{abs}_{X_1}(f) = \mathrm{abs}_X(f) < \infty$ and $\int_0^\infty e^{-\lambda t} f(t)\,dt$ is a limit in both X and X_1 for $\lambda \in \mathbf{C}_\omega$ with $\omega \ge \mathrm{abs}_{X_1}(f)$. For $f : [0,\infty) \to X$ which is of local bounded variation on or continuous in $[0,\infty)$, we define $\mathrm{abs}(df)$ as the extended real number $\inf\{\mathrm{Re}\,\lambda : \widehat{df}(\lambda) := \lim_{T\to\infty} \int_0^T e^{-\lambda t}\,df(t) \text{ exists}\}$ and f is said to be Laplace-Stieltjes transformable if $\mathrm{abs}(df) < \infty$. If $f : [0,\infty) \to X$ is of local bounded variation or continuous with $f(0) = 0$ and $\omega(f) < \infty$ and if $\omega(f) \le \omega \in \mathbf{R}$, then

$$\widehat{df}(\lambda) = \lambda\widehat{f}(\lambda) \tag{2.6}$$

for all $\lambda \in \mathbf{C}_\omega$.

Proposition 2.2. *Suppose that $f \in C([0,\infty); X)$ with $\omega(f) < \infty$ and that $g \in BV_\varepsilon([0,\infty); \mathbf{C})$ for some $\varepsilon \ge 0$. Let ω be a number such that $\omega \ge \max\{\omega(f),\ \varepsilon\}$. Then $\mathrm{abs}(f * dg) \le \omega$ and for $\lambda \in \mathbf{C}_\omega$,*

$$\widehat{f * dg}(\lambda) = \widehat{f}(\lambda)\widehat{dg}(\lambda).$$

Theorem 2.3 (Uniqueness Theorem). *Let $f \in L^1_{\mathrm{loc}}([0,\infty); X)$ with $\mathrm{abs}(f) < \infty$. If there exists an $\omega \ge \mathrm{abs}(f)$ such that $\widehat{f} \equiv 0$ on (ω,∞), then $f(t) = 0$ for almost all $t \ge 0$.*

By $\mathrm{Lip}_\omega([0,\infty); X)$ for $\omega \in \mathbf{R}$ we denote the space consisting of those functions $F : [0,\infty) \to X$ such that $F(0) = 0$ and $\|F\|_{\mathrm{Lip}_\omega}$ defined as $\inf\{M : \|F(t+h) - F(t)\| \le M \int_t^{t+h} e^{\omega r}\,dr \text{ for } t,\ h \ge 0\}$ is finite. If $\omega \ge 0$ and $F \in \mathrm{Lip}_\omega([0,\infty); X)$, then clearly $\omega(F) \le \omega$. If

$f \in L^1_{\text{loc}}([0,\infty);X)$ with $\omega(f) < \infty$, then for any number $\omega > \omega(f)$, $f^{[1]} \in \text{Lip}_\omega([0,\infty);X)$.

Theorem 2.4 (Phragmén-Doetsch Inversion Theorem). *Let $F \in \text{Lip}_\omega([0,\infty);X)$ and let $r := \widehat{dF}$ on (ω,∞). Then*

$$\Big\| F(t) - \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!}\, e^{tnj} r(nj) \Big\| \le \frac{2}{n} \|F\|_{\text{Lip}_\omega}$$

for all $n \in \mathbf{N}$ with $n > \omega$ and all $t \ge 0$.

Let $\mathcal{F} : [0,\infty) \to X$ be multivalued. A single-valued function $f : [0,\infty) \to X$ is called a section of $\mathcal{F}$ if $f(t) \in \mathcal{F}(t)$ for all $t \ge 0$ (see [**3**] or [**4**]). We denote the set of all sections of $\mathcal{F}$ by $\sec(\mathcal{F})$. We extend the definition of Laplace transformable functions to multivalued ones as follows. Let $\mathcal{F} : [0,\infty) \to X$ be a multivalued function for which $f \in L^1_{\text{loc}}([0,\infty);X)$ with $\text{abs}(f) < \infty$ for all $f \in \sec(\mathcal{F})$. We define $\text{abs}(\mathcal{F})$ as the extended real number $\sup_{f\in\sec(\mathcal{F})} \text{abs}(f)$. $\mathcal{F}$ is said to be Laplace transformable if $\text{abs}(\mathcal{F}) < \infty$. By $\widehat{\mathcal{F}}(\lambda)$ we denote the set $\{\widehat{f}(\lambda) : f \in \sec(\mathcal{F})\}$ for $\lambda \in \mathbf{C}_{\text{abs}(\mathcal{F})}$. If a multivalued function $\mathcal{F}$ is exponentially bounded on $[0,\infty)$, i.e., there exist constants M and ω for which $\|f(t)\| \le Me^{\omega t}$ for every $t \ge 0$ and $f \in \sec(\mathcal{F})$, then $\text{abs}(\mathcal{F}) \le \omega$.

3. Existence and uniqueness. Let X be a Banach space throughout this section. The following characterization of exponentially bounded solutions to (VI) extends Theorem 2.1 in [**6**].

Theorem 3.1. *Let $\mathcal{A} \subseteq X \times X$ be a multi- or single-valued $(X_{\mathcal{A}} \hookrightarrow X)$-closed linear operator and $\mu \in BV_\varepsilon([0,\infty);\mathbf{C})$ for some $\varepsilon \ge 0$. Let $\mathcal{F} : [0,\infty) \to X$ be a multi- or single-valued, Laplace transformable function. Let $v \in C([0,\infty);X)$ be an exponentially bounded function such that $v * d\mu \in L^1_{\text{loc}}([0,\infty) : X_{\mathcal{A}})$ with $\text{abs}_{X_{\mathcal{A}}}(v * d\mu) < \infty$. Let $\omega \ge \max\{\varepsilon,\ \text{abs}(\mathcal{F}),\ \omega(v),\ \text{abs}_{X_{\mathcal{A}}}(v * d\mu)\}$. Then the following are equivalent.*

(i) *v is a solution to* (VI).

(ii) $\widehat{d\mu}(\lambda)\widehat{v}(\lambda) \in D(\mathcal{A})$ *and* $0 \in (I - \widehat{d\mu}(\lambda)\mathcal{A})\widehat{v}(\lambda) - \widehat{\mathcal{F}}(\lambda)$ *for all* $\lambda \in \mathbf{C}_\omega$.

(iii) $\widehat{d\mu}(l)\widehat{v}(l) \in D(\mathcal{A})$ *and* $0 \in (I - \widehat{d\mu}(l)\mathcal{A})\widehat{v}(l) - \widehat{\mathcal{F}}(l)$ *for all* $l \in \mathbf{N}$ *with* $l > \omega$.

Proof. Suppose that (i) holds. Then $v(t) - f(t) \in \mathcal{A}\int_0^t v(t-s)\,d\mu(s)$, $t \geq 0$, for some $f \in \sec(\mathcal{F})$. Let $\lambda \in \mathbf{C}_\omega$. It follows from Propositions 2.1 and 2.2 that

$$\begin{aligned}\widehat{v}(\lambda) - \widehat{f}(\lambda) &= \int_0^\infty e^{-\lambda t} v(t)\,dt - \int_0^\infty e^{-\lambda t} f(t)\,dt \\ &= \int_0^\infty e^{-\lambda t}(v(t) - f(t))\,dt \\ &\in \mathcal{A}\int_0^\infty e^{-\lambda t} v * d\mu(t)\,dt \\ &= \mathcal{A}\widehat{v}(\lambda)\widehat{d\mu}(\lambda).\end{aligned}$$

Hence, $\widehat{v}(\lambda) \in \widehat{d\mu}(\lambda)\mathcal{A}\widehat{v}(\lambda) + \widehat{\mathcal{F}}(\lambda)$. Thus, (i) implies (ii). Clearly, (ii) implies (iii). We show that (iii) $\Longrightarrow$ (i) by the Phragmén inversion formula (Theorem 2.4):

$$G(t) = \lim_{n\to\infty}\sum_{j=1}^\infty \frac{(-1)^{j+1}}{j!}\, e^{tnj} r(nj)$$

for $t \geq 0$ and $n > \omega$ where $G \in \operatorname{Lip}\omega([0,\infty); X)$ and $r(\cdot) = \widehat{dG}(\cdot)$ on (ω, ∞). Suppose that (iii) holds. Then

$$\widehat{v}(l) - \widehat{f}(l) \in \mathcal{A}\widehat{v * d\mu}(l) \tag{3.1}$$

for some $f \in \sec(\mathcal{F})$ and for all $l \in \mathbf{N}$ with $l > \omega$. Let $\omega' > \omega$. It follows from $\omega(v^{[1]})$, $\omega(f^{[1]}) < \omega'$ that $v^{[2]}$, $f^{[2]} \in \operatorname{Lip}_{\omega'}([0,\infty); X)$ and from $\omega_{X_\mathcal{A}}((v*d\mu)^{[1]}) < \omega'$ that $(v*d\mu)^{[2]} \in \operatorname{Lip}_{\omega'}([0,\infty) : X_\mathcal{A})$. Hence it follows from (3.1) and the relations (2.5) and (2.6) that

$$\widehat{dv^{[2]}}(l) - \widehat{df^{[2]}}(l) \;\in\; \mathcal{A}d\widehat{(v*d\mu)}^{[2]}(l)$$

for every $l \in \mathbf{N}$ with $l > \omega'$. It follows from the Phragmén inversion formula and the $(X_\mathcal{A} \hookrightarrow X)$-closedness of $\mathcal{A}$ that

$$v^{[2]}(t) - f^{[2]}(t) \;\in\; \mathcal{A}(v * d\mu)^{[2]}(t), \quad t \geq 0.$$

It follows from the twice differentiability of $(v * d\mu)^{[2]}(t)$ in $X_{\mathcal{A}}$ and the $(X_{\mathcal{A}} \hookrightarrow X)$-closedness of $\mathcal{A}$ that

$$v(t) - f(t) \in \mathcal{A}(v * d\mu)(t), \quad t \geq 0.$$

Thus, $v(t) \in \mathcal{A}(v * d\mu)(t) - \mathcal{F}(t)$, $t \geq 0$. □

Remark. (i) One can see that Theorem 3.1 does not depend on the choice of the auxiliary Banach space $X_{\mathcal{A}}$. The existence of the space $X_{\mathcal{A}}$ satisfying the hypothesis in Theorem 3.1, especially when $\mathcal{A}$ is not closed in X, will be enough for the theorem to hold.

(ii) Theorem 3.1 can be applied to finding the solutions of (VI). Let y be a solution of the characteristic inclusion

$$0 \in (I - \widehat{d\mu}(\lambda)\mathcal{A})y(\lambda) - \widehat{\mathcal{F}}(\lambda), \quad \lambda > \omega$$

for some $\omega \geq \max\{\varepsilon, \operatorname{abs}(\mathcal{F})\}$. Suppose that y has a Laplace transform representation $y = \widehat{v}$ on (ω, ∞) for some $v \in C([0, \infty); X)$ such that $\omega(v) \leq \omega$ and $\operatorname{abs}_{X_{\mathcal{A}}}(v * d\mu) \leq \omega$. Then by Theorem 3.1, v is a solution to (VI).

We take simple problems and show how Theorem 3.1 is applied to finding their solutions in the following.

Example 1. Let $X = C[0, 1]$. We consider the problem

$$\frac{\partial}{\partial r} v(t, r) = a(r) \int_0^t v(t - s, r)\, ds + f(t), \quad t \geq 0, \tag{3.2}$$

where f is a continuous function from $[0, \infty)$ to X with $\operatorname{abs}(f) < \infty$ and $a(r)$ is a function in $C^1[0, 1]$. Let $Bu(r) = (d/dr)u(r)$ for every $u \in C^1[0, 1]$ and define $Au(r) = a(r)u(r)$ for every $u \in C[0, 1]$. Let $\mu(t) = t$. Then the equation (3.2) becomes

$$Bv(t) = A \int_0^t v(t - s)\, d\mu(s) + f(t), \quad t \geq 0. \tag{3.3}$$

B is a closed linear operator with domain $C^1[0, 1]$ and range $C[0, 1]$, and $B^{-1}u(r) = \int u(r)\, dr = \{\int_0^r u(s)\, ds + C : C \in \mathbf{R}\}$ which is a multivalued closed linear operator from $C[0,1]$ to $C^1[0,1]$. Obviously, A is a continuous linear operator on $C[0,1]$. $\mu \in BV_\varepsilon([0,\infty);\mathbf{C})$ for all $\varepsilon > 0$ and $\widehat{d\mu}(\lambda) = 1/\lambda$ for every $\lambda > 0$. A function $v \in C([0,\infty);X)$ is a solution to the equation (3.3) if and only if it satisfies the inclusion

$$v(t) \in \mathcal{A}\int_0^t v(t-s)\,d\mu(s) + \mathcal{F}(t), \quad t \geq 0, \tag{3.4}$$

where $\mathcal{A}u(r) := B^{-1}Au(r) = \int a(r)u(r)\,dr$ is a multivalued closed linear operator from $C[0,1]$ to $C^1[0,1]$ (see [**7**]), and $\mathcal{F}(t)(r) = B^{-1}f(t)(r) = \int f(t)(r)\,dr = \{f(t)^{[1]}(r) + C : C \in \mathbf{R}\}$ is a multivalued function from $[0,\infty)$ to $C[0,1]$. If $\omega(v) < \infty$, then $\omega(v * dt) \leq \max\{0,\ \omega(v)\}$. Let ω be any number such that $\omega \geq \max\{0,\ \mathrm{abs}(f)\}$. In order to find a solution of (3.4), we consider the characteristic inclusion

$$0 \in (I - \widehat{d\mu}(\lambda)\mathcal{A})y(\lambda) - \widehat{\mathcal{F}}(\lambda), \quad \lambda > \omega, \tag{3.5}$$

where $y(\lambda)$ is an unknown function.

$$\begin{aligned}(I - \widehat{d\mu}(\lambda)\mathcal{A})y(\lambda) &= y(\lambda) - \frac{1}{\lambda}\int a(r)y(\lambda)(r)\,dr \\ &= \Big\{y(\lambda)(r) - \frac{1}{\lambda}((a\cdot y(\lambda))^{[1]}(r) + K) : K \in \mathbf{R}\Big\} \\ &= \Big\{y(\lambda)(r) - \frac{1}{\lambda}(a\cdot y(\lambda))^{[1]}(r) - \frac{K}{\lambda} : K \in \mathbf{R}\Big\}.\end{aligned}$$

$\widehat{\mathcal{F}}(\lambda)(r) = B^{-1}\widehat{f}(\lambda)(r) = \int \widehat{f}(\lambda)(r)\,dr = \{\widehat{f}(\lambda)^{[1]}(r) + C : C \in \mathbf{R}\}$ since B^{-1} is closed. Hence y is an analytic solution of (3.5) if and only if $(I - \widehat{d\mu}(\lambda)\mathcal{A})y(\lambda) \cap \widehat{\mathcal{F}}(\lambda) \neq \varnothing$, $\lambda > \omega$, which is equivalent to

$$y(\lambda)(r) - \frac{1}{\lambda}(a\cdot y(\lambda))^{[1]}(r) - \frac{K}{\lambda} = \widehat{f}(\lambda)^{[1]}(r) + C, \quad \lambda > \omega. \tag{3.6}$$

for some $K, C \in \mathbf{R}$. Differentiating (3.6) with respect to r we get

$$\begin{gathered}\frac{d}{dr}y(\lambda)(r) - \frac{1}{\lambda}a(r)y(\lambda)(r) = \widehat{f}(\lambda)(r), \\ 0 \leq r \leq 1, \quad \lambda > \omega,\end{gathered} \tag{3.7}$$

which is a linear first order differential equation of $y(\lambda)$. If $y(\lambda)$ is a solution of (3.7), then it is a solution of the characteristic inclusion (3.5). If v is a function with $\omega(v) < \infty$, then $\omega(v * dt) \leq \max\{0, \omega(v)\}$. Hence if there exists a function v in $C([0,\infty); X)$ with $\omega(v) < \infty$ such that $\widehat{v}(\lambda) = y(\lambda)$, $\lambda > \omega$, then $\omega(v * dt) \leq \max\{0, \omega(v)\} \leq \omega$, and so from the remark following Theorem 3.1, v is a solution of the inclusion (3.4).

The following trivial inclusion is solved directly.

Example 2. Let X be a Banach space, and let $\mathcal{A} := 0_\infty : \{0\} \to X$ be an operator defined by $\mathcal{A}0 = X$ which is the inverse of the zero operator in X (see [**4**] for example). Then $\mathcal{A}$ is non-single valued, closed and linear. Let μ be a unit step function

$$\mu(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0. \end{cases}$$

For a point x in X, define $\mathcal{F}(t) := x$ for all $t \geq 0$. Then $\mu \in BV_0([0,\infty); \mathbf{C})$ and $\omega(\mathcal{F}) = 0$. For $v \in C([0,\infty); X)$, $v * d\mu(t) = v(t)$. Hence $v * d\mu(t) \in D(\mathcal{A})$ implies that $v \equiv 0$ on $[0,\infty)$. Under the above assumptions, we have the inclusion

$$v(t) \in \mathcal{A}v * d\mu(t) + x = X + x = X. \tag{3.8}$$

Clearly, $v \equiv 0$ satisfies (3.7). Hence $v \equiv 0$ is a unique solution to (3.8).

If the function $\mathcal{F}$ in (VI) is single-valued, Theorem 3.1 and the linearity of $\mathcal{A}$ give a uniqueness of the solutions of (VI): the inclusion (VI) has at most one exponentially bounded solution if and only if for any $\omega \geq \varepsilon$, the characteristic inclusion

$$0 \in (I - \widehat{d\mu}(\lambda)\mathcal{A})y(\lambda), \quad \lambda > \omega$$

has no nonzero solution y which has a Laplace transform representation $y(\lambda) = \widehat{v}(\lambda)$ of an exponentially bounded function $v \in C([0,\infty); X)$ such that $\mathrm{abs}_{X_\mathcal{A}}(v * d\mu) < \infty$ and $v * d\mu(t) \in D(\mathcal{A})$. If $\mathcal{F}$ is single-valued another uniqueness for (VI) is possible. The following extends Theorem 2.2 in [**6**].

Theorem 3.2. *Let $\mathcal{A} \subseteq X \times X$ be a multi- or single-valued ($X_\mathcal{A} \hookrightarrow X$)-closed linear operator and $\mu \in BV_\varepsilon([0,\infty); \mathbf{C})$ for some $\varepsilon \geq 0$. Let*

$\mathcal{F} \in L^1_{\mathrm{loc}}([0,\infty);X)$ *be a single-valued, Laplace transformable function. Suppose that there exists a sequence* $\{\lambda_k\}_k$ *in* $\mathbf{C}_\varepsilon$ *such that* $\operatorname{Re}\lambda_k \to \infty$ *as* $k \to \infty$*, and for which it holds for all* $k \in \mathbf{N}$ *that* $\widehat{d\mu}(\lambda_k)^{-1}x \notin \mathcal{A}x$ *for any nonzero* x *in* $D(\mathcal{A})$*. Then* (VI) *has at most one exponentially bounded solution* v *for which* $\operatorname{abs}_{X_\mathcal{A}}(v * d\mu) < \infty$*.*

Proof. Since $\mathcal{F}$ is single-valued and $\mathcal{A}$ is linear, it suffices to show that $v \equiv 0$ is the only exponentially bounded solution with $\operatorname{abs}_{X_\mathcal{A}}(v * d\mu) < \infty$ to the inclusion

$$v(t) \in \mathcal{A}\int_0^t v(t-s)\,d\mu(s), \quad t \geq 0. \tag{3.9}$$

Suppose that $v \in C([0,\infty);X)$ is a solution of (3.9) such that $\omega(v) < \infty$ and $\operatorname{abs}_{X_\mathcal{A}}(v * d\mu) < \infty$. Let $\omega \geq \max\{\varepsilon,\ \omega(v),\ \operatorname{abs}_{X_\mathcal{A}}(v * d\mu)\}$. Then it follows from Theorem 3.1 that there exists a $K \in \mathbf{N}$ such that

$$\widehat{v}(\lambda_k) \in \mathcal{A}\widehat{d\mu}(\lambda_k)\widehat{v}(\lambda_k)$$

for all $k \geq K$. Hence, $\widehat{v}(\lambda_k) = 0$ for all $k \geq K$. Fix a $k \geq K$. We claim that $\widehat{v} \equiv 0$ on $\mathbf{C}_\omega$. Assume not. Let $m \in \mathbf{N}$ be the order of the zero λ_k of the analytic function $\widehat{v}$ on $\mathbf{C}_\omega$. Since $(\widehat{d\mu}\ \widehat{v})(\lambda_k) = \widehat{v * d\mu}(\lambda_k)$, it follows from the $(X_\mathcal{A} \hookrightarrow X)$-closedness and linearity of $\mathcal{A}$ that $(\widehat{d\mu}\ \widehat{v})^{(m)}(\lambda_k) \in D(\mathcal{A})$ and

$$\begin{aligned}\widehat{v}^{(m)}(\lambda_k) \in \mathcal{A}(\widehat{d\mu}\ \widehat{v})^{(m)}(\lambda_k) &= \mathcal{A}\sum_{j=0}^m \binom{m}{j}\widehat{d\mu}^{(j)}(\lambda_k)\widehat{v}^{(m-j)}(\lambda_k) \\ &= \widehat{d\mu}(\lambda_k)\mathcal{A}\widehat{v}^{(m)}(\lambda_k)\end{aligned}$$

where $\widehat{v}^{(m)}(\lambda_k) \neq 0$. This contradicts the hypothesis. Thus, $\widehat{v} \equiv 0$. It follows from Theorem 2.3 and the continuity of v that $v \equiv 0$ on $[0,\infty)$. □

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