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ON SIZE MAPPINGS

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ABSTRACT. A real-valued mapping r from the hyperspace of all compact subsets of a given metric space X is called a size mapping if $r(\{x\}) = 0$ for $x \in X$ and $r(A) \leq r(B)$ if $A \subset B$. We investigate what continua admit an open or a monotone size mapping. Special attention is paid to the diameter mappings.

1. Introduction. Size is a natural notion: it is a nonnegative real number informing how "large" a given object is. The notion of category allows us to define small and large sets (in topological terms)—sets of the first category (size zero) and of the second category (size one). Also volume, area and their generalization—Lebesgue measure in $\mathbf{R}^n$, are examples of size. Many prominent mathematicians (for instance Henri Lebesgue, Felix Hausdorff) contributed to the development of measure theory, created with connection to the need of defining size of sets. So we can say that a set is large if its measure (or external measure) is a big number. On the other hand, we can say that a set is large if it contains two points which are far from each other. Therefore we have a few examples of size: category, Lebesgue measure (and in general nonatomic measures) and diameter. All of them have two properties which agree with our intuition of size: any single point is of size zero and the size of a subset is not bigger than the size of a set containing it. Those properties can be accepted as the definition of size. In the following we will consider size mappings (compare [**16**]) defined on the hyperspace of all compact nonempty subsets of a given metric space equipped with the Hausdorff metric (we will denote it by $(2^X, d_H)$).

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Definition 1.1. A mapping $r : (2^X, d_H) \to \mathbf{R}$ is said to be *size* provided that

1) $r(\{x\}) = 0$ for each point $x \in X$;

2) if $A \subset B$ then $r(A) \leq r(B)$ for every set $A, B \in 2^X$.

Let us notice that diameter and nonatomic measures do not "distinguish" some sets and their proper subsets. For instance, the diameters of the interval $[0,1]$ and the set $\{0,1\}$ are equal. It is the same about the Lebesgue measure of the Cantor set and its two-point subset. Moreover, nonatomic measures fail to be continuous.

Definition 1.2. A continuous mapping $\omega : 2^X \to \mathbf{R}$ is said to be a *Whitney map* provided that it satisfies the following conditions:

1) $\omega(\{x\}) = 0$ for each point $x \in X$;

2) if $A \subset B$ and $A \neq B$, then $\omega(A) < \omega(B)$ for every set $A, B \in 2^X$.

For the existence of a Whitney map, see [**15**, pp. 25–27].

Nadler, Jr., asked whether every continuum admits a monotone or open Whitney map [**15**, pp. 468–469]. These questions were answered in the negative by W.J. Charatonik (see [**1**] and [**3**, p. 215]; compare Theorem 4.4 and Example 4.5 below). Further, Nadler asked whether the circle S^1 admits a metric (equivalent to the Euclidean one) such that the diameter mapping is open [**15**, p. 472]. In this paper we give a positive answer to this question (see Example 5.6).

Problem 1.3. For what continua X does there exist, for some equivalent metric on X, an open (monotone, confluent) diameter mapping?

More generally, one can pose the following problem.

Problem 1.4. For what continua does there exist an open (monotone, confluent) nontrivial size mapping?

In this paper we deal with those problems and we present some partial results.

Investigating openness of size mappings, we have seen that there are two kinds of problems concerning openness: whether the size of a set can be decreased or increased. Therefore we introduce the notion of lower and upper semi-openness of a mapping into reals.

Then in Chapter 3 we investigate basic properties and relations between notions of lower (upper) semi-openness, monotoneity and confluence of size mappings. In Chapter 4 we show that for some continua there is no confluent size mapping such that only singletons have size zero. Chapter 5 deals with the diameter mapping. We prove that some special kinds of continua admit an open diameter mapping.

2. Preliminaries. In this paper we consider metric spaces only. By a *continuum* we mean a compact connected space consisting of more than one point, and by a *mapping* we mean a continuous function. A mapping $f : X \to Y$ between spaces X and Y is said to be

i) *open* if images of open subsets of X are open in $f(X)$;

ii) *monotone* if preimages of points are connected;

iii) *confluent* if for every continuum $K \subset f(X)$ and for every component C of $f^{-1}(K)$ we have $f(C) = K$.

It is well known that open mappings of compact spaces and monotone ones are confluent but not conversely.

Note that if Y is a subset of $\mathbf{R}$ equipped with the Euclidean metric, then $f : X \to Y$ is open if and only if for every point $x \in X$ and its neighborhood U there is a number $\varepsilon > 0$ such that $[f(x)-\varepsilon, f(x)+\varepsilon] \cap f(X) \subset f(U)$. Therefore, one can define further classes of mappings having similar properties.

Definition 2.1. A mapping $f : X \to \mathbf{R}$ is *lower* (*upper*) *semi-open at a point* $x \in X$ if for every neighborhood U there is a number $\varepsilon > 0$ such that $[f(x)-\varepsilon, f(x)] \cap f(X) \subset f(U)$ (or $[f(x), f(x)+\varepsilon] \cap f(X) \subset f(U)$, respectively). It is called *lower* (*upper*) *semi-open* if it is lower (upper) semi-open at every one of its points.

Now let us recall the definitions of some classes of spaces. By a *dendrite* we mean a locally connected continuum containing no simple closed curve. A *local dendrite* is defined as a continuum, every point of which has a neighborhood being a dendrite. It is well known that a continuum is a local dendrite if and only if it is locally connected and contains at most a finite number of simple closed curves (see [**14**, p. 304]). A *tree* is a finite dendrite, i.e., a dendrite with finitely many end points. The *cone over a compact space* X is the quotient space $(X \times [0,1])/(X \times \{1\})$ obtained by shrinking $X \times \{1\}$ in $X \times [0,1]$ to a point. The *suspension of a compact space* X is the quotient space $(X \times [-1,1])/(X \times \{1\})/(X \times \{-1\})$ obtained by shrinking each of $X \times \{-1\}$ and $X \times \{1\}$ to different points (vertices of the suspension). Let us notice that cones and suspensions (of compact spaces) are continua. We will use coordinates of points of cones and suspensions as in the appropriate Cartesian products; in particular, $A \times \{1\}$ denotes the one-point set for any $A \subset X$.

Recall that, for a given sequence of sets $\{C_n : n \in \mathbf{N}\}$ in a space X, we denote by $\operatorname{Li} C_n$ its *lower limit*, i.e., the set of points $x \in X$ such that each open neighborhood of x intersects all but finitely many of the sets C_n; by $\operatorname{Ls} C_n$ its *upper limit*, i.e., the set of points $x \in X$ such that each open neighborhood of x intersects infinitely many of the sets C_n. If $\operatorname{Li} C_n = \operatorname{Ls} C_n$, we say that the sequence is *convergent* and denote the common value of $\operatorname{Li} C_n$ and $\operatorname{Ls} C_n$ by $\operatorname{Lim} C_n$. Note that this notion of convergence does agree with the one defined by the Hausdorff metric ([**15**, p. 4]).

3. Some properties of size mappings. It is obvious that every monotone mapping is confluent but, in general, not conversely. If we restrict our considerations to size mappings, the situation will slightly change. Theorems 3.2 and 3.3 describe connections between lower semi-open, monotone and confluent size mappings. To prove them we need the following lemma.

Lemma 3.1. *Let X be a continuum and $r : 2^X \to \mathbf{R}$ be a size mapping. Then the set $r^{-1}([t, r(X)])$ is arcwise connected for every $t \in [0, r(X)]$. Moreover, if for every $t \in [0, r(X)]$ the set $r^{-1}([0,t])$ is connected, then the mapping r is monotone.*

Proof. Let $t \in [0, r(X)]$. The set $r^{-1}([t, r(X)])$ is arcwise connected because X belongs to it and for any element $A \in r^{-1}([t, r(X)])$ there exists an ordered (by inclusion) arc in 2^X joining A to X ([**15**, p. 59]). Now assume that for every $t \in [0, r(X)]$ the set $r^{-1}([0,t])$ is connected. The space 2^X is unicoherent ([**15**, p. 178]), therefore $r^{-1}(t) = r^{-1}([0,t]) \cap r^{-1}([t, r(X)])$ is connected and the mapping r is monotone. $\square$

The following theorem states an equivalence between monotone and confluent size mappings (for a Whitney map, see [**4**, p. 93]).

Theorem 3.2. *Let X be a continuum and let $r : 2^X \to \mathbf{R}$ be a size mapping such that $r^{-1}(0)$ is connected. Then r is monotone if and only if r is confluent.*

Proof. Since every monotone mapping is confluent, it suffices to show the converse implication. Let $t \in [0, r(X)]$ and C be components of $r^{-1}([0,t])$. Since r is confluent, then $r^{-1}(0) \cap C \neq \varnothing$. So the set $r^{-1}([0,t])$ is connected because $r^{-1}(0)$ is connected. Therefore, by Lemma 3.1, the mapping r is monotone. $\square$

Now we show a relationship between lower semi-openness and monotoneity of size mappings.

Theorem 3.3. *Let X be a continuum and $r : 2^X \to \mathbf{R}$ be a size mapping such that $r^{-1}(0)$ is connected. If r is lower semi-open, then r is monotone.*

Proof. Fix $t \in [0, r(X)]$. We prove that $r^{-1}([0,t])$ is connected. Suppose that there exist closed nonempty sets $\mathcal{M}$ and $\mathcal{N}$ in 2^X such that $r^{-1}([0,t]) = \mathcal{M} \cup \mathcal{N}$ and $\mathcal{M} \cap \mathcal{N} = \varnothing$. We consider two cases.

Case 1. $r^{-1}(0) \cap \mathcal{M} \neq \varnothing$ and $r^{-1}(0) \cap \mathcal{N} \neq \varnothing$. Then $r^{-1}(0) = (r^{-1}(0) \cap \mathcal{M}) \cup (r^{-1}(0) \cap \mathcal{N})$ and $(r^{-1}(0) \cap \mathcal{M}) \cap (r^{-1}(0) \cap \mathcal{N}) = \varnothing$, in contradiction to connectedness of $r^{-1}(0)$.

Case 2. $r^{-1}(0) \cap \mathcal{M} = \varnothing$, i.e., $r^{-1}(0) \subset \mathcal{N}$. Let $t_0 = \inf\{r(P) : P \in \mathcal{M}\}$. Since $\mathcal{M}$ is compact, t_0 is a positive number. The mapping

r is lower semi-open, so for any set $A \in \mathcal{M} \cap r^{-1}(t_0)$ and for every neighborhood $\mathcal{U}$ of A we can find $B \in \mathcal{U}$ such that $r(B) < r(A)$. Obviously, such a set B belongs to $\mathcal{N}$, so for every such A we can find a sequence $\{A_n : n \in \mathbf{N}\}$, with $A_n \in \mathcal{N}$ converging to A. Therefore, $A \in \mathcal{M} \cap \mathcal{N}$ is in contradiction to the condition $\mathcal{M} \cap \mathcal{N} = \varnothing$. We have proven that $r^{-1}([0,t])$ is connected for every $t \in [0, r(X)]$. Then, by Lemma 3.1, r is monotone. $\square$

Remark 3.4. The converse implication to that of Theorem 3.3 is not true. Namely, Illanes has shown in [**12**, p. 285] that there is a continuum X having a monotone Whitney map (with the preimage of zero connected) and such that no Whitney map on 2^X is lower semi-open. As another (simpler) example, one can consider any local dendrite containing a simple closed curve equipped with a metric d described in Theorem 5.20 below.

Theorem 3.3 implies that every open diameter or open Whitney map is monotone. The following example shows that the assumption of connectedness of the preimage of zero is essential.

Example 3.5. Let us consider the circle S^1 equipped with such a metric d that the diameter mapping $\mathrm{diam}_d : (2^{S^1}, d_H) \to \mathbf{R}$ is open (see 5.6) and the mappings $g : S^1 \to S^1$ and $r : (2^{S^1}, d_H) \to \mathbf{R}$ given by the formulas $g(z) = z^2$ and $r(A) = \mathrm{diam}_d(g(A))$. Note that r is an open size mapping (see Proposition 3.6 below), but it is not monotone because $r^{-1}(0) = \{\{z\} : z \in S^1\} \cup \{\{z, -z\} : z \in S^1\}$ is not connected.

The following statement was used in Example 3.5.

Proposition 3.6. *Assume that there exist a surjective mapping $f : X \to Y$ and an open (monotone) size mapping $r : 2^Y \to \mathbf{R}$. Then there exists an open (monotone) size mapping $\tilde{r} : 2^X \to \mathbf{R}$ defined by the formula $\tilde{r}(A) = r(f(A))$.*

Proof. The induced mapping $2^f : 2^X \to 2^Y$ between the hypersurfaces of compact spaces X and Y is open (monotone) provided that $f : X \to Y$ is [**9**, Theorems 3.2 and 3.5]. Thus the superposition of r and 2^f is an open (monotone) size mapping. $\square$

Illanes [**10**, p. 517] proved the following theorem.

Theorem 3.7 (Illanes). *For every locally connected continuum X, there exists an open Whitney map for 2^X.*

So for every continuum which can be mapped onto a locally connected continuum by an open (monotone) mapping, there exists an open (monotone) size mapping.

Example 3.8. The Lelek fan ([**5**]) cannot be mapped onto any locally connected continuum by an open or monotone mapping, because confluent images of this continuum are homeomorphic to it (see [**5**]). However, the Lelek fan admits an open Whitney map by [**8**, p. 678].

4. Size mappings for some special continua. Now assume that $r : 2^X \to \mathbf{R}$ is a size mapping such that $r^{-1}(0) = F_1(X) = (\{x\} : x \in X\}$. We answer Problem 1.4 restricted to the class of mappings being a common generalization of diameter and Whitney map, and we describe some continua which have no confluent size mapping. The following concept is a modification of one due to Czuba ([**6**]).

Definition 4.1. A proper nonempty closed subset A of a continuum X is said to be an R^3-set provided that there exists an open set U containing A and a sequence $\{C_n : n \in \mathbf{N}\}$ of components of U such that $\operatorname{Li} C_n = A$.

Example 4.2. Consider a plane continuum X consisting of the segments joining the plane $\langle 0, 2\rangle$ to the points $\langle 1/n, -1\rangle$, the segments joining the point $\langle 0, -2\rangle$ to the points $\langle -1/n, 1\rangle$ and the segment $\{0\} \times [-2, 2]$. The segment $\{0\} \times [-1, 1]$ is an R^3-set (even an R^3-continuum) in X.

Example 4.3. Take the Cantor ternary set $\mathcal{C}$ in the segment $[0, 1] \times \{0\}$. Replace every contiguous segment by a circle of the same diameter described on it. The continuum consists of $\mathcal{C}$, of all those circles (let us denote their union by B) and two rays (i.e., one-to-one continuous images of $[0,\infty))$ approximating the corresponding upper and lower halves of $B \cup \mathcal{C}$ lying in the upper or lower half-planes, respectively. This continuum, constructed by Czuba (unpublished), contains an R^3-set $\mathcal{C} \cap ([0,1/3] \times \{0\})$, but no R^3-continuum.

Let us notice that containing R^3-sets is a topological property, independent on the choice of metric. This property is connected to admitting confluent diameter and Whitney mapping. A link between them is described in Theorem 4.4 below. The theorem is a generalization of the results concerning Whitney maps only (compare [**3**, p. 215], and its proof is similar to the one in [**13**, p. 213]).

Theorem 4.4. *If a continuum X contains an R^3-set and $r : 2^X \to \mathbf{R}$ is a size mapping such that $r^{-1}(0) = F_1(X)$, then r is not confluent.*

Proof. Let X be a continuum, let B be an R^3-set in X, and let U be an open subset of X containing B. Let $\{C_n : n \in \mathbf{N}\}$ b a sequence of components of U such that $\operatorname{Li} C_n = B$. Choose an arbitrary point $p \in B$ and a sequence $\{p_n : n \in \mathbf{N}\}$ such that $p_n \in C_n$, converging to p. Then the set $A_n = \{p, p_n, p_{n+1}, \dots\}$ is a compact subset of X and $\operatorname{Lim} A_n = \{p\}$.

Since the set B is closed, there is a number $\varepsilon > 0$ such that the closure of the ε-ball V about B is contained in U. Let D_n be the component of $\operatorname{cl} V$ containing p_n. For a given point x in $\operatorname{bd} V$, define a subsequence $\{D_n(x) : n \in \mathbf{N}\}$ of the sequence $\{D_n : n \in \mathbf{N}\}$ such that $x \notin \operatorname{Ls} D_n(x)$.

Put $B(x) = \operatorname{Ls} D_n(x)$. Since $B(x)$ is compact and does not contain x, then there is a number $\varepsilon(x) > 0$ such that the $\varepsilon(x)$-ball about x and the set $B(x)$ are disjoint. Define $V(x)$ as the $\varepsilon(x)/2$-ball about x. The family $\{V(x) : x \in \operatorname{bd} V\}$ is an open cover of the compact set $\operatorname{bd} V$, so one can choose a finite subcover $\{V(x_1), V(x_2), \dots, V(x_i)\}$ of $\operatorname{bd} V$. Let

$$\begin{aligned} \varepsilon_0 &= \min\{\varepsilon, \varepsilon(x_1), \dots, \varepsilon(x_i)\}, \\ \mathcal{B} &= \{A \in 2^X : \operatorname{diam}_d(A) \geq \varepsilon_0/2\}, \\ t &= \inf\{r(A) : A \in \mathcal{B}\}. \end{aligned}$$

Note that $t > 0$, since $r^{-1}(0) = F_1(X)$. Consider the interval $[0, t/2]$. Clearly, $r^{-1}([0,t/2])$ and $\mathcal{B}$ are disjoint. Let k be such an index that $r(A_k) \in (0, t/2]$.

Suppose that r is confluent. Then, by Theorem 3.2, the mapping r is monotone. In particular, $r^{-1}([0,t/2])$ is connected.

For each $j \in \{1,\dots,i\}$, choose such an index $n_j \geq k$ that $C_{n_j} \cap \operatorname{cl} B(x_j,\varepsilon_0) = \varnothing$. Then $D_{n_j} \cap \operatorname{cl} B(x_j,\varepsilon_0) = \varnothing$. By [**13**, p. 101], there exist disjoint compact sets H_j and K_j such that $H_j \cup K_j = \operatorname{cl} V$, $D_{n_j} \subset H_j$ and $\operatorname{cl} V \cap \operatorname{cl}(B(x_j,\varepsilon_0)) \subset K_j$. Define $W_j = H_j \setminus \operatorname{bd} V = (X \setminus K_j) \cap V$. Note that W_j is an open set in X.

Let

$$\mathcal{L} = \{A \in r^{-1}([0,t/2]) : A \subset H_1 \cup \dots \cup H_i \text{ and } A \cap H_j \neq \varnothing \text{ for each } j \leq i\}.$$

The set $\mathcal{L}$ is closed in $r^{-1}([0,t/2])$ and nonempty (because $\{p_{n_1},\dots,p_{n_i}\} \in \mathcal{L}$). Define the set

$$\mathcal{M} = \{A \in r^{-1}([0,t/2]) : A \subset W_1 \cup \dots \cup W_i \text{ and } A \cap W_j \neq \varnothing \text{ for each } j \leq i\}.$$

Notice that $\mathcal{M}$ is open in $r^{-1}([0,t/2])$ and $\mathcal{M} \subset \mathcal{L}$. We will show that $\mathcal{M} = \mathcal{L}$. Let $A \in \mathcal{L}$. Suppose that $A \notin \mathcal{M}$, i.e., $A \cap \operatorname{bd} V \neq \varnothing$. Then there exists $j \leq i$ such that $A \cap \operatorname{cl} B(x_j,\varepsilon_0/2) \neq \varnothing$. But $\operatorname{diam}_d(A) < \varepsilon_0/2$, so $A \subset \operatorname{cl} V \cap \operatorname{cl} B(x_j,\varepsilon_0) \subset K_j$, which contradicts the assumption that $A \cap H_j \neq \varnothing$. Thus $\mathcal{L} = \mathcal{M}$. Therefore $\mathcal{L}$ is both open and closed in $r^{-1}([0,t/2])$. On the other hand, $r^{-1}([0,t/2]) \setminus \mathcal{L}$ is nonempty, since it contains the set $\{\{x\} : x \in X \setminus \operatorname{cl} V\}$. Thus, $r^{-1}([0,t/2])$ is not connected, so r is not confluent. $\square$

The condition that the continuum contains no R^3-sets is necessary but not sufficient for the existence of a confluent size mapping with the preimage of zero consisting of the singletons only.

Example 4.5. The double spiral continuum is the union of the sets $S^1 = \{\exp(it) : t \in [0,2\pi]\}$, $M = \{(1+1/t)\exp(-it) : t \in [1,\infty)\}$ and $K = \{(1-1/t)\exp(it) : t \in [1,\infty)\}$. The double spiral continuum X contains no R^3-sets. In spite of this, every size mapping $r : 2^X \to \mathbf{R}$ such that $r^{-1}(0) = F_1(X)$ is not confluent. Indeed, an argument from [**1**] can be applied to any size mapping with this property (not only to a Whitney map).

Containing R^3-sets does not interfere with an existence of open, monotone or confluent size mappings which have preimages of zero different from the set of singletons.

Example 4.6. Consider the continuum X described in Section 4.2. It can be mapped under an open mapping onto an arc, so there exists an open size mapping on 2^X.

5. Some properties of diameter. Diameter plays a special role in various attempts to "measure" sets: this notion is used for instance in defining Lebesgue and Hausdorff measures. From a certain point of view, diameter is more complicated than a Whitney map since its properties (except continuity, of course) closely depend on the choice of the metric. Nadler, Jr., has observed that the diameter mapping of the unit circle equipped with the Euclidean metric is not open (see [**15**, p. 471] and Example 5.1 below). He asked in [**15**, p. 472] if there is an open diameter mapping for a circle. We answer this question in the affirmative in Theorem 5.5.

Example 5.1 [**15**, p. 472]. Consider a circle S^1 equipped with the Euclidean metric d. Let $\{a, b, c\}$ be the set of vertices of an equilateral triangle inscribed in the circle. The set $\mathcal{U} = \{A \in 2^X : d_H(A, \{a, b, c\}) < 1/2\}$ is open in $(2^{S^1}, d_H)$, but $\operatorname{diam}_d(\mathcal{U})$ is not open in $\operatorname{diam}_d(2^{S^1}) = [0, 2]$ because for each $A \in \mathcal{U}$ the diameter of A is not less than $\operatorname{diam}_d(\{a, b, c\}) = \sqrt{3}$.

The question as to what continua admit metrics such that the diameter mapping is open (monotone, confluent) is answered only partially. A class of such continua are cones.

Theorem 5.2. *Let (X, d) be a compact metric space of diameter 1, and let Y be a cone over X equipped with the metric*

$$\rho(\langle x, s\rangle, \langle y, t\rangle) = 2|t - s| + d(x, y) \min\{1 - s, 1 - t\}$$

for $x, y \in X$ and $s, t \in [0, 1]$. Then the diameter mapping $\operatorname{diam}_\rho : (2^Y, \rho_H) \to \mathbf{R}$ is open.

Proof. Let us notice that $\operatorname{diam}_\rho(Y) = 2$, so $\operatorname{diam}_\rho(2^Y) = [0,2]$ by connectedness of 2^Y. We have to show that for every $A \in 2^Y$ and an open neighborhood $\mathcal{U}$ of A in 2^Y, there exist positive numbers a_1 and a_2 such that $(\operatorname{diam}_\rho(A) - a_1, \operatorname{diam}_\rho(A) + a_2) \cap [0,2] \subset \operatorname{diam}_\rho(\mathcal{U})$. Let us fix $A \in 2^Y$, $\varepsilon > 0$ and $\mathcal{U} = \{B \in 2^Y : \rho_H(A,B) < \varepsilon\}$. Define the family of mappings $f_\alpha : Y \to Y$ for each $\alpha \in [0,1]$ by the formula $f_\alpha(\langle x,s\rangle) = \langle x, 1-\alpha(1-s)\rangle$. Take $\alpha \in (1-\varepsilon/2, 1]$ and define $A_\alpha = f_\alpha(A)$. Then $\rho_H(A, A_\alpha) \le 2(1-\alpha) < \varepsilon$ and $\operatorname{diam}_\rho(A_\alpha) = \alpha \operatorname{diam}_\rho(A)$, so

$$\begin{aligned}\operatorname{diam}_\rho(\mathcal{U}) &\supset \{\operatorname{diam}_\rho(A_\alpha) : \alpha \in (1-\varepsilon/2, 1]\} \\ &= (\operatorname{diam}_\rho(A) - (\varepsilon/2)\operatorname{diam}_\rho(A), \operatorname{diam}_\rho(A)].\end{aligned}$$

Put $a_1 = (\varepsilon/2)\operatorname{diam}_\rho(A)$. Now we find a number $a_2 > 0$ such that

$$[\operatorname{diam}_\rho(A), \operatorname{diam}_\rho(A) + a_2) \subset \operatorname{diam}_\rho(\mathcal{U}).$$

Since Y is compact, there are $\langle x,s\rangle$ and $\langle y,t\rangle$ such that $\operatorname{diam}_\rho(A) = \rho(\langle x,s\rangle, \langle y,t\rangle)$. Assume that $t \ge s$. Then

$$\rho(\langle x,s\rangle, \langle y,t\rangle) = d(x,y)(1-t) + 2(t-s).$$

If $t < 1$, then choose a number $\eta > 0$ such that $\eta < \min\{\varepsilon/2, 1-t\}$. For each $\alpha \in [0,\eta]$ define $A_\alpha = A \cup \{\langle y, t+\alpha\rangle\}$. Then $\rho_H(A, A_\alpha) \le 2\alpha < \varepsilon$ and $\operatorname{diam}_\rho(A_\alpha) \ge \rho(\langle x,s\rangle, \langle y, t+\alpha\rangle) = 2(t+\alpha-s) + d(x,y)(1-(t+\alpha)) = d(x,y)(1-t) + 2(t-s) - \alpha d(x,y) + 2\alpha \ge \operatorname{diam}_\rho(A) + \alpha$. If $t = 1$, then we can assume that $s > 0$ (otherwise $\operatorname{diam}_\rho(A) = 2$). In this case, choose a number $\eta > 0$ such $\eta < \min\{\varepsilon/2, s\}$. For each $\alpha \in [0,\eta]$, define $A_\alpha = A \cup \{\langle x, s-\alpha\rangle\}$. Then $\rho_H(A, A_\alpha) \le 2\alpha < \varepsilon$ and $\operatorname{diam}_\rho(A_\alpha) \ge \operatorname{diam}_\rho(A) + \alpha$. The set $\mathcal{A}_\varepsilon = \{A_\alpha : \alpha \in [0,\eta]\}$ is a continuum in 2^Y, so the diameter on A_ε has the Darboux property. Therefore, $\operatorname{diam}_\rho(\mathcal{U}) \supset \operatorname{diam}_\rho(\mathcal{A}_\varepsilon) \supset [\operatorname{diam}_\rho(A), \operatorname{diam}_\rho(A) + \eta]$. Putting $a_2 = \eta$ we are done. □

Recall that, for $n \in \{3,4,5,\dots\}$, the *n-od* is the cone over an n-point set. The cones over $\{0\} \cup \{1/n : n \in \mathbf{N}\}$ and over the Cantor set are called the *harmonic fan* and the *Cantor fan*, respectively.

Corollary 5.3. *An n-od, for $n \in \{3,4,5,\dots\}$, the harmonic fan and the Cantor fan admit open diameter mappings.*

The next examples of continua with open diameters are suspensions of some spaces. To simplify notation, put $T = [-1, 1]$.

Theorem 5.4. *Let (X, d) be a compact space of diameter 1, and let Y be a suspension of X equipped with the metric*

$$(*) \qquad \rho(\langle x, s\rangle, \langle y, t\rangle) = 2|t - s| + d(x, y)\min\{1 - |s|, 1 - |t|\}$$

for $x, y \in X$ and $s, t \in T$. Then the diameter mapping $\mathrm{diam}_\rho : (2^Y, \rho_H) \to \mathbf{R}$ is upper semi-open.

Proof. Let $A \in 2^X$ and a number $\varepsilon > 0$ be given. We prove that if $\mathrm{diam}_\rho(A) < \mathrm{diam}_\rho(Y) = 4$, then the image of the ε-neighborhood of A in 2^X contains an interval of the form $[\mathrm{diam}_\rho(A), \mathrm{diam}_\rho(A)+a]$ for some number $a > 0$. Let $\langle x, s\rangle, \langle y, t\rangle$ be two points of A with the greatest distance, i.e., $\rho(\langle x, s\rangle, \langle y, t\rangle) = \mathrm{diam}_\rho(A)$ and $t \leq s$. Assume $s < 1$, and put $\eta = \min\{\varepsilon, 1-s\}$. For any $\alpha \in [0, \eta/2)$, define $A_\alpha = A \cup \{\langle x, s+\alpha\rangle\}$. (If $s = 1$, then let $\eta = \min\{\varepsilon, t + 1\}$ and $A_\alpha = A \cup \{\langle y, t - \alpha\rangle\}$.) Notice that $\rho_H(A, A_\alpha) \leq 2\alpha < \varepsilon$, so $\mathcal{A} = \{A_\alpha : \alpha \in [0, \eta/2]\}$ is a subcontinuum in 2^X contained in the ε-ball about A. Since $\mathrm{diam}_\rho(A_\alpha) \geq \rho(\langle x, s + \alpha\rangle, \langle y, t\rangle) \geq \rho(\langle x, s\rangle, \langle y, t\rangle) + 2\alpha - \alpha d(x, y) \geq \mathrm{diam}_\rho(A) + \alpha$, then the image of $\mathcal{A}$ under diam_ρ contains the interval $[\mathrm{diam}_\rho(A), \mathrm{diam}_\rho(A) + \eta/2]$. □

Theorem 5.5. *If X is a discrete space, i.e., the distance of any two distinct points is 1, then the diameter mapping for the suspension of X equipped with the metric $(*)$ is open.*

Proof. By Theorem 5.4, the diameter mapping diam_ρ is upper semi-open, so it is enough to prove that it is lower semi-open. Let $A \in 2^Y$ be such that $\mathrm{diam}_\rho(A) > 0$. Let $\mathcal{U}$ be an open neighborhood of A. Consider two cases.

Case 1. $A \subset X \times [0, 1]$ or $A \subset X \times [-1, 0]$. By the symmetry we assume $A \subset X \times [0, 1]$. For $s \in [0, 1]$, let $A_s = \{\langle x, 1 - s(1 - t)\rangle \in Y : \langle x, t\rangle \in A\}$.

Case 2. $A \cap (X \times (0, 1]) \neq \varnothing \neq A \cap (X \times [-1, 0))$. Then, for $s \in [0, 1]$, we put $A_s = \{\langle x, st\rangle \in Y : \langle x, t\rangle \in A\}$.

Thus, in both cases, there exists $\varepsilon > 0$ such that $A_s \in \mathcal{U}$ for $s \in [1-\varepsilon, 1]$. By the definition of ρ, we have $\operatorname{diam}_\rho(A_s) < \operatorname{diam}_\rho(A)$ for $s < 1$, so $\operatorname{diam}_\rho(\{A_s : s \in [1-\varepsilon, 1]\})$ is of the form $[\operatorname{diam}_\rho(A) - a, \operatorname{diam}_\rho(A)]$ for some $a > 0$. So diam_ρ is lower semi-open at A. □

Example 5.6. The circle S^1 is homeomorphic to the suspension of a two-point space. In this way we obtain a construction of an open diameter on the hyperspace of the circle.

Example 5.7. The assumption that the distances between any two points of the space X are equal is essential in Theorem 5.5 for openness of the metric ρ defined by $(*)$.

Indeed, let v_1, v_2, v_3, v_4 be the four vertices of a square of diameter 1 with their usual distances, i.e., $d(v_1, v_2) = d(v_2, v_3) = d(v_3, v_4) = d(v_4, v_1) = \sqrt{2}/2$ and $d(v_1, v_3) = d(v_2, v_4) = 1$. Take the suspension Y of $\{v_1, v_2, v_3, v_4\}$ equipped with the metric ρ defined by $(*)$. Then there is a number $s > 0$ such that the distances between any two distinct points of the set $A = \{\langle v_1, s\rangle, \langle v_2, -s\rangle, \langle v_3, s\rangle, \langle v_4, -s\rangle\}$ are the same. Denote the common distance by δ. Let ε be a small positive number, and let A' be a set with $\rho_H(A, A') < \varepsilon$, such that $\operatorname{diam}_\rho(A') < \operatorname{diam}_\rho(A)$. Then there are two positive numbers t_1, t_3 such that $\langle v_1, t_1\rangle, \langle v_3, t_3\rangle \in A'$ and $\rho(\langle v_1, t_1\rangle, \langle v_3, t_3\rangle) < \delta$. Thus, either $t_1 > s$ or $t_3 > s$. Similarly, there are two positive numbers t_2 and t_4 such that $\langle v_2, -t_2\rangle, \langle v_4, -t_4\rangle \in A'$ and $\rho(\langle v_2, -t_2\rangle, \langle v_4, -t_4\rangle) < \delta$. Thus, either $t_2 > s$ or $t_4 > s$. By the symmetry, assume that $t_1 > s$ and $t_2 > 2$, and observe that

$$\rho(\langle v_1, t_1\rangle, \langle v_2, -t_2\rangle) > \rho(\langle v_1, s\rangle, \langle v_2, -s\rangle) = \delta,$$

contrary to the assumption $\operatorname{diam}_\rho(A') < \delta$. The proof is finished.

Problem 5.10. Does the suspension of any compact metric space (continuum) admit an open diameter mapping? In particular, does the n-dimensional sphere admit such a mapping?

Remark 5.11. If (X, d) is a continuum, then $\operatorname{diam}_d : (2^X, d_H) \to \mathbf{R}$ is upper semi-open at $\{x\}$ for every $x \in X$.

Proof. Let $\varepsilon < \operatorname{diam}_d(X)$ and $\mathcal{U} = \{A \in 2^X : d_H(\{x\}, A) < \varepsilon\}$. Then the set $\mathcal{V} = \{\{x, y\} : d(x, y) < \varepsilon\}$ is contained in $\mathcal{U}$ and $\operatorname{diam}_\delta(\mathcal{V}) = [0, \varepsilon)$. Suppose that there is a number $t \in [0, \varepsilon)$ which does not belong to $\operatorname{diam}_d(\mathcal{V})$. Then the sets $\{y \in X : d(x, y) < t\}$ and $\{y \in X : d(x, y) > t\}$ are open in X, disjoint and their union is X, in contradiction to connectedness of X. □

The next result concerns Cartesian products of continua.

Theorem 5.12. *Let (X, d) and (Y, ρ) be continua, both of diameter 1, and let $d \times \rho$ be the metric on $X \times Y$ defined by the formula*

$$(d \times \rho)(\langle x, y\rangle, \langle u, v\rangle) = \max\{d(x, u), \rho(y, v)\}.$$

Then the diameters $\operatorname{diam}_d : (2^X, d_H) \to \mathbf{R}$ and $\operatorname{diam}_\rho : (2^Y, \rho_H) \to \mathbf{R}$ are both lower (upper) semi-open if and only if the diameter $\operatorname{diam}_{d\times\rho} : (2^{X\times Y}, (d \times \rho)_H) \to \mathbf{R}$ is lower (upper) semi-open, respectively.

Proof. Denote the projections of a set $C \subset X \times Y$ onto X and Y by $P_X(C)$ and $P_Y(C)$. Note that $\operatorname{diam}_{d\times\rho}(C) = \max\{\operatorname{diam}_d(P_X(C)), \operatorname{diam}_\rho(P_Y(C))\}$ for every $C \in 2^{X\times Y}$.

Let $A \in 2^{X\times Y}$, $\varepsilon > 0$ and $\mathcal{U} = \{C \in 2^{X\times Y} : (d \times \rho)_H(A, C) < \varepsilon\}$. Since $X \times Y$ is compact, there is a finite set $B \subset A$ such that $\rho_H(A, B) < \varepsilon/2$ and $\operatorname{diam}_{d\times\rho}(A) = \operatorname{diam}_{d\times\rho}(B)$. We can assume that $\operatorname{diam}_{d\times\rho}(A) = \operatorname{diam}_d(P_X(A))$ and $\operatorname{diam}_{d\times\rho}(B) = \operatorname{diam}_d(P_X(B))$.

Necessity. Assume that diam_d and diam_ρ are upper semi-open. We will prove that $\operatorname{diam}_{d\times\rho}$ is. To this aim it is enough to find a number $\eta_1 > 0$ such that

$$[\operatorname{diam}_{d\times\rho}(A), \operatorname{diam}_{d\times\rho}(A) + \eta_1] \cap [0, 1] \subset \operatorname{diam}_{d\times\rho}(\mathcal{U}).$$

We can assume that $\operatorname{diam}_{d\times\rho}(A) < 1$. Let $\mathcal{V}$ be the $\varepsilon/4$-ball about $P_X(B)$ in X. By upper semi-openness of diam_d, there is a number η_1 such that $[\operatorname{diam}_d(P_X(B)), \operatorname{diam}_\rho(P_X(B)) + \eta_1] \subset \operatorname{diam}_d(\mathcal{V})$. For each $\alpha \in [0, \eta_1]$, choose finite sets $B_{\alpha,X} \in \mathcal{V}$ such that $\operatorname{diam}_d(B_{\alpha,X}) = \operatorname{diam}_d(P_X(B)) + \alpha$. Define the sets

$$A_\alpha = \big\{\langle x, y\rangle \in B_{\alpha,X} \times P_Y(B) : \\ (\exists \langle u, v\rangle \in B)(d(x, u) < \varepsilon/4 \text{ and } \rho(y, v) < \varepsilon/4)\big\}. \qquad \square$$

Note that $(d \times \rho)_H(A, A_\alpha) < \varepsilon$ and $\operatorname{diam}_{d\times\rho}(A_\alpha) = \operatorname{diam}_{d\times\rho}(A) + \alpha$ for each $\alpha \in [0, \eta_1]$. Therefore, $[\operatorname{diam}_{d\times\rho}(A), \operatorname{diam}_{d\times\rho}(A) + \eta_1] \subset \operatorname{diam}_{d\times\rho}(\mathcal{U})$.

Now we assume that diam_d and diam_ρ are both lower semi-open and prove that $\operatorname{diam}_{d\times\rho}$ is lower semi-open, too. To do it we find a number $\eta_2 > 0$ such that $[\operatorname{diam}_{d\times\rho}(A) - \eta_2, \operatorname{diam}_{d\times\rho}(A)] \cap [0, 1] \subset \operatorname{diam}_{d\times\rho}(\mathcal{U})$.

Consider two cases.

Case 1. $\operatorname{diam}_\rho(P_Y(B)) > 0$. Let $\mathcal{V}$ be the $\varepsilon/4$-ball about $P_X(B)$ in X, and let $\mathcal{W}$ be the $\varepsilon/4$-ball about $P_Y(B)$ in Y. By lower semi-openness of diam_d and diam_ρ, there is a number η_2 such that $[\operatorname{diam}_d(P_X(B)) - \eta_2, \operatorname{diam}_\rho(P_X(B))] \subset \operatorname{diam}_d(\mathcal{V})$ and $[\operatorname{diam}_\rho(P_Y(B)) - \eta_2, \operatorname{diam}_\rho(P_Y(B))] \subset \operatorname{diam}_\rho(\mathcal{W})$. For each $\alpha \in [0, \eta_2]$, choose finite sets $B_{\alpha,X} \in \mathcal{V}$ and $B_{\alpha,Y} \in \mathcal{W}$ such that $\operatorname{diam}_d(B_{\alpha,X}) = \operatorname{diam}_d(P_X(B)) - \alpha$ and $\operatorname{diam}_\rho(B_{\alpha,Y}) = \operatorname{diam}_\rho(P_Y(B)) - \alpha$. Define the sets

$$A_\alpha = \big\{\langle x, y\rangle \in B_{\alpha,X} \times B_{\alpha,Y} : \\ (\exists \langle u, v\rangle \in B)(d(x, u) < \varepsilon/4 \text{ and } \rho(y, v) < \varepsilon/4)\big\}.$$

Case 2. $\operatorname{diam}_\rho(P_Y(B)) = 0$, i.e., $P_Y(B) = \{y\}$. By lower semi-openness of diam_d there is a number $\eta_2 > 0$ such that, for each $\alpha \in [0, \eta_2]$, we can find a set $B_{\alpha,X}$ in X such that $d_H(B_{\alpha,X}, P_X(B)) < \varepsilon$ and $\operatorname{diam}_d(B_{\alpha,X}) = \operatorname{diam}_d(B) - \alpha$. Put $A_\alpha = B_{\alpha,X} \times \{y\}$.

Note that in both cases $(d \times \rho)_H(A, A_\alpha) < \varepsilon$ and $\operatorname{diam}_{d\times\rho}(A_\alpha) = \operatorname{diam}_{d\times\rho}(A) - \alpha$ for each $\alpha \in [0, \eta_2]$. Therefore, $[\operatorname{diam}_{d\times\rho}(A) - \eta_2, \operatorname{diam}_{d\times\rho}(A)] \subset \operatorname{diam}_{d\times\rho}(\mathcal{U})$.

Sufficiency. Suppose that $\operatorname{diam}_d : (2^X, d_H) \to \mathbf{R}$ is not lower (upper) semi-open at $C \in 2^X$, i.e., for every $\varepsilon > 0$ there is a sequence of negative (positive) numbers $\{t_n : n \in \mathbf{N}\}$ converging to 0 and such that $d_H(C, D) < \varepsilon$ follows $\operatorname{diam}_d(D) \notin \{\operatorname{diam}_d(C) + t_n : n \in \mathbf{N}\}$. Put $\varepsilon = \operatorname{diam}_d(C)/4$ (C is not a singleton by Remark 5.11). Let y be an arbitrary point of Y and $\mathcal{U}$ be the ε-ball about $C \times \{y\}$ in $2^{X \times Y}$. Observe that $\operatorname{diam}_{d\times\rho}(C \times \{y\}) = \operatorname{diam}_d(C)$ and $\operatorname{diam}_{d\times\rho}(B) = \operatorname{diam}_d P_X(B) \notin \{\operatorname{diam}_d(C) + t_n : n \in \mathbf{N}\} = \{\operatorname{diam}_{d\times\rho}(C \times \{y\}) + t_n : n \in \mathbf{N}\}$ for each B belonging to the ε-ball about $C \times \{y\}$ in $X \times Y$, so $\operatorname{diam}_{d\times\rho}$ is not lower (upper) semi-open at $C \times \{y\}$. □

Example 5.13. Tori $T^n = (S^1)^n$ and spaces of the form $T^m \times I^k$, where $k, m, n \in \mathbf{N}$, admit open diameter mappings.

Now we consider some special kinds of continua that admit open diameter mappings. To this aim recall a notation and a definition. For a given continuum X, the symbol $C(X)$ denotes the subspace of 2^X consisting of all nonempty subcontinua of X.

Definition 5.14 [**7**, p. 556]. A continuum X is called *arc-smooth at* q provided that there is a mapping $\alpha : X \to C(X)$ satisfying

(a) $\alpha(q) = \{q\}$;

(b) for each $x \in X \setminus \{q\}$, the set $\alpha(x)$ is an arc from q to x;

(c) if $x \in \alpha(y)$, then $\alpha(x) \subset \alpha(y)$.

Illanes proved that if X is an arc-smooth continuum, then there exists an open Whitney map for 2^X [**11**].

Theorem 5.15. *If a continuum X is arc-smooth at a point $q \in X$, then there is a metric ρ on X such that*

(a) $\operatorname{diam}_\rho : 2^X \to \mathbf{R}$ *is lower semi-open*;

(b) *if $A \in 2^X$ and $q \notin A$, then* diam_ρ *is upper semi-open at A.*

Proof. Let $\omega : 2^X \to \mathbf{R}$ be a Whitney map satisfying the condition

(1) if $A, B, C \in 2^X$ and $A \subset B$, then $\omega(B) - \omega(A) \geq \omega(B \cup C) - \omega(A \cup C)$.

Such mappings do exist, as was observed in [**2**, p. 536].

For $A, B \in 2^X$, let $D_\omega(A, B) = \omega(A \cup B) - \min\{\omega(A), \omega(B)\}$. Then

D_ω is a metric on 2^X (see [**2**, p. 536]). Denote by $\Gamma(X)$ the space of all order arcs in 2^{2^X}. Therefore we can consider the Hausdorff metric D on $\Gamma(X)$ induced by the metric D_ω. For any $x \in X$, put $\mathcal{A}_x = \{\alpha(y) : y \in \alpha(x)\}$, and observe that $\mathcal{A}_x$ is an order arc from $\{q\}$ to $\alpha(x)$. Let ρ_1 be a metric on X defined by $\rho_1(x,y) = D(\mathcal{A}_x, \mathcal{A}_y)$. Note that Fact 7 of [**2**, p. 537] implies that if $y \in \alpha(x)$, then

(2) $\rho_1(x,y) = D_\omega(\alpha(x), \alpha(y)) = \omega(\alpha(x)) - \omega(\alpha(y))$.

In particular $\rho_1(q,x) = \omega(\alpha(x))$. Further, since for every order arc $\mathcal{A} \subset 2^X$ the partial mapping $\omega|\mathcal{A} : \mathcal{A} \to [0,\infty)$ is an isometry (see [**2**, p. 537]), the metric ρ_1 is radially convex at q, i.e., if $y \in \alpha(x)$, then $\rho_1(q,x) = \rho_1(q,y) + \rho_1(y,x)$. Moreover, by [**2**, p. 537], it satisfies the following condition

(3) if $\omega(\alpha(x)) = \omega(\alpha(y))$ and $x' \in \alpha(x)$, $y' \in \alpha(y)$ with $\omega(\alpha(x')) = \omega(\alpha(y'))$, then $\rho_1(x',y') \le \rho_1(x,y)$.

Now we will define the needed metric ρ. Assume $\operatorname{diam}_{\rho_1}(X) \le 1$ (we can achieve this by multiplying the metric ρ_1 by a constant). Let $x, y \in X$ with $\omega(\alpha(y)) \le \omega(\alpha(x))$. Denote by y' the only point of $\alpha(x)$ satisfying $\omega(\alpha(y')) = \omega(\alpha(y))$. Then we put

$$\rho(x,y) = 4(\omega(\alpha(x)) - \omega(\alpha(y))) + \rho_1(y,y')\omega(\alpha(y)).$$

We show that ρ is a metric. We need to check the triangle inequality

(4) $\rho(x,z) \le \rho(x,y) + \rho(y,z)$

only (the other two axioms are easy consequences of the definitions). To this aim observe the following property of ρ.

(5) If x, y and y' are as above, then $\rho(x,y) = \rho(x,y') + \rho(y,y')$.

Assuming $\omega(\alpha(z)) \le \omega(\alpha(x))$, we consider three cases.

Case 1. $\omega(\alpha(y)) \ge \omega(\alpha(x))$. Let z' be the point of $\alpha(x)$ such that $\omega(\alpha(z')) = \omega(\alpha(z))$; let $z'' \in \alpha(y)$ with $\omega(\alpha(z'')) = \omega(\alpha(z))$, and let $x' \in \alpha(y)$ with $\omega(\alpha(x')) = \omega(\alpha(x))$. By (5) we have to check that

$$\rho(x,z') + \rho(z,z') \le \rho(y,x') + \rho(x,x') + \rho(y,z'') + \rho(z,z'').$$

It is enough to show that $\rho(z,z') \le \rho(x,x') + \rho(z,z'')$. This inequality is equivalent, by the definition of ρ, to

(6) $\rho_1(z,z')\omega(\alpha(z)) \le \rho_1(x,x')\omega(\alpha(x)) + \rho_1(z,z'')\omega(\alpha(z))$.

By (3) we see that $\rho_1(z', z'') \leq \rho_1(x, x')$, and since $\omega(\alpha(x)) \geq \omega(\alpha(z))$, (6) follows.

Case 2. $\omega(\alpha(z)) \leq \omega(\alpha(y)) < \omega(\alpha(x))$. Let $y' \in \alpha(x)$ with $\omega(\alpha(y')) = \omega(\alpha(y))$; $z' \in \alpha(x)$ with $\omega(\alpha(z')) = \omega(\alpha(z))$; and $z'' \in \alpha(y)$ with $\omega(\alpha(z'')) = \omega(\alpha(z))$. By (5) we have to check that

(7) $\rho(x, z') + \rho(z, z') \leq \rho(x, y') + \rho(y, y') + \rho(y, z'') + \rho(z, z'')$.

Because $\rho(x, z') = \rho(x, y') + \rho(y, z'')$, it is enough to show that $\rho(z, z') \leq \rho(y, y') + \rho(z, z'')$. By (3) and by the definition of ρ we have $\rho(y, y') \geq \rho(z', z'')$, so (7) follows from the triangle inequality for ρ_1 and the definition of ρ.

Case 3. $\omega(\alpha(y)) < \omega(\alpha(z))$. Let $z' \in \alpha(x)$ with $\omega(\alpha(z')) = \omega(\alpha(z))$; let $y' \in \alpha(x)$ with $\omega(\alpha(y')) = \omega(\alpha(y))$, and $y'' \in \alpha(z)$ with $\omega(\alpha(y'')) = \omega(\alpha(y))$. Again, by (5), we have to show

$$\rho(x, z') + \rho(z, z') \leq \rho(x, y') + \rho(y, y') + \rho(y, y'') + \rho(z, y'').$$

By (5) we have to prove

$$\rho(x, z') + \rho(z, z') \leq \rho(x, z') + \rho(y', z') + \rho(y, y\prime) + \rho(y, y'') + \rho(y', z'),$$

or

$$\rho(z, z') \leq 2\rho(y', z') + \rho(y, y') + \rho(y, y'').$$

By the triangle inequality for ρ_1, we have $\rho(y', y'') \leq \rho(y, y') + \rho(y, y'')$, so it is enough to show that $\rho(z, z') \leq 2\rho(z', y') + \rho(y', y'')$, which is equivalent to

$$\rho_1(y', y'')\omega(\alpha(y)) - \rho_1(z, z')\omega(\alpha(z)) + 8(\omega(\alpha(z)) - \omega(\alpha(y))) \geq 0.$$

Using again the triangle inequality for ρ_1, we have

$$\rho_1(z, z') \leq \rho_1(y', y'') + 2(\omega(\alpha(z)) - \omega(\alpha(y))),$$

so it is enough to show that

$$\rho_1(y', y'')\omega(\alpha(y)) - \rho_1(y', y'')\omega(\alpha(z)) - 2(\omega(\alpha(z)) - \omega(\alpha(y)))\omega(\alpha(z)) \\ + 8(\omega(\alpha(z)) - \omega(\alpha(y))) \geq 0.$$

This inequality is equivalent to

$$(\omega(\alpha(z)) - \omega(\alpha(y)))(8 - \rho_1(y', y'') - 2\omega(\alpha(z))) \geq 0.$$

Note that the first factor is nonnegative by the assumption, and the second one is positive since $\operatorname{diam}_{\rho_1}(X) \leq 1$. The proof that ρ is a metric is complete.

To show that the mapping diam_ρ is lower semi-open, define a homotopy $H : X \times [0,1] \to X$ by the condition: $H(x,t)$ is the only point of the arc $\alpha(x)$ such that $\omega(\alpha(H(x,t))) = t\omega(\alpha(x))$.

Observe that $H(x,0) = q$ and $H(x,1) = x$. Moreover, for $x \neq y$ and $t < 1$, condition (3) and the definition of ρ imply

(8) $\rho(H(x,t), H(y,t)) < \rho(x,y)$.

Let $A \in 2^X$ be fixed, and let $\mathcal{U}$ be a neighborhood of A in 2^X. Then there is an $\varepsilon > 0$ such that $\{H(A \times \{s\}) : s \in [1-\varepsilon, 1]\} \subset \mathcal{U}$, and by (8) we infer that $\operatorname{diam}_\rho(\{H(A \times \{s\}) : s \in [1-\varepsilon, 1]\})$ is of the form $[\operatorname{diam}_\rho(A) - \eta, \operatorname{diam}_\rho(A)]$ for some $\eta > 0$, i.e., diam_ρ is lower semi-open, so a) is shown. To show b), assume that $A \in 2^X$ and $q \notin A$. Let $x, y \in A$ be two points such that $\operatorname{diam}_\rho(A) = \rho(x,y)$ and $\omega(\alpha(y)) \leq \omega(\alpha(x))$. Let $\mathcal{U}$ be an open neighborhood of A in 2^X. There is an $\varepsilon > 0$ such that $\{A \cup \{H(y,s)\} : s \in [1-\varepsilon, 1]\} \subset \mathcal{U}$. Put $y_s = H(y,s)$. We will check that $\rho(x, y_s) > \rho(x,y)$ for $s < 1$. By the definition of ρ, we have

$$\rho(x, y_s) - \rho(x,y) = \omega(\alpha(y))(4(1-s) + s\rho_1(y_s, y'_s) - \rho_1(y, y')).$$

By the triangle inequality for ρ_1, we have

$$\rho(x, y_s) - \rho(x,y) \geq \omega(\alpha(y))(4(1-s) - (1-s)\rho_1(y_s, y'_s) - 2\rho_1(y, y_s)),$$

so

$$\rho(x, y_s) - \rho(x,y) \geq \omega(\alpha(y))(1-s)(4 - \rho_1(y_s, y'_s) - 2\omega(\alpha(y))).$$

Since $s < 1, \omega(\alpha(y)) \in (0,1]$ and $\operatorname{diam}_{\rho_1}(X) \leq 1$, then all the factors are positive and $\rho(x, y_s) > \rho(x,y)$. Therefore $\operatorname{diam}_\rho(A \cup \{H(y,s)\}) > \operatorname{diam}_\rho(A)$ for $s < 1$, so $\operatorname{diam}_\rho(\{A \cup \{H(y,s)\} : s \in [1-\varepsilon, 1]\})$ is of the form $[\operatorname{diam}_\rho(A), \operatorname{diam}_\rho(A) + \eta]$ for some $\eta > 0$. So, diam_ρ is upper semi-open at A, and b) is proved. □

Theorem 5.16. *Let X_1 and X_2 be two continua that are arc-smooth at points $q_1 \in X_1$ and $q_2 \in X_2$, respectively. Let $X = X_1 \cup q_1q_2 \cup X_2$, where q_1q_2 is an arc such that $q_1q_2 \cap X_i = \{q_i\}$ for $i \in \{1,2\}$. Then there exists a metric ρ on X such that $\operatorname{diam}_\rho : 2^X \to \mathbf{R}$ is open.*

Proof. For $i \in \{1,2\}$, let ρ_i be a metric on X_i satisfying conditions (a) and (b) of Theorem 5.15 and such that $\operatorname{diam}_{\rho_i}(X_i) \leq 1/2$. For $i \in \{1,2\}$, let $\alpha_i : X_i \to C(X_i)$ be a mapping as in Definition 5.14 for the arc-smooth continuum X_i. Choose a point $p \in q_1q_2 \setminus \{q_1, q_2\}$. Note that X is an arc-smooth continuum with the mapping $\alpha : X \to C(X)$ defined by the following conditions:

(i) if $x \in pq_i$, then $\alpha(x)$ is the only arc with end points p and x;

(ii) if $x \in X_i$, then $\alpha(x) = pq_i \cup \alpha_i(x)$.

We will define the needed metric ρ in several steps.

If $x, y \in X_i$, then $\rho(x,y) = \rho_i(x,y)$.

Define ρ on the arcs pq_i as the convex metric on arcs pq_1 and pq_2 with $\rho(p, q_i) = 1$. If $x \in pq_i$ and $y \in X_i$, put $\rho(x,y) = \rho(x,q_i) + \rho(x,p)\rho(y,q_i)$.

For $x \in X$, define $w : X \to \mathbf{R}$ by

$$w(x) = \begin{cases} \rho(x,p) & \text{if } x \in pq_i, \\ 1 + \rho(x, q_i) & \text{if } x \in X_i. \end{cases}$$

If $x \in X_i \cup pq_i$ and $y \in X_j \cup pq_j$ for $i \neq j$ and $w(y) \leq w(x)$, then let y' be the only point of $\alpha(x)$ with $w(y') = w(y)$. Define

$$\rho(x,y) = \rho(x,y') + w(y)/4.$$

To show that it is a metric, we need to check the triangle inequality

(1) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$

only since the other two axioms are easy consequences of the definitions. We shall consider several cases.

Case 1. $x \in pq_i$.

1a) If $z \in pq_i$, then (1) is a consequence of the triangle inequality for the metric on pq_i.

1b) Assume now that $z \in X_i$. If $y \in xq_i$, then (1) is easy to check. Assume $y \in px$. We have to verify that

$$\rho(x, q_i) + \rho(x, p)\rho_i(z, q_i) \leq \rho(x, y) + \rho(y, q_i) + \rho(y, p)\rho(z, q_i)$$

which is equivalent to

$$\rho(x, p)\rho_i(z, q_i) \leq 2\rho(x, y) + \rho(y, p)\rho_i(z, q_i).$$

The last inequality is a consequence of the fact that $\operatorname{diam}_{\rho_i}(X_i) \leq 1/2$.

The inequality (1) for the case if $y \in pq_j \cup X_j$ for $i \neq j$ is a consequence of the above consideration.

1c) Let $z \in pq_j \cup X_j$ for $i \neq j$. Then (1) for this case follows from inequality (1) for cases 1a) and 1b).

Case 2. $x \in X_i$.

2a) If $z \in pq_i$, then (1) is a consequence of inequality (1) for the case 1b).

2b) Let $z \in X_i$. If $y \in X_i$, then (1) follows from the triangle inequality for ρ_i. If $y \in pq_j \cup X_j$ for $i \neq j$, then $\rho(x, y) \geq 1/4$ and $\rho(y, z) \geq 1/4$, while $\rho(x, z) \leq 1/2$.

If $y \in pq_i$, then we have to show

$$\rho_i(x, z) \leq 2\rho(y, q_i) + \rho(y, p)(\rho_i(x, q_i) + \rho_i(z, q_i)).$$

By the triangle inequality for ρ_i, it is enough to show that

$$2\rho(y, q_i) + \rho_i(x, z)(\rho(y, p) - 1) \geq 0.$$

This inequality follows from the fact that $1 - \rho(y, p) = \rho(y, q_i)$ and $\rho_i(x, z) \leq 1/2$.

2c) If $z \in pq_j \cup X_j$ for $i \neq j$, then (1) is a consequence of the previously considered cases. This finishes the proof that ρ is a metric.

In the proof of upper semi-openness of diam_ρ, we will need a homotopy $H : X \times [0, 1] \to X$ defined by the condition: $H(x, t)$ is the only point of $\alpha(x)$ with $w(H(x, t)) = t\, w(x)$.

To show upper semi-openness of diam_ρ at any $A \in 2^X$, take an open neighborhood $\mathcal{U}$ of A in 2^X and consider two cases.

Case 1. $p \notin A$. Let $x, y \in A$ be two points such that $\operatorname{diam}_\rho(A) = \rho(x, y)$ and $w(x) \le w(y)$. For $s \in [0,1]$, put $A_s = A \cup \{H(y,s)\}$.

Case 2. $p \in A$. If $A \cap (X_1 \cup X_2) \neq \varnothing$, then $\operatorname{diam}_\rho(A) = 1 = \operatorname{diam}_\rho(X)$, and thus diam_ρ is upper semi-open at A. Otherwise, there is a point $x \in pq_i$ (for some $i \in \{1,2\}$) such that $\operatorname{diam}_\rho(A) = \rho(x,p)$. For $s \in [0,1]$ denote by x_s the point of xq_i such that $\rho(x_s, q_i) = s\,\rho(x, q_i)$ and put $A_s = A \cup \{x_s\}$.

In each of the two cases there is an $\varepsilon > 0$ such that $A_s \in \mathcal{U}$ for $s \in [1-\varepsilon, 1]$. By the definition of ρ we have $\operatorname{diam}_\rho(A_s) > \operatorname{diam}_\rho(A)$ for $s < 1$, so $\operatorname{diam}_\rho(\{A_s : s \in [1-\varepsilon,1]\})$ is of the form $[\operatorname{diam}_\rho(A), \operatorname{diam}_\rho(A) + \eta]$ for some $\eta > 0$, so diam_ρ is upper semi-open at A.

To prove lower semi-openness of diam_ρ at any $A \in 2^X$, take again an open neighborhood $\mathcal{U}$ of A in 2^X and consider three cases.

Case 1. $A \subset pq_1 \cup pq_2$. For $s \in [0,1]$ define $A_s = H(A \times \{s\})$.

Case 2. $p \notin A$ and $A \cap (X_1 \cup X_2) \neq \varnothing$. For $s \in [0,1]$ define $A_s = (A \cup (pq_1 \cup pq_2)) \cup H(A \cap (X_1 \cup X_2) \times \{s\})$.

Case 3. $p \in A$ and $A \cap (X_1 \cup X_2) \neq \varnothing$. Then $\operatorname{diam}_\rho(A) = 1$. For any $s \in [0,1]$ and for $i \in \{1,2\}$ we define $x_s^i \in pq_i$ by $w(x_s^i) = 1 - s$. Put $A_s = (A \cap \{x \in X : w(x) \ge s\}) \cup \{x_s^1, x_s^2\}$.

Observe that, in any case, there is an $\varepsilon > 0$ such that $A_s \in \mathcal{U}$ for $s \in [1-\varepsilon, 1]$ and $\operatorname{diam}_\rho(A_s) < \operatorname{diam}_\rho(A)$ for $s \in [1-\varepsilon, 1)$. Therefore, $\operatorname{diam}_\rho(\{A_s : s \in [1-\varepsilon, 1]\})$ is of the form $[\operatorname{diam}_\rho(A) - \eta, \operatorname{diam}_\rho(A)]$ for some $\eta > 0$, so diam_ρ is lower semi-open at A.

In this way the proof of openness of diam_ρ is complete. □

Remark 5.17. Note that if X_i is a star-shaped continuum in a linear space or a smooth dendroid (see [**15**, p. 117] for the definition), then

X_i satisfies assumptions of Theorem 5.15. Also one can take one-point spaces as X_1 or X_2.

Corollary 5.18. *If X is either a dendrite containing a free arc or a subcontinuum of the harmonic fan, then X admits an open diameter mapping.*

We gave some examples of spaces which admit open diameter mappings. However, the following question remains open.

Question 5.19. Does every dendrite (local dendrite, graph, locally connected continuum) admit an open diameter mapping?

The theorem below is a contribution to this question. It shows monotoneity and lower semi-openness only of a diameter mapping on some local dendrites. For the existence of the needed convex metric on any locally connected continuum, see [**15**, p. 38].

Theorem 5.20. *Let X be a local dendrite equipped with a convex metric ρ. Denote by t the smallest of diameters of simple closed curves contained in X, and define a new metric d putting $d = \rho$ if X is a dendrite and $d(x, y) = \min\{\rho(x, y), t/2\}$ for each $x, y \in X$ otherwise. Then the diameter mapping $\mathrm{diam}_d : (2^X, d_H) \to \mathbf{R}$ is monotone. Moreover, it is lower semi-open if and only if X is a dendrite.*

Proof. First we show that diam_d is monotone. To prove that for each number $s \in [0, \mathrm{diam}_d(X)]$ the set $\mathrm{diam}_d^{-1}(s)$ is connected, it suffices to show connectedness of $\mathrm{diam}_d^{-1}([0, s])$ and use Lemma 3.1.

Let A be a closed subset of X. Choose a point p in A. For every point $x \in X$ find an arc $\overline{px}$ of diameter equal to $d(x, p)$. If there were two arcs with this property, then their union would contain a simple closed curve of the diameter less than t, so this arc is unique. For each $\alpha \in [0, 1]$ define the mapping $h_{\alpha,p}(x) = x_\alpha$, where $x_\alpha \in \overline{px}$ and $d(p, x_\alpha) = \alpha\, d(p, x)$. Observe that $d(x, x_\alpha) = (1 - \alpha)d(p, x)$. Let $A_\alpha = h_{\alpha,p}(A)$. Then $\mathrm{diam}_d(A_\alpha) = \alpha\, \mathrm{diam}_d(A)$ and $d_H(A, A_\alpha) \le (1 - \alpha)\mathrm{diam}_d(A)$. Thus every set A can be joined to the singleton $\{p\}$, where $p \in A$, with the arc $\{h_{\alpha,p}(A) : \alpha \in [0,1]\}$. Obviously, this arc is contained in $\mathrm{diam}_d^{-1}([0,t])$. Since the set $F_1(X) = \mathrm{diam}_d^{-1}(0)$ is connected, $\mathrm{diam}_d^{-1}([0,t])$ is connected, too. Observe that if X is a dendrite, then the set $\{A_\alpha : \alpha \in (1-\varepsilon, 1]\}$ is contained in the $\varepsilon/2$-ball about A, and the image of this ball contains the interval $((1-\varepsilon)\mathrm{diam}_d(A), \mathrm{diam}_d(A)]$ so diam_d is lower semi-open. If X is not a dendrite, then any set which is close to X has the diameter $t/2$, so diam_d is not lower semi-open at X. $\square$

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