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TOTAL VERSUS SINGLE POINT BLOW-UP FOR A NONLOCAL GASEOUS IGNITION MODEL

STEPHEN BRICHER

ABSTRACT. In this paper we investigate an integro-parabolic equation that may be considered as a mathematical model for the temperature within the ignition period of a gaseous fuel. For radially symmetric, non-increasing initial data, we determine where classical solutions become unbounded in finite time as well as describe the asymptotic behavior of these hot-spots. The method of analysis is based on maximum principle techniques and the method of stationary states.

1. Introduction.

1.1 Statement of the problem: Gaseous ignition models. The thermal combustion process in a solid fuel, where heat transfer by conduction is constant and the reaction rate depends on temperature, can be modeled [**4**] by the semi-linear parabolic equation

$$u_t = \Delta u + f(u),$$

where typically $f(u)$ is either $\exp(u)$ or u^p with $p > 1$.

For an ideal gaseous fuel in a bounded container, the motion caused by the compressibility of the gas leads to the addition of a *nonlocal* integral term that complicates the model. For example, the ignition period of a thermal event can be described by the following integro-

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parabolic problem [**4**, Chapters 1 and 5]

$$u_t = \Delta u + e^u + \frac{\gamma - 1}{|\Omega|} \int_\Omega e^u \, dy, \quad (x,t) \in \Omega \times (0, \infty) \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in \Omega \tag{2}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{3}$$

where $\Omega \subset R^n$ is a bounded open container, u is the temperature perturbation of the gas, $\gamma \geq 1$ is the gas parameter and $|\Omega| \equiv \mathrm{vol}\,(\Omega)$.

In this paper we concentrate on the case where $\gamma > 1$. We point out that if $\gamma = 1$, which corresponds to a solid fuel, then (1) loses its nonlocal dependence and we obtain the semi-linear equation $u_t = \Delta u + \exp(u)$ that has received much attention. See [**3**], [**4**] and [**6**] as well as the references therein for a discussion of the behavior of solutions to the solid fuel ignition model.

In addition to equation (1), the nonlocal parabolic equation

$$u_t = \Delta u + e^u + \frac{\gamma - 1}{\gamma |\Omega|} \int_\Omega u_t \, dy \tag{4}$$

is used as a gaseous-diffusive ignition model [**4**, Chapter 5]. By integrating (4) over Ω and applying Green's first identity to $\int_\Omega \Delta u \, dy$, we observe that (4) is equivalent to

$$u_t = \Delta u + e^u + \frac{\gamma - 1}{|\Omega|} \left[\int_{\partial\Omega} \frac{\partial u}{\partial n} \, d\sigma + \int_\Omega e^u \, dy \right].$$

By deleting the nonlocal gradient term, we obtain (1). This suggests that the gaseous ignition model (1) can be considered as a description of the thermal behavior for a gaseous fuel whose flux of the temperature's spatial rate of change across the container's boundary is negligible.

In this paper we are primarily concerned with analyzing the behavior of the unique solution (see [**5**]) to IBVP (1)–(3) in the radially symmetric case that we now identify. Let $\Omega = B_R \equiv \{x \in R^n : |x| < R\}$ and assume the initial temperature profile satisfies the compatibility condition $u_0 = 0$ for $x \in \partial B_R$ and is radially decreasing. Specifically, we assume

$$\begin{cases} u_0 = u_0(r) & \text{with } r = |x|, \\ u_0'(0) = 0 & u_0(R) = 0 \\ u_0'(r) < 0 & \text{for } r > 0. \end{cases} \tag{5}$$

At times, we also assume u_0 is a lower solution of (1). That is,

$$\Delta u_0 + e^{u_0} + \frac{\gamma - 1}{|B_R|} \int_\Omega e^{u_0}\, dy \geq 0. \tag{6}$$

It is possible to show that both models (1) and (4) satisfy an *invariance property* with respect to radially symmetric, non-increasing functions. Indeed, if u_0 satisfies (5) and u solves IBVP (1)–(3) or IBVP (4), (2)–(3), then u is radially symmetric and non-increasing in r for all t in the maximal time interval of existence for the appropriate problem. For a proof of this invariance property, see [**1**, Theorem 9] or [**4**, Theorem 5.14].

Throughout we will assume that the initial data u_0 satisfies (5) with $u_0(0) = \sup_{B_R} u(r)$ sufficiently large to guarantee finite time blow-up; that is, there exists $T < \infty$ such that $\lim_{t\to T^-} \sup_{B_R} u(x,t) = \infty$. Such an assumption is possible since the solution u of IBVP (1)–(3) is an upper solution to the semi-linear problem

$$\begin{aligned} w_t &= \Delta w + e^w, \quad (x,t) \in B_R \times (0,\infty) \\ w(x,0) &= u_0(x), \quad x \in B_R \\ w(x,t) &= 0, \quad x \in \partial B_R, \quad t > 0, \end{aligned}$$

which is known to have finite time blow-up when the initial data is sufficiently large (see [**4**]). Hence, since $u(x,t) \geq w(x,t)$, the solution u blows up in finite time as well. In fact, the corresponding maximal times satisfy the inequality $T_u < T_w$. Thus, any comparison between u and w for times $t > T_u$ has no meaning. Because of this, it's not possible to use the known information about the asymptotic behavior of w to gain a precise description of the blow-up behavior of u.

1.2 Main results. In this paper we determine the blow-up set for IBVP (1)–(3) with initial data satisfying (5), which is defined as

$$\begin{aligned} B(u_0) \equiv \{x \in B_R : \exists\{(x_m,t_m)\} \text{ with } (x_m,t_m) \longrightarrow (x,T) \\ \text{and } u(x_m,t_m) \longrightarrow \infty \text{ as } m \longrightarrow \infty\}. \end{aligned}$$

We find that the structure of the blow-up set depends on the spatial dimension with $n = 3$ being the set's bifurcation value. When $n \leq 2$

we observe *total blow-up*, and when $n \geq 3$ with $\gamma - 1 > 0$ sufficiently small, we observe *single point blow-up*. We precisely state our results in the following two theorems, which are proven in Sections 4 and 3, respectively.

Theorem 1.1 (Total blow-up for $n \leq 2$). *Let u be the solution of* IBVP (1)–(3) *with initial data satisfying* (5) *and* (6). *If $n \leq 2$, then the solution has total blow-up*; *that is, $B(u_0) = B_R$.*

Theorem 1.2 (Single point blow-up for $n \geq 3$). *Let u be the solution of* IBVP (1)–(3) *with initial data satisfying* (5) *and* (6). *If $n \geq 3$, then there exists $\beta > 0$ such that $1 < \gamma < 1 + \beta$ implies the solution has single point blow-up*; *that is, $B(u_0) = \{0\}$.*

It's interesting to observe that, in the radially symmetric case (5), the blow-up set when $\gamma = 1$, which corresponds to the solid fuel model, is $B(u_0) = \{0\}$ for both the IBVP and CP in *any* dimension. Thus, the bifurcation in the blow-up set for IBVP (1)–(3) is a consequence of its nonlocal structure. The proof when $\gamma = 1$ is based on maximum principle techniques presented in [**9**], which also includes a discussion of non-symmetric domains. See also [**4**] and [**6**], which includes the CP.

For the case $n \geq 3$ with $\gamma - 1 > 0$ sufficiently small, we are able to obtain an upper bound on the final-time profile and estimate the asymptotic behavior near the blow-up point within the so-called hot-spot variable domain. We state these results in the following two theorems, which are proven in Section 3.

Theorem 1.3 (Final-time profile bound). *Let u satisfy the hypotheses of Theorem* 1.2 *with $1 < \gamma < 1 + \beta$. Then the final-time profile, $u(r, T)$, satisfies*

$$u(r, T) \leq -2 \log r + \log |\log r| + C$$

for all $0 < r \ll 1$.

Theorem 1.4 (Hot-spot behavior). *Let u satisfy the hypotheses of Theorem* 1.2 *with $1 < \gamma < 1 + \beta$. Then*

$$\lim_{t \to T^-} [u(x, t) + \log(T - t)] = 0$$

uniformly on sets of the form $|x| \leq C\sqrt{T-t}$.

We point out that these estimates are precisely those observed for the behavior of blow-up in the solid fuel model, where $\gamma = 1$ in equation (1). See [**3**], [**4**] and [**6**]. Thus we may regard our results in Theorems 1.3 and 1.4 as statements on the *stability* of the asymptotic behavior near blow-up with respect to certain *nonlocal* additive perturbations of the elliptic operator $\Delta u + \exp(u)$.

1.3 Related work. Currently, gaseous ignition models as well as other nonlocal problems are receiving attention. We briefly mention some of the recent results related to problems considered in this paper.

Using comparison techniques for nonlocal problems, which are discussed in Section 2, the existence of a unique solution to IBVP (1)–(3) was proven in [**5**] for more general domains than the ball B_R.

An investigation of steady-state solutions to IBVP (1)–(3) was carried out in [**15**] where the author's main result is the following. If the initial data is below the minimal steady-state solution $\underline{u}_s(x)$, then the solution to the IBVP exists for all time and will converge to $\underline{u}_s(x)$.

In [**17**] the author considers, among other problems, the nonlocal parabolic equation

$$u_t - \Delta u = \int_\Omega \exp(u(y,t))\, dy$$

and proves that

$$\lim_{t \to T^-} \frac{\|u(t)\|_\infty}{|\log(T-t)|} = 1.$$

We observe similar behavior (Theorem 1.4) for equation (1), which includes the same nonlocal term as well as the ignition term, $\exp(u)$.

Regarding IBVP (4), (2)–(3) with $u_0(x) \equiv 0$, the authors of [**1**] use semi-group techniques to prove the solution is non-negative, radially symmetric and non-decreasing on its maximal interval of existence. The fact that the blow-up set is the origin for all n was established in [**2**], and a proof that the solution satisfies the conclusions of Theorems 1.3 and 1.4 was given in [**7**].

The outline of the remaining sections is as follows. In Section 2 we briefly review comparison methods for nonlocal problems as discussed in [**5**] and use them to obtain monotonicity results for a class of integro-parabolic problems that includes (1) as well as a comparison result for solutions to (1) and (4).

Section 3 is devoted to IBVP (1)–(3) in the case $n \geq 3$ with $\gamma - 1 > 0$ sufficiently small, where single point blow-up (Theorem 1.2) and estimates on the behavior near blow-up (Theorems 1.3 and 1.4) are established using maximum principle techniques similar to [**7**] and [**9**].

A proof of total blow-up for IBVP (1)–(3) when $n \leq 2$ (Theorem 1.1) is given in Section 4. The proof is based on intersection comparison techniques and the method of stationary states as discussed in [**16**, Chapter VII].

2. Comparison methods for nonlocal problems. Before reviewing comparison methods for nonlocal problems, we first identify our notation. Let D be an open bounded domain in R^n whose boundary is of class $C^{2+\alpha}$, $0 < \alpha < 1$. Let $D_T = D \times (0,T)$ and $B_T = (\partial D \times [0,T)) \cup (D \times \{0\})$ denote the parabolic boundary for D_T.

It is well-known that the standard definitions of lower and upper solutions to an IBVP do not necessarily yield comparison techniques for nonlocal problems. For a discussion that includes counterexamples, we refer the reader to [**5**]. Thus we need to modify the definitions of lower and upper solutions for applications to integro-parabolic problems. For convenience, we include a summary of the ideas found in [**5**], which we state for autonomous equations such as our gaseous models under consideration.

Consider the parabolic functional IBVP

$$u_t - \Delta u = F(x, u, \phi(u)), \quad (x,t) \in D_T \tag{7}$$

$$u(x,t) = \Psi(x,t), \quad (x,t) \in B_T, \tag{8}$$

where $\phi(u)$ denotes a nonlocal functional, for example, $\phi(u) = a\int_D f(u)\,dy$.

Let $\alpha, \beta \in C(\overline{D}_T)$ with $\alpha(x,t) \leq \beta(x,t)$. Relative to these functions, we define

$$S_\alpha(x,t) \equiv \{u \in C(\overline{D}_T) : \alpha(y,s) \le u(y,s) \le \beta(y,s) \\ \text{for } (y,s) \in \overline{D}_T \text{ with } u(x,t) = \alpha(x,t)\}.$$

That is to say, functions in $S_\alpha(x,t)$ lie between α and β on $\overline{D}_T$ and equal α at the point (x,t). Similarly, we define $S_\beta(x,t)$. We are now in a position to define our lower and upper solutions to IBVP (7)–(8) as in [**5**].

Definition 2.1. The functions $\alpha, \beta \in C^{2,1}(D_T)$ are a *lower-upper solution pair* for IBVP (7)–(8) provided

$$\alpha(x,t) \le \Psi(x,t) \le \beta(x,t), \quad (x,t) \in B_T \\ \alpha(x,t) \le \beta(x,t), \quad (x,t) \in \overline{D}_T$$

and, for each $(x,t) \in D_T$,

$$\alpha_t - \Delta\alpha \le F(x, u(x,t), \phi(u)) \quad \text{for all } u \in S_\alpha(x,t) \\ \beta_t - \Delta\beta \ge F(x, u(x,t), \phi(u)) \quad \text{for all } u \in S_\beta(x,t).$$

The existence of such a lower-upper pair implies that a solution of IBVP (7)–(8) exists and lies between the pair, which is proven in [**5**, Theorem 3].

If the righthand side of (7) satisfies a *monotonicity property*:

$$(9) \qquad u_1 \le u_2 \quad \text{implies} \quad F(x, u_1, \phi(u_1)) \le F(x, u_2, \phi(u_2)),$$

then we have the existence of maximum principle and comparison techniques for solutions to the nonlocal IBVP (7)–(8) via the lower-upper solution pairs given in Definition 2.1. See [**5**] for the details.

Motivated by the gaseous ignition model (1)–(3), we consider the functional Dirichlet problem

$$(10) \qquad u_t - \Delta u = F(x, u, \phi(u)), \quad (x,t) \in D \times (0,\infty)$$

$$(11) \qquad u(x,0) = u_0(x), \quad x \in D$$

$$(12) \qquad u(x,t) = 0, \quad x \in \partial D, \quad t > 0,$$

where $F \geq 0$ satisfies (9). Observe that the righthand side of (1) is an example of the functional

$$F = f(u) + \int_D g(u)\,dy,$$

where f and g are both non-decreasing, which satisfies (9). Hence the following theorems regarding solutions to IBVP (10)–(12) also hold for IBVP (1)–(3).

Theorem 2.1. *Let $u_0 \geq 0$ satisfy $-\Delta u_0 \leq F(x, u_0, \phi(u_0))$. Then the solution u to* IBVP (10)–(12) *satisfies $u(x,t) \geq u_0(x)$.*

Proof. Let $\alpha(x,t) = u_0(x)$ which is a lower solution to IBVP (10)–(12) due to (9) and so, by comparison, $u(x,t) \geq u_0(x)$. □

Theorem 2.2. *Let u and v be solutions to* IBVP (10)–(12) *with corresponding initial data u_0 and v_0 satisfying $0 \leq u_0 \leq v_0$. Then $u(x,t) \leq v(x,t)$ on their common interval of existence.*

Proof. Let $z(x,t) = v(x,t) - u(x,t)$. Then z satisfies the parabolic functional IBVP

$$\begin{aligned} z_t - \Delta z &= F(x, z+u, \phi(z+u)) - F(x, u, \phi(u)), \quad (x,t) \in D \times (0,\infty) \\ z(x,0) &= z_0(x) \equiv v_0(x) - u_0(x), \quad x \in \overline{D}, \\ z(x,t) &= 0, \quad x \in \partial D, \quad t > 0. \end{aligned}$$

It is not difficult to observe that $\alpha(x,t) \equiv 0$ is a lower solution to the above problem since $F \geq 0$ satisfies the monotonicity property (9). Hence, by comparison, $z(x,t) \geq 0$. □

Corollary 2.3 (Monotonicity in time). *Let $u_0 \geq 0$ satisfy*

$$-\Delta u_0 \leq F(x, u_0, \phi(u_0))$$

and let u be the solution to IBVP (10)–(12). *Then $u_t \geq 0$.*

Proof. Let $u^\delta(x,t) = u(x,t+\delta)$. Then u^δ solves the problem

$$\begin{aligned} u_t^\delta - \Delta u^\delta &= F(x, u^\delta, \phi(u^\delta)), \quad (x,t) \in D \times (0,\infty), \\ u^\delta(x,0) &= u(x,\delta), \quad x \in \overline{D}, \\ u^\delta(x,t) &= 0, \quad x \in \partial D, \quad t > 0. \end{aligned}$$

By Theorem 2.1, $u^\delta(x,0) \geq u_0(x)$ and from Theorem 2.2 we may conclude that $u^\delta(x,t) \geq u(x,t)$, which yields that $u_t \geq 0$. □

There is a comparison property between solutions to models (1) and (4) that we give in the following theorem, which is also discussed in [**1**].

Theorem 2.4. *Let u_0 satisfy* (5) *and u be the solution to* IBVP (1)–(3)*, and let v be the solution to* IBVP (4), (2)–(3)*. Then $u(x,t) \geq v(x,t)$ on their common interval of existence.*

Proof. As mentioned in Section 1.1, the solutions u and v are radially symmetric and non-increasing. Thus, $v_r(r,t) \leq 0$. Hence, for any $w \geq v$ we have that

$$\begin{aligned} v_t - \Delta v &= \exp(v) + \frac{\gamma - 1}{\gamma |B_R|} \int_{B_R} v_t \, dy \\ &= \exp(v) + \frac{\gamma - 1}{|B_R|} \left[\int_{\partial B_R} \frac{\partial v}{\partial n} \, d\sigma + \int_{B_R} \exp(v) \, dy \right] \\ &\leq \exp(v) + \frac{\gamma - 1}{|B_R|} \int_{B_R} \exp(v) \, dy \\ &\leq \exp(w) + \frac{\gamma - 1}{|B_R|} \int_{B_R} \exp(w) \, dy. \end{aligned}$$

Therefore, by Definition 2.1, we see that v is a lower solution of (1) and so $v(x,t) \leq u(x,t)$ by comparison. □

As mentioned in [**1**] and [**4**], not much is known about IBVP (4), (2)–(3) for nonzero initial data. However, because of Theorem 2.4, we are able to use solutions to IBVP (1)–(3) as upper estimates for solutions to IBVP (4), (2)–(3) on their common interval of existence.

3. Behavior for $n \geq 3$: Single point blow-up. In this section we consider IBVP (1)–(3) for the case $n \geq 3$ and prove Theorems 1.2–1.4. We will use the notation

$$g(t) \equiv \frac{\gamma - 1}{|\Omega|} \int_{\Omega} \exp(u)\, dy \quad \text{and} \quad G(t) \equiv \int_0^t g(s)\, ds \tag{13}$$

throughout Sections 3 and 4.

Lemma 3.1. *Let u be the solution to* IBVP (1)–(3) *with initial data satisfying* (5) *and* (6)*. Then we have the following upper bound*

$$u(r,t) \leq -\frac{2}{\alpha} \log r + C + G(t) \tag{14}$$

for any $0 < \alpha < 1$.

Proof. The proof is based on a maximum principle technique similar to [**2**] and [**9**]. Let us define the following auxiliary function

$$J(r,t) \equiv r^{n-1} u_r(r,t) + \varepsilon r^n \exp(\alpha(u - G(t))).$$

Since $u_t \geq 0$ (Corollary 2.3) and $u_r < 0$ (maximum principle) on $(0, R) \times (0, T)$, we are able to use a maximum principle argument to show $J \leq 0$ on $(r,t) \in [0, R] \times [0, T)$ for any $0 < \alpha < 1$ and $\varepsilon > 0$ that's sufficiently small. See [**2**], [**3**], [**6**] and [**9**] for an illustration of the details. Estimate (14) then follows by integrating the inequality $J \leq 0$ from 0 to r. The constant C will depend on α and ε, but not on γ. Specifically, $C = -\alpha^{-1} \log(\alpha\varepsilon/2)$ for all $0 < \alpha < 1$ and $0 < \varepsilon < \min\{1/n, (1-\alpha)/2\alpha\}$. □

Lemma 3.2. *Let u be the solution to* IBVP (1)–(3) *with initial data satisfying* (5) *and* (6)*. If $n \geq 3$, then*

$$G(t) \leq (\gamma - 1) T C \exp(G(t)). \tag{15}$$

Proof. From estimate (14) we have that

$$\exp(u) \leq C r^{-2/\alpha} \exp(G(t)).$$

Since $n \geq 3$, we may choose $\alpha \in (0,1)$ such that $2/n < \alpha < 1$. Thus, integrating over B_R yields

$$g(t) \leq (\gamma - 1)C \exp(G(t)) \tag{16}$$

because the exponent for r will be $n - 1 - 2/\alpha > -1$. Now estimate (16) implies

$$G(t) = \int_0^t g(s)\, ds \leq (\gamma - 1)C \int_0^t \exp(G(s))\, ds,$$

which yields inequality (15) since G is increasing. □

Lemma 3.3. *Let u satisfy the assumptions of Lemma* 3.2 *and assume $n \geq 3$. Then there exists $\beta > 0$ such that $1 < \gamma < 1 + \beta$ implies $g \in L^1(0,T)$.*

Proof. By Lemma 3.2, we see that

$$G(t) \exp(-G(t)) \leq (\gamma - 1)TC. \tag{17}$$

Now the function $h(v) \equiv v \exp(-v)$ attains its global maximum on the interval $0 \leq v < \infty$ at $v = 1$. Specifically,

$$h(v) \leq h(1) = e^{-1} \quad \text{for } 0 \leq v < \infty. \tag{18}$$

Now the blow-up time T will depend on γ, and by our comparison results we have the monotonicity result, $\gamma_1 \leq \gamma_2$ implies $T(\gamma_2) \leq T(\gamma_1)$. Therefore, $\gamma > 1$ implies $T(\gamma) < T(1)$, and thus we may choose $\beta > 0$ sufficiently small so that

$$\beta CT = \beta CT(\gamma) \leq \beta CT(1) < e^{-1},$$

where the constant C does not depend on γ as mentioned in the proof of Lemma 3.1. Hence, by inequalities (17) and (18), we see that $0 < \gamma - 1 < \beta$ implies $h(G(t))$ does not attain its global maximum since

$$h(G(t)) = G(t) \exp(-G(t)) \leq \beta CT < e^{-1}. \tag{19}$$

Suppose, for the moment, that $1 < G(T) \leq \infty$. Since $G(t)$ is increasing and $G(0) = 0$, there exists $t_0 \in (0, T)$ such that $G(t_0) = 1$ by continuity. Thus the function $h(G(t))$ attains its global maximum, e^{-1}, at time $t = t_0$, which contradicts (19). Therefore, $G(t)$ is bounded for all $0 \leq t \leq T$. In fact, $G(t) \leq G(T) < 1$, which yields the desired result, $g \in L^1(0, T)$. □

Using these lemmas, we may prove Theorem 1.2. Indeed, Lemma 3.3 implies $G(t)$ is bounded for all $t \leq T$ and so inequality (14) provides the upper bound

$$u(r, T) \leq -\frac{2}{\alpha} \log r + C,$$

which implies the only blow-up point is the origin.

By modifying the choice of the auxiliary function $J(r, t)$ in the proof of Lemma 3.1, it is possible to obtain the upper bound on the final-time profile

$$u(r, T) \leq -2 \log r + \log |\log r| + C, \quad 0 < r \ll 1 \tag{20}$$

that is the conclusion of Theorem 1.3. For example, the choice

$$J(r, t) = r^{n-1} u_r(r, t) + \varepsilon F(u, t),$$

where

$$F(u, t) = \frac{1}{a + u} \exp(u + k(t)), \quad k(t) = \int_t^T g(s)\, ds$$

in the proof of Lemma 3.1 will produce the estimate

$$\int_{u(r,t)}^{u(0,t)} (a + s) \exp(-s)\, ds \geq \frac{\varepsilon r^2}{2} \exp(k(t)),$$

from which the upper bound (20) can be obtained. See the references [**3**] and [**7**] for details.

In addition to the final-time profile bound (20), we are able to obtain important L^∞ bounds on the rate of blow-up with respect to time. Specifically, if u satisfies the hypotheses of Lemma 3.2 and we assume

$n \geq 3$ together with $1 < \gamma < 1 + \varepsilon$, then, since $g \in L^1(0,T)$ by Lemma 3.3, we can show that

$$u(r,t) \leq -\log(T-t) + C \tag{21}$$

$$u(0,t) \geq |\log(T-t)| \left[1 + \frac{\varepsilon\, n \exp(k(t))}{|\log(T-t)|^2}\,(1+o(1))\right] \tag{22}$$

$$u_r^2 \leq 2\exp(u(0,t))\,[1+G(t)] \tag{23}$$

for all $(r,t) \in [0,R) \times [0,T)$. Proofs of these bounds for IBVP (1)–(3) are similar to those given in [**7**] and so we do not provide the details. See also [**3**], [**4**] and [**6**] for similar L^∞ bounds of solutions to quasilinear parabolic problems with no functional dependence.

To begin our discussion of Theorem 1.4, we make the *hot-spot* change of variables

$$\begin{cases} \tau = -\log(T-t) \qquad y = x(T-t)^{-1/2}, \\ w(y,\tau) = u(x,t) + \log(T-t), \end{cases}$$

where w solves the parabolic equation

$$w_\tau = \Delta w - \frac{1}{2}\, y \cdot \nabla w + e^w - 1 + e^{-\tau} g(T - e^{-\tau})$$

on the set $(y,\tau) \in B(\tau) \times (\tau_0, \infty)$ with $B(\tau) \equiv \{y \in R^n : |y| < R\exp(\tau/2)\}$. The initial condition and moving boundary condition are given by

$$\begin{gathered} w(y,\tau_0) \equiv w_0(y) = u_0(ye^{-\tau/2}), \quad y \in B(\tau) \\ w(y,\tau) = 0, \quad (y,\tau) \in \partial B(\tau) \times (\tau_0, \infty). \end{gathered}$$

From the L^∞ bounds in (21)–(23), it follows that

$$\begin{cases} -c_1 - c_2|y| \leq w \leq c_3, & |\nabla w| \leq c_4, \\ |\Delta w| \leq c_5, & |w_\tau| \leq c_6 + c_7|y|. \end{cases} \tag{24}$$

We now use a stabilization technique [**14**] to show $w \to 0$ as $\tau \to \infty$. We do not provide the details as the proofs given in [**3**] and [**6**] can be adapted to this problem, but we do outline the main argument for the readers' convenience. Using the bounds in (24), we are able to use the energy functional

$$E(t) = \int_{B_{\sqrt{\tau}}} \exp(-|y|^2/4)\left[\frac{1}{2}\,|\nabla w|^2 + w - \exp(w)\right] dy$$

to show that

$$\int_a^\infty \int_{B_\delta} w_\tau^2 \exp(-|y|^2/4)\, dy\, d\tau < \infty$$

for any constants $\delta > 0$ and $a \gg 1$. This will imply that the omega limit set consists of solutions to a stationary problem. That is,

$$\omega(w_0) = \left\{ w \in C^2 : \Delta w - \frac{1}{2}\, y \cdot \nabla w + e^w - 1 = 0 \right\}.$$

Finally, the only allowable function in the omega limit set satisfying the bounds in (24) and that intersects the singular solution $S(r) \equiv \log(2(n-2)/r^2)$ exactly once is $w \equiv 0$. See [**4**] for details. Thus, $w(y,\tau) \to 0$ as $\tau \to \infty$ uniformly on compact subsets in y. Translating back to the original variables (x,t) yields the conclusion of Theorem 1.4.

4. Behavior for $n \leq 2$: Total blow-up. In this section we consider IBVP (1)–(3) with radially symmetric, non-increasing initial data $u_0(r)$ that satisfies (5) for the case $n \leq 2$. We prove Theorem 1.1 using the *method of stationary states*, which is described in the text [**16**, Chapter VII]. See, for example, the references [**10**], [**11**] [**12**] and [**13**] for additional applications of this method.

We begin by describing some important properties of stationary solutions to (1) in the radially symmetric case. Let $U = U(r; U_0)$ be a solution to the stationary nonlocal initial-value problem

$$r^{1-n}(r^{n-1}U_r)_r + e^U + \frac{n(\gamma-1)}{R^n}\int_0^R e^U r^{n-1}\, dr = 0 \tag{25}$$

$$U(0) = U_0 \tag{26}$$

$$U'(0) = 0. \tag{27}$$

By integrating (25) from 0 to r, it is easy to see that

$$U_r \geq -\frac{\gamma r \exp(U_0)}{n} \quad \text{and} \quad U(r;U_0) \geq U_0\left(1 - \frac{r^2}{r_0^2}\right)_+, \tag{28}$$

where $r_0^2 = 2nU_0 \exp(-U_0)$. In addition, let $0 < \varepsilon \ll 1$, then

$$U_r \leq -r^{1-n} c(\varepsilon), \tag{29}$$

for $r \geq \varepsilon$, where $c(\varepsilon) = \int_0^\varepsilon s^{n-1} \exp(U(s))\, ds$.

Integrating (29) from ε to r and using the bound

$$c(\varepsilon) \geq \frac{\varepsilon^n}{n} \exp(U(\varepsilon))$$

yields the upper bound

$$U(r) \leq U(\varepsilon) - \frac{\varepsilon^n}{n} \exp(U(\varepsilon)) \int_\varepsilon^r s^{1-n}\, ds.$$

We observe that if $U(\varepsilon)$ is sufficiently large, the righthand side of this inequality is decreasing relative to the variable $U(\varepsilon)$. The bounds in (28) imply that $U(\varepsilon) = U(\varepsilon; U_0) \geq U_0(1-\varepsilon^2/r_0^2)_+$. Thus, when $U_0 \gg 1$ and $0 < \varepsilon \ll 1$, we obtain the estimate

$$U(r; U_0) \leq U_0(1-\varepsilon^2/r_0^2)_+ - \frac{\varepsilon^n}{n} \int_\varepsilon^r s^{1-n}\, ds \exp(U_0(1-\varepsilon^2/r_0^2)_+) \tag{30}$$

for $0 < \varepsilon \ll 1$ and $U_0 \gg 1$.

We point out that, when U_0 is sufficiently large, estimate (30) implies that

$$\overline{\operatorname{supp} U(r; U_0)} \subset [0, R)$$

for any n; hence, we conclude that steady-state solutions to IBVP (1)–(3) fail to exist when the initial data u_0 satisfies (5) with $u_0(0)$ sufficiently large.

We now recall a well-known property regarding the intersections between the solution $u(r,t)$ of IBVP (1)–(3) with a stationary solution $U(r; U_0)$ of IVP (25)–(27).

Lemma 4.1. *Let $N(t; U_0)$ denote the number of intersections of the functions $u(r,t)$ and $U(r; U_0)$. Then $N(t; U_0)$ is a non-increasing function of t. In particular,*

$$N(t; U_0) \leq N(0; U_0)$$

for all $0 \le t < T$.

The proof of this lemma relies on maximum principle and comparison techniques, which hold for our nonlocal problem as mentioned in Section 2. We do not provide the details of the proof, which can be found in the references [**10**]–[**13**], but outline the argument. Indeed, observe that the difference $w \equiv u - U$ satisfies a linear parabolic equation

$$w_t = \Delta w + b(x,t)w + \frac{\gamma - 1}{|B_R|}\int_{B_R} b(y,t)w\,dy,$$

where $b(x,t) = \int_0^1 \exp(su + (1-s)U)\,ds$. The number of intersections between u and U is not more than the number of zeros of the difference w. The lemma then follows from the well-known comparison property that the number of sign changes of w is not more than the number of sign changes of w on the parabolic boundary.

Using the properties of the stationary solutions mentioned in (28)–(30), we may use Lemma 4.1 to establish the following comparison between the final-time profile $u(r,T)$ and stationary solutions $U(r;U_0)$. See [**16**, pp. 421+] or the references [**10**]–[**13**] for a proof as well as other applications.

Lemma 4.2. *Let* $u(r,t)$ *be an unbounded solution to* IBVP (1)–(3) *with initial function* $u_0(r)$ *satisfying* (5). *Moreover, assume there exists* U_0^* *such that for all* $U_0 > U_0^*$, *the functions* $u_0(r)$ *and* $U(r;U_0)$ *intersect at most one point. Then, for all sufficiently small* $r > 0$, *we have that*

$$u(r,T^-) \equiv \varliminf_{t\to T^-} u(r,t) \ge \sup_{U_0 \ge U_0^*} U(r;U_0). \tag{31}$$

Briefly, the idea of the proof is to use Lemma 4.1 to show that the number of intersections, $N(t;U_0)$, between u and $U(r;U_0)$ is at most one for $U_0 > U_0^*$ and $t \in (t_{U_0}, T)$. Since u becomes unbounded as t approaches T, it is possible to rule out the case $N(t;U_0) = 1$. Thus, by comparison, $u(r,t) \ge U(r;U_0)$ for all $U_0 > U_0^*$ and $t \in (t_{U_0}, T)$ from which estimate (31) will follow.

We observe that the hypothesis $N(0; U_0) \leq 1$ for large U_0 is satisfied due to the assumptions on $u_0(r)$ and properties (28)–(30) for the stationary solutions $U(r; U_0)$.

For our problem, it is possible to estimate the *stationary-solution envelope*,

$$L(r) \equiv \sup_{U_0 \geq U_0^*} U(r; U_0).$$

Indeed, let $K(r)$ satisfy

$$K(r) = U_0 - \frac{\gamma e^{U_0} r^2}{2n} \quad \text{and} \quad K' = -\frac{\gamma e^{U_0} r}{n},$$

which imply, by eliminating the parameter U_0, that K satisfies the differential equation

$$K = \log\left(\frac{-nK'}{\gamma r}\right) + \frac{r}{2} K'. \tag{32}$$

By (28) we see that $L(r) \geq K(r)$. We are interested in monotone decreasing solutions to (32) that satisfy $K(r) \to \infty$ as $r \to 0^+$. It is easy to verify that, for an appropriate choice of the constant, $K(r) = -2\log r + C$ is such a solution. Hence, we have from (31) that

$$u(r, T^-) \geq -2\log r + C \quad \text{for } 0 < r \ll 1. \tag{33}$$

We now discuss how this lower bound on the final-time profile can be used to show the solution $u(r, t)$ of IBVP (1)–(3) blows up everywhere in B_R. To begin, we observe that since $r^{-2} \notin L^1(B_R)$ when $n \leq 2$, estimate (33) implies that

$$g(t) = \frac{\gamma - 1}{|B_R|} \int_{B_R} \exp(u)\, dy \longrightarrow \infty \quad \text{as } t \to T^-. \tag{34}$$

Suppose, for the moment, that $g \in L^1(0, T)$. Then by an argument similar to Section 3, we have single point blow-up at the origin. Let $b \in (0, R)$ then $u(r, t)$ is bounded on the set $[b, R] \times [0, T)$, which implies that Δu solves a linear parabolic PDE with bounded coefficients on $[b, R] \times [0, T)$; hence, by Schauder estimates [**8**], Δu is bounded on $[b, R] \times [0, T)$. However, estimate (34) implies that

$$\Delta u(R, t) \longrightarrow -\infty \quad \text{as } t \to T^-,$$

which yields a contradiction. Therefore, it must be that $g \notin L^1(0,T)$.

Now, choose $b \in (0,R)$ and let $\rho = \min\{b, R-b\}$. On the ball $B(b;\rho) \equiv \{x \in R^n : |x-b| < \rho\}$, consider the linear parabolic IBVP

$$\begin{aligned} w_t &= \Delta w + g(t), \quad (x,t) \in B(b;\rho) \times (0,T) \\ w(x,0) &= 0, \quad x \in B(b;\rho) \\ w(x,t) &= 0, \quad x \in \partial B(b;\rho), \quad t > 0. \end{aligned}$$

Using Green's functions, the solution to this problem evaluated at $\bar{x}$, with $|\bar{x}| = b$, can be written as

$$\begin{aligned} w(\bar{x},t) &= \int_0^t \int_{B(b;\rho)} G(\bar{x},y,t-s) g(s)\, dy\, ds \\ &\geq \int_0^t g(s) \int_{B(b;\rho)} G(\bar{x},y,T-0)\, dy\, ds \\ &\geq K(\rho) \int_0^t g(s)\, ds, \end{aligned} \tag{35}$$

where $K(\rho) = \int_{B(b;\rho)} G(\bar{x},y,T)\, dy$. Since $g \notin L^1(0,T)$, inequality (35) implies that $w(\bar{x},t) \to \infty$ as $t \to T^-$. The solution u of IBVP (1)–(3) is an upper solution to the above linear problem and therefore $u(\bar{x},t) \to \infty$ as $t \to T^-$ for any $\bar{x} \in B_R$ since the choice of b was arbitrary. This proves we have *total blow-up* for $n \leq 2$, which is the conclusion of Theorem 1.1.

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Department of Mathematics, Linfield College, McMinnville, OR 97128
E-mail address: `sbricher@linfield.edu`