ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 1, Spring 2002

ABELIAN GROUPS WITH SELF-INJECTIVE QUASI-ENDOMORPHISM RINGS

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1. Introduction. One of the most important concepts in the discussion of endomorphism rings of torsion-free abelian groups is that of faithfulness. A left R-module A is *fully faithful* (*faithful*) if $M \otimes_R A \neq 0$ for all (finitely generated) right R-modules M. It is easy to see that a faithful module which is flat is fully faithful. On the other hand, faithful modules exist which are not fully faithful: $\oplus_p \mathbf{Z}/p\mathbf{Z}$ where the direct sum is taken over all primes is a faithful $\mathbf{Z}$-module, but is not fully faithful.

Abelian groups which are faithful or fully faithful as modules over their endomorphism ring share some of the homological properties of torsion-free groups of rank 1 which Baer discussed in 1937 in [**11**]. For instance, a self-small abelian group A is (faithful) fully faithful as a module over its endomorphism ring if and only if an exact sequence $0 \to B \xrightarrow{\alpha} G \to P \to 0$ splits if P is A-projective (of finite A-rank) and $G = S_A(G) + \alpha(B)$, for details, see [**2**] and [**9**]. Here P is *A-projective* (*of finite A-rank*) if it is a direct summand of $\oplus_J A$ for some (finite) index-set J, and $S_A(G) = \mathrm{Hom}\,(A, G)A$. The group A is *self-small* if, for every index-set I and all $\alpha \in \mathrm{Hom}\,(A, \oplus_I A)$ there is a finite subset I' of I such that $\alpha(A) \subseteq \oplus_{I'} A$. For example, every torsion-free group of finite rank is self-small, but self-small torsion-free groups of arbitrary cardinality exist.

In this paper the concept of faithfulness is extended to the quasi-category of torsion-free abelian groups: A torsion-free group A is said to be *quasi-fully faithful* if $\mathbf{Q}A = \mathbf{Q} \otimes_{\mathbf{Z}} A$ is a fully faithful $\mathbf{Q}E$-module where $E = E(A)$ denotes the *endomorphism ring of A* and $\mathbf{Q}E = \mathbf{Q} \otimes_{\mathbf{Z}} E$ is its *quasi-endomorphism ring*. Theorem 2.1 and its corollaries give additional characterizations of quasi-fully faithful groups. It is shown that every quasi-fully faithful group A has the finite *quasi-Baer-splitting property*, i.e., an exact sequence $0 \to B \xrightarrow{\alpha}$

Received by the editors on December 19, 1997, and in revised form on August 18, 1998.

$G \to P \to 0$ quasi-splits if P is a quasi-summand of an A-projective group of finite A-rank and $G \doteq S_A(G) + \alpha(B)$ [**4**]. The converse holds if A is *almost flat*, Theorem 2.4, i.e., there is a nonzero integer m such that $m\mathrm{Tor}_1^E(M, A) = 0$ for all right E-modules M (for details on almost flatness, see [**6**]). This equivalence may fail if the almost flatness of A is replaced by the weaker requirement that A is quasi-flat since there is a torsion-free group A of rank 4 which is quasi-flat and has the finite quasi-Baer-splitting property but is not quasi-fully faithful, Example 2.6. Here A is *quasi-flat*, see [**7**], if $\mathrm{Tor}_1^E(M, A)$ is torsion for all right E-modules M. Every torsion-free abelian group G arises as a direct summand of a quasi-flat group since $\mathbf{Z} \oplus G$ is flat and hence quasi-flat.

Problem 84 in [**13**] seeks a characterization of the abelian groups with self-injective endomorphism ring. Though one was given by the author in [**1**], the only torsion-free groups in the class which arises as an answer are divisible. However, asking the same question for quasi-endomorphism rings instead of endomorphism rings results in a large class of abelian groups which contains several well-known classes of torsion-free groups of finite rank, e.g., the groups with a semi-simple Artinian quasi-endomorphism ring. Section 3 addresses this quasi-version of [**13**, Problem 84]. The results of Section 2 are used to give several characterizations of the quasi-flat abelian groups of finite rank whose quasi-endomorphism ring is self-injective, Theorem 3.5.

2. Quasi-fully faithful abelian groups. Associated with every abelian group A is an adjoint pair (H_A, T_A) of functors between the category of abelian groups and the category of right E-modules. These functors are defined as $H_A(G) = \mathrm{Hom}\,(A, G)$ and $T_A(M) = M \otimes_E A$ for all abelian groups G and all right E-modules M, and induce natural maps $\theta_G^A : T_A H_A(G) \to G$ and $\phi_M^A : M \to H_A T_A(M)$ by $\theta_G^A(\alpha \otimes a) = \alpha(a)$ and $[\phi_M^A(x)](a) = x \otimes a$ for all $a \in A$, $x \in M$ and $\alpha \in H_A(G)$. Usually, the superscripts referring to A are omitted. The class of A-solvable groups arises as the largest full subcategory of the category of abelian groups on which θ_G induces a natural equivalence between $T_A H_A$ and the identity functor. For instance, all A-projective groups are A-solvable if A is self-small.

The concept of A-solvability is extended to the quasi-category of abelian groups by calling an abelian group G is $\mathbf{Q}A$-solvable if the induced map $\mathbf{Q}\theta_G : \mathbf{Q}T_AH_A(G) \to \mathbf{Q}G$ is an isomorphism. Because $\operatorname{im}\theta_G = S_A(G)$, the group G is $\mathbf{Q}A$-solvable if and only if $\ker\theta_G$ and $G/S_A(G)$ are torsion. Finally, an exact sequence $0 \to B \xrightarrow{\alpha} C \xrightarrow{\beta} G \to 0$ of abelian groups is called *quasi-A-balanced* if $H_A(G)/\operatorname{im} H_A(\beta)$ is torsion. In this case, $C/[\alpha(B) + S_A(G)]$ is torsion: For every $c \in C$, a nonzero integer k exists, elements $a_1, \dots, a_n$ of A and maps $\phi_1, \dots, \phi_n \in H_A(G)$ with $k\beta(c) = \sum_{i=1}^n \phi_i(a_i)$. Since $H_A(G)/\operatorname{im} H_A(\beta)$ is torsion, one can choose a nonzero integer m and maps $\psi_1, \dots, \psi_n \in H_A(C)$ such that $m\phi_i = \beta\psi_i$ for $i = 1, \dots, n$. Then $\beta(kmc) = \beta(\sum_{i=1}^n \psi_i(a_i))$, and $kmc \in S_A(C) + \ker\beta$.

Since every $\mathbf{Q}E$-module M has a torsion-free divisible additive group, $\operatorname{Tor}_1^E(M, \mathbf{Q}A/A) = 0$ and $M \otimes_E [\mathbf{Q}A/A] = 0$. Thus, $M \otimes_{\mathbf{Q}E} \mathbf{Q}A \cong T_A(M)$. In particular, A is quasi-fully faithful if and only if $T_A(M) \neq 0$ for all nonzero right E-modules M whose additive group is torsion-free divisible. The first result of this section discusses the relation between quasi-full faithfulness and quasi-A-balanced sequences.

Theorem 2.1. *The following conditions are equivalent for a self-small torsion-free abelian group A:*

a) *A is quasi-fully faithful.*

b) *If M is a nonzero right E-module with a torsion-free additive group, then $T_A(M) \neq 0$.*

c) *A right E-module M such that $T_A(M)$ is torsion is itself torsion as an abelian group.*

d) *An exact sequence $0 \to B \xrightarrow{\alpha} C \xrightarrow{\beta} G \to 0$ such that G is $\mathbf{Q}A$-solvable is quasi-A-balanced if and only if $C/[\alpha(B)+S_A(C)]$ is torsion.*

Proof. a) $\Rightarrow$ b). Suppose that $T_A(M) = 0$ for some right E-module M whose additive group is torsion-free. The inclusion $M \subseteq \mathbf{Q}M$ induces the exact sequence $T_A(M) \to T_A(\mathbf{Q}M) \to T_A(\mathbf{Q}M/M) \to 0$. Since $T_A(M) = 0$, one has $T_A(\mathbf{Q}M) = 0$. Hence, $\mathbf{Q}M = 0$ because of a), and $M = 0$ too since M^+ is torsion-free.

b) $\Rightarrow$ c). Suppose that M is a right E-module such that $T_A(M)$ is torsion. If the additive group of M is not torsion, then one may assume that it is torsion-free. As before, an exact sequence $T_A(M) \to$

$T_A(\mathbf{Q}M) \to T_A(\mathbf{Q}M/M) \to 0$ exists. Since the first and third term are torsion groups, $T_A(\mathbf{Q}M) = 0$, which is impossible by b) because $\mathbf{Q}M \neq 0$.

c) $\Rightarrow$ d). Let G be a $\mathbf{Q}A$-solvable group, and consider an exact sequence $0 \to B \xrightarrow{\alpha} C \xrightarrow{\beta} G \to 0$. If $C/[S_A(C) + \alpha(B)]$ is torsion, let $M = \operatorname{im} H_A(\beta) \subseteq H_A(G)$. The map $\theta : T_A(M) \to G$ which is defined by $\theta(\phi \otimes a) = \phi(a)$ for all $\phi \in M$ and $a \in A$ fits into the commutative diagram

$$\begin{tikzcd}
T_AH_A(C) \arrow[r, "T_AH_A(\beta)"] \arrow[d, "\theta_C"'] & T_A(M) \arrow[r] \arrow[d, "\theta"'] & 0 \\
C \arrow[r, "\beta"] & G \arrow[r] & 0.
\end{tikzcd}$$

Since coker $(\beta\theta_C)$ is torsion as an epimorphic image of the torsion group $C/[\alpha(B) + S_A(C)]$, the group $\operatorname{coker}\theta$ has to be torsion. Moreover, θ fits into the commutative diagram

$$\begin{tikzcd}
T_A(M) \arrow[r, "T_A(\iota)"] \arrow[d, "\theta"] & T_AH_A(G) \arrow[r] \arrow[d, "\theta_G"] & T_A(H_A(G)/M) \arrow[r] & 0 \\
G \arrow[r, "1_G"'] & G & &
\end{tikzcd}$$

where $\iota : M \to H_A(G)$ denotes the inclusion map. The Snake-Lemma induces an exact sequence $\ker\theta_G \to T_A(H_A(G)/M) \to \operatorname{coker}\theta$ in which the first and third term are torsion. Therefore, $T_A(H_A(G)/M)$ is torsion. The same holds for the additive group of $H_A(G)/M$ by c); and the given sequence is quasi-A-balanced.

d) $\Rightarrow$ a). Let M be the right $\mathbf{Q}E$-module with $T_A(M) = 0$. Consider an exact sequence $P \xrightarrow{\alpha} F \xrightarrow{\beta} M \to 0$ in which P and F are projective E-modules. An application of the functor T_A yields the exact sequence $0 \to U \to T_A(P) \xrightarrow{T_A(\alpha)} T_A(F) \to T_A(M) = 0$ for some suitable subgroup U of $T_A(P)$. The last sequence is quasi-A-balanced by d) since $T_A(F)$ is A-solvable in view of the self-smallness of A. It induces the top-row of the commutative diagram

$$\begin{tikzcd} H_AT_A(P) \arrow[r, "H_AT_A(\alpha)"] & H_AT_A(F) \arrow[r] & T \arrow[r] & 0 \\ P \arrow[u, "\phi_P"', "\wr"] \arrow[r, "\alpha"] & F \arrow[u, "\phi_F"', "\wr"] \arrow[r] & M \arrow[r, "\phi"'] & 0 \end{tikzcd}$$

in which T is a right E-module whose additive group is torsion. The induced map ϕ is an isomorphism by the 5-lemma. Hence, M^+ is torsion. Since M is a $\mathbf{Q}E$-module, $M = 0$. $\square$

The property of A to be quasi-fully faithful can be characterized in terms of the right ideals of E if A is quasi-flat. As a reminder for the reader, U_* denotes the $\mathbf{Z}$-purification of U in G whenever U is a subgroup of a torsion-free group G.

Corollary 2.2. *The following conditions are equivalent for a quasi-flat torsion-free abelian group A:*

a) *A is quasi-fully faithful.*

b) *$H_A([IA]_*) = I_*$ for all right ideals I of E.*

c) *If I is a right ideal of E such that A/IA is torsion, then E/I is torsion as an abelian group.*

Proof. a) $\Rightarrow$ b). If I is a right ideal of E, then $J = H_A([IA]_*)$ is a pure right ideal of E containing I_*. Let $\phi : E/I \to E/J$ be the natural projection. Since A is quasi-flat, the first term in the induced exact sequence $\mathrm{Tor}_1^E(E/J, A) \to T_A(J/I) \to T_A(E/I) \overset{T_A(\phi)}{\to} T_A(E/J) \to 0$ is a torsion group. Moreover, the map $T_A(\phi)$ fits into the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & IA \arrow[r] \arrow[d, "\iota"] & A \arrow[r] \arrow[d, "1_A"] & T_A(E/I) \arrow[r] \arrow[d, "T_A(\phi)"] & 0 \\ 0 \arrow[r] & JA \arrow[r] & A \arrow[r] & T_A(E/J) \arrow[r] & 0 \end{tikzcd}$$

in which ι is the inclusion map. By the Snake-Lemma, $\ker T_A(\phi) \cong JA/IA \subseteq IA_*/IA$ is a torsion group. Thus, $T_A(J/I)$ is torsion, and the same holds for the additive group of J/I by Theorem 2.1, which is impossible unless $J = I_*$.

b) $\Rightarrow$ c). If A/IA is torsion, $I_* = \mathrm{Hom}\,(A, IA_*) = E$.

c) $\Rightarrow$ a). By Theorem 2.1, it suffices to show that a right E-module M with a torsion-free additive group vanishes if $T_A(M) = 0$. For every $x \in M$, there is a right ideal I of E such that $xE \cong E/I$. Since A is quasi-flat, the first group in the exact sequence $\mathrm{Tor}\,_1^E(M/xE, A) \to T_A(xE) \to T_A(M) = 0$ is torsion. Therefore, $A/IA \cong T_A(E/I) \cong T_A(xE)$ is a torsion group. By c), E/I is torsion as an abelian group. Since M has a torsion-free additive group, $x = 0$. $\square$

Corollary 2.3. *Let A be a quasi-flat torsion-free abelian group.*

a) *If every right ideal of $\mathbf{Q}E$ is the right annihilator of some subset of $\mathbf{Q}E$, then A is quasi-fully faithful.*

b) *If A is strongly indecomposable and $\mathbf{Q}E$ is a finite dimensional $\mathbf{Q}$-algebra, then A is quasi-fully faithful.*

Proof. a) Consider a right ideal I of E and choose $\alpha \in H_A(IA_*)$. For every $a \in A$, there are a nonzero integer m, $a_1, \ldots, a_n \in A$ and $\beta_1, \ldots, \beta_n \in I$ such that $m\alpha(a) = \sum_{j=1}^n \beta_j(a_j)$. Since $\mathbf{Q}I$ is the right annihilator of some $S \subseteq \mathbf{Q}E$ and $\mathbf{Q}E/E$ is torsion as an abelian group, $\mathbf{Q}I$ actually is the right annihilator of a set $S' \subseteq E$ whose elements are nonzero integer multiples of the elements of S. Thus, $m\sigma\alpha(a) = \sum_{j=1}^n \sigma\beta_j(a_j) = 0$ for all $\sigma \in S'$. This shows $S'\alpha = 0$ and $\alpha \in \mathbf{Q}I \cap E = I_*$. By Corollary 2.2, A is quasi-fully faithful.

b) Since $\mathbf{Q}A$ is a finitely generated flat module over the local Artinian ring $\mathbf{Q}E$, it is free. However, nonzero free modules are fully faithful and so A is quasi-fully faithful. $\square$

In particular, A satisfies condition a) in the last corollary if it is a quasi-flat torsion-free abelian group whose quasi-endomorphism ring is a *quasi-Frobenius ring*, i.e., it is a right and left Artinian self-injective ring.

A self-small abelian group A has the finite quasi-Baer splitting property if and only if every finitely generated right E-module M such that

$T_A(M)$ is bounded is itself bounded as an abelian group [**4**, Theorem 2.3]. Using this result it is easy to see that every quasi-fully faithful group A has the finite quasi-Baer splitting property since a finitely generated module whose additive group is torsion has to be bounded. The next result shows that the converse is true if A is almost flat:

Theorem 2.4. *The following conditions are equivalent for a self-small almost flat abelian group A:*

a) *A is quasi-fully faithful.*

b) *A has the finite quasi-Baer-splitting property.*

c) *The class of right E-modules M such that* $\operatorname{coker}\phi_M$ *is torsion is closed with respect to submodules.*

Proof. Throughout this proof, let m be a nonzero integer such that $m\operatorname{Tor}_1^E(-, A) = 0$.

a) $\Rightarrow$ c). Let M be a right E-module such that $\operatorname{coker}\phi_M$ is torsion as an abelian group. Consider a submodule U of M, and assume that it has already been shown that $\ker\phi_{M/U}$ is torsion. The inclusion $U \subseteq M$ induces an exact sequence $0 \to U \xrightarrow{\alpha} M \xrightarrow{\pi} M/U \to 0$ which gives the exact sequence $\operatorname{Tor}_1^E(M/U, A) \xrightarrow{\Delta} T_A(U) \xrightarrow{T_A(\alpha)} T_A(M)$. Setting $V = \operatorname{im}\Delta$ yields the exact sequence $0 \to H_A(V) \to H_AT_A(U) \xrightarrow{H_AT_A(\alpha)} H_AT_A(M)$ in which $H_A(V)$ is bounded by m as an abelian group since the same holds for V as an epimorphic image of the group $\operatorname{Tor}_1^E(M/U, A)$.

For $x \in H_AT_A(U)$, there are $y \in M$ and a nonzero integer k with $H_AT_A(\alpha)(kx) = \phi_M(y)$ since ϕ_M has a torsion cokernel. Because of the naturality of ϕ, one obtains

$$\phi_{M/U}\pi(y) = H_AT_A(\pi)\phi_M(y) = H_AT_A(\pi\alpha)(kx) = 0.$$

Therefore, $l\pi(y) = 0$ for some nonzero integer l since $\ker\phi_{M/U}$ is assumed to be torsion as an abelian group. Thus, write $ly = \alpha(u)$ for some $u \in U$, and obtain $lkH_AT_A(\alpha)(x) = l\phi_M(y) = \phi_M\alpha(u) = H_AT_A(\alpha)\phi_U(u)$. Since $\ker H_AT_A(\alpha)$ is bounded by m, one has $mlk = m\phi_U(u)$. Consequently, ϕ_U has a torsion cokernel.

It remains to show that $\ker\phi_{M/U}$ is torsion as an abelian group. Consider an exact sequence $0 \to W \xrightarrow{\sigma} F \xrightarrow{\delta} M/U \to 0$ of right E-modules in which F is free. It induces the exact sequence

$$0 \to \operatorname{Tor}_1^E(M/U, A) \longrightarrow T_A(W) \xrightarrow{T_A(\sigma)} T_A(F) \xrightarrow{T_A(\delta)} T_A(M/U) \longrightarrow 0.$$

If one sets $K = \operatorname{im} T_A(\sigma)$ and denotes the inclusion $K \subseteq T_A(F)$ by ι, then the previous sequence induces the exact sequences $0 \to \operatorname{Tor}_1^E(M/U, A) \to T_A(W) \xrightarrow{\overline{T_A(\sigma)}} K \to 0$ and $0 \to K \xrightarrow{\iota} T_A(F) \xrightarrow{T_A(\delta)} T_A(M/U) \to 0$ such that $T_A(\sigma) = \iota\overline{T_A(\sigma)}$. Since $\operatorname{Tor}_1^E(M/U, A)$ is bounded, and K is torsion-free as a subgroup of the A-projective group $T_A(F)$, the first of these sequences splits. Consequently, $H_A(\overline{T_A(\sigma)})$ is onto, and $T_A(W) \cong K \oplus T$ for some bounded group T. The second sequence induces the top-row of the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_A(K) & \xrightarrow{H_A(\iota)} & H_AT_A(F) & \xrightarrow{H_AT_A(\delta)} & H_AT_A(M/U) & & \\
 & & \uparrow{\scriptstyle \phi} & & \wr\uparrow{\scriptstyle \phi_F} & & \uparrow{\scriptstyle \phi_{M/U}} & & \\
0 & \longrightarrow & W & \xrightarrow{\sigma} & F & \xrightarrow{\delta} & M/U & \longrightarrow & 0.
\end{array}$$

By the Snake-Lemma, $\ker\phi_{M/U} \cong \operatorname{coker}\phi$, and it suffices to show that the latter is torsion.

To see this, observe that

$$H_A(\iota)\phi = \phi_F\sigma = H_AT_A(\sigma)\phi_W = H_A(\iota)H_A(\overline{T_A(\sigma)})\phi_W$$

yields $\phi = H_A(\overline{T_A(\sigma)})\phi_W$. Since the map $H_a(\overline{T_A(\sigma)})$ is onto, $\operatorname{coker}\phi$ is torsion once it has been established that $\operatorname{coker}\phi_W$ is torsion as an abelian group.

Consider an exact sequence $P \xrightarrow{\lambda} W \to 0$ of right E-modules with P projective. It induces an epimorphism $T_A(P) \xrightarrow{T_A(\lambda)} T_A(W)$. Because $T_A(W) \cong K \oplus T$, one obtains $\ker\theta_{T_A(W)} \cong \ker\theta_K \oplus \ker\theta_T$. Since T is bounded, the same holds for $T_AH_A(T)$, and $\ker\theta_T$ is bounded. Moreover, the commutative diagram

$$\begin{array}{ccccc}
\operatorname{Tor}_1^E(H_AT_A(F)/H_A(K), A) & \xrightarrow{\Delta} & T_AH_A(K) & \xrightarrow{T_AH_A(\iota)} & T_AH_AT_A(F) \\
 & & \downarrow{\scriptstyle \theta_K} & & \wr\downarrow{\scriptstyle \theta_{T_A(F)}} \\
0 & \longrightarrow & K & \xrightarrow{\iota} & T_A(F)
\end{array}$$

yields that $\ker \theta_K = \ker T_A H_A(\iota) = \operatorname{im} \Delta$ is bounded by m. Therefore, $T_A(W)$ is $\mathbf{Q}A$-solvable since $S_A(T_A(W)) = T_A(W)$. By Theorem 2.1, the sequence $T_A(P) \stackrel{T_A(\lambda)}{\to} T_A(W) \to 0$ is quasi-A-balanced. There is a right E-module T_1 which is torsion as an abelian group and makes the top row of the commutative diagram

$$\begin{tikzcd}
H_A T_A(P) \arrow[r, "H_A T_A(\lambda)"] & H_A T_A(W) \arrow[r] & T_1 \arrow[r] & 0 \\
P \arrow[u, "\phi_P", "\wr"'] \arrow[r, "\lambda"] & W \arrow[u, "\phi_W"] \arrow[r] & 0 &
\end{tikzcd}$$

exact. Since λ is onto, $\operatorname{coker} \phi_W = \operatorname{coker} \phi_W \lambda = \operatorname{coker} H_A T_A(\lambda)\phi_P = \operatorname{coker} H_A T_A(\lambda) \cong T_1$ and ϕ_W has a torsion cokernel.

c) $\Rightarrow$ b). Let M be a nonzero finitely generated right E-module such that $T_A(M)$ is bounded. An exact sequence $0 \to U \stackrel{\alpha}{\to} F \to M \to 0$ of right E-modules where F is a finitely generated free module induces the sequence $0 \to \operatorname{Tor}_1^E(M, A) \to T_A(U) \stackrel{T_A(\alpha)}{\to} T_A(F) \to T_A(M) \to 0$ in which $\operatorname{Tor}_1^E(M, A)$ and $T_A(M)$ are bounded abelian groups. Therefore, $T_A(\alpha)$ and hence $H_A T_A(\alpha)$ are quasi-isomorphisms. There is a nonzero integer l such that $l \operatorname{coker} H_A T_A(\alpha) = 0$. One has $l\phi_F(x) = H_A T_A(\alpha)(z)$ for some $z \in H_A T_A(U)$ whenever $x \in F$. By c), ϕ_U has a torsion cokernel, and therefore, $kz = \phi_U(u)$ for some nonzero integer k and $u \in U$. Thus, $lk\phi_F(x) = H_A T_A(\alpha)\phi_U(u) = \phi_F \alpha(u)$ since $H_A T_A(\alpha)\phi_U = \phi_F \alpha$. Consequently, $lkx = \alpha(u)$ and M^+ is torsion. Since M is finitely generated, M^+ is bounded.

b) $\Rightarrow$ a). If A is not quasi-fully faithful, then there is a right E-module M such that $T_A(M)$ torsion but $tM \neq M$. No generality is lost, if one assumes that the additive group of M is torsion-free. An exact sequence $0 \to M \to \mathbf{Q}M$ of right E-modules exists, which induces the exact sequence $\operatorname{Tor}_1^E([\mathbf{Q}M]/M, A) \stackrel{\Delta}{\to} T_A(M) \to T_A(\mathbf{Q}M)$. Since $T_A(\mathbf{Q}M)$ is torsion-free and divisible and $T_A(M)$ is torsion, Δ is onto, and $T_A(M)$ is bounded by m as an image of $\operatorname{Tor}_1^E([\mathbf{Q}M]/M, A)$. If U is a finitely generated submodule of M, then the inclusion $U \subseteq M$, induces an exact sequence $\operatorname{Tor}_1^E(M/U, A) \to T_A(U) \to T_A(M)$ which yields $m^2 T_A(U) = 0$. Since A has the finite quasi-Baer splitting property, U is bounded as an abelian group which is not possible. $\square$

This section concludes with an example that the implication b) $\Rightarrow$ a) in the last theorem may fail if A is not almost flat:

Example 2.5. a) There exists a quasi-flat torsion-free abelian group A of rank 4 which has the finite quasi-Baer-splitting property but is not quasi-fully faithful.

b) There exists a quasi-flat quasi-fully faithful torsion-free group A of rank 4 which is not almost flat.

Proof. Select subgroups X and Y of $\mathbf{Q}$ containing $\mathbf{Z}$ with the property that 1 has height sequences $(0,1,0,1,\dots)$ in X and $(1,0,1,0,1,0,\dots)$ in Y. As in [**12**], choose an isomorphism $\theta : \mathbf{Q}/X \to \mathbf{Q}/Y$, and set $G = \{(a,b) \in \mathbf{Q} \oplus \mathbf{Q} \mid \theta(a+X) = b+Y\}$. Since $X \cong X \oplus \{0\}$ and $Y \cong \{0\} \oplus Y$ are pure in G, the typeset of G has at least three elements. By [**8**, Theorem 3.3], $E(G) \subseteq \mathbf{Q}$. Since $G/(X \oplus Y) \cong \mathbf{Q}/\mathbf{Z}$, one can find a subgroup H of G containing $X \oplus Y$ such that $G/H \cong \mathbf{Z}(2^\infty)$ (see also [**6**] and [**7**]). In the same way as for G, one shows $E(H) \subseteq \mathbf{Q}$. Since $r_2(H) = 2$, one obtains $\mathbf{Z}_2 \otimes H \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$. Therefore, the group $(f(H)+H)/H$ is finite for any $f \in \mathrm{Hom}\,(H,G)$. Consequently, $\mathrm{Hom}\,(H,G)$ has rank 1.

Any homomorphism from H into G induces an endomorphism of X since X is pure and fully invariant in G. Hence, $\mathrm{Hom}\,(H,G) \cong \mathbf{Z}$ which implies that $S_H(G) \doteq H$. In particular, $G/S_H(G)$ is an infinite torsion group. On the other hand, $\mathrm{Hom}\,(G,H) = 0$ since G is not quasi-isomorphic to H.

a) By [**4**, Corollary 3.3], the group $A = G \oplus H$ has the finite quasi-Baer splitting property if $S_H(G) \neq G$ and $S_G(H) \neq H$, and these conditions are indeed satisfied.

The left ideal $I = \{\alpha \in E \mid \alpha(G) = 0\}$ of $E = E(A)$ is a two-sided ideal since $\mathrm{Hom}\,(G,H) = 0$ yields that G is fully invariant in A. Furthermore, $IA = IH \supseteq H \oplus \mathrm{Hom}\,(H,G)H = H \oplus S_H(G)$ yields that A/IA is torsion as an image of the torsion group $G/S_H(G)$. On the other hand, since E/I is nonzero and torsion-free, A is not quasi-fully faithful by Theorem 2.1.

It remains to show that A is quasi-flat. By [**7**, Corollary 3.2] (Ulmer's theorem), it suffices to show that $[\ker\phi]/S_A(\ker\phi)$ is torsion for all maps $\phi \in \mathrm{Hom}\,(A^n, A)$ and all $n < \omega$. Consider a homomorphism $\phi : U \oplus V \to A$ where $U \cong G^n$ and $V \cong H^n$ for some $n < \omega$. Set $K = \ker \phi$, and observe that U contains a subgroup $U' \cong H^n$ such that U/U' is torsion. Moreover, $\phi(U' \oplus V) \subseteq S_H(G \oplus H) = S_H(G) \oplus H$. Since $S_H(G) \doteq H$, one obtains that $\phi(U' \oplus V)$ is isomorphic to a subgroup of $H \oplus H$. Because of $E(H) \subseteq \mathbf{Q}$, one has $S_A(K) \supseteq S_H(K \cap [U' \oplus V]) = K \cap [U' \oplus V]$. Consequently, $K/S_A(K)$ is an epimorphic image of $K/(K \cap [U' \oplus V]) \cong (K + [U' \oplus V])/[U' \oplus V] \subseteq (U \oplus V)/(U' \oplus V)$. Since the latter group is torsion, $K/S_A(K)$ is torsion.

b) On the other hand, if the exact sequence $0 \to G \to A \to H \to 0$ represents an element of infinite order of $\mathrm{Ext}\,(H, G)$, then it was shown in [**6**] and [**7**] that A is strongly indecomposable, quasi-flat group which is not almost flat. Because of part a) of Corollary 2.3, A is quasi-fully faithful. $\square$

Interestingly, 4 is the smallest rank for which a quasi-flat group A of finite rank exists which has the finite quasi-Baer splitting property but is not quasi-fully faithful. By [**7**, Theorem 4.3], every quasi-flat torsion-free group A of rank at most 3 is quasi-isomorphic to a group which is flat as a module over its endomorphism ring. By [**6**, Corollary 2.5], A is almost flat. Therefore, A has the finite quasi-Baer splitting property if and only if A is quasi-fully faithful.

3. Annihilator conditions. Write $r_R(S)$ and $l_R(S)$ for the right and left annihilator of a subset S of a ring R, respectively. A right and left Noetherian ring R is a quasi-Frobenius ring if and only if every right (left) ideal of R is the right (left) annihilator of a finite subset of R. It is the goal of this section to study torsion-free abelian groups whose quasi-endomorphism rings satisfy one (both) of these annihilator conditions.

Lemma 3.1. *The following are equivalent for a torsion-free abelian group A:*

a) *Every right ideal of $\mathbf{Q}E$ is the right annihilator of a subset of $\mathbf{Q}E$.*

b) *$r_E l_E(I)/I$ is $\mathbf{Z}$-torsion for all right ideals I of E.*

Proof. a) $\Rightarrow$ b). Let I be a right ideal of E whose $\mathbf{Z}$-purification in E is denoted by I_*. If S is a subset of $\mathbf{Q}E$ with $\mathbf{Q}I_* = r_{\mathbf{Q}E}(S)$, then no generality is lost if one assumes $S \subseteq E$. Then, $r_E(S) = r_{\mathbf{Q}E}(S) \cap E = \mathbf{Q}I_* \cap E = I_*$. Thus, $r_E l_E(I)/I = r_E l_E(I_*)/I = I_*/I$ is torsion.

b) $\Rightarrow$ a). Observe that $r_E l_E(I) = I$ for all pure right ideals I of E because of b). If J is any right ideal of $\mathbf{Q}E$, then $l_{\mathbf{Q}E}(J) = l_{\mathbf{Q}E}(J \cap E)$ since $J/J \cap E$ is torsion. But then, $J = r_{\mathbf{Q}E}(l_E(J \cap E)) \supseteq r_{\mathbf{Q}E} l_{\mathbf{Q}E}(J) \supseteq J$. □

A torsion-free group G such that $G/S_A(G)$ is torsion has *finite A-rank* if every descending chain $U_0 \supseteq U_1 \supseteq \cdots$ of pure subgroups of G such that $U_n/S_A(U_n)$ is torsion for all $n < \omega$ has finite length. If A has finite rank, then a torsion-free group G such that $G/S_A(G)$ is torsion has finite A-rank if and only if G has finite rank.

Theorem 3.2. *Let A be a quasi-flat, quasi-fully faithful torsion-free abelian group whose quasi-endomorphism ring is right Artinian.*

a) *The following conditions are equivalent for a torsion-free $\mathbf{Q}A$-solvable group G:*

i) *G has finite A-rank.*

ii) *$\mathbf{Q}H_A(G)$ is a finitely generated right $\mathbf{Q}E$-module.*

iii) *G contains a finitely A-generated subgroup U such that G/U is torsion.*

b) *Let G be a torsion-free abelian group. Then $\overline{G} = G/R_A(G)$ is a $\mathbf{Q}A$-solvable group of finite A-rank if and only if the following two conditions are satisfied*

i) *$\overline{G}/S_A(\overline{G})$ is torsion.*

ii) *For every homomorphism $\sigma : G \to A^I$, there is a finite subset J of I such that $\ker \sigma = \ker \pi_J \sigma$ where $\pi_J : A^I \to A^J$ is the projection with kernel $A^{I \setminus J}$.*

Proof. i) $\Rightarrow$ ii). Since $\mathbf{Q}E$ is a right Artinian ring, $\mathbf{Q}H_A(G)$ is finitely generated if and only if it is Artinian. Consider submodules U_1 and U_2 of $\mathbf{Q}H_A(G)$ such that U_1 is a proper submodule of U_2. The $\mathbf{Z}$-pure E-submodules $V_1 = U_1 \cap H_A(G) \subseteq U_2 \cap H_A(G) = V_2$ of $H_A(G)$

satisfy $V_1 \neq V_2$ since $\mathbf{Q}V_i = U_i$ for $i = 1, 2$. The inclusion maps $i_1 : V_1 \to H_A(G)$, $i_2 : V_2 \to H_A(G)$ and $\iota : V_1 \to V_2$ induce the commutative diagram

$$\begin{tikzcd}
 & \operatorname{Tor}_1^E(V_2/V_1, A) \arrow[d] & \\
\operatorname{Tor}_1^E(H_A(G)/V_1, A) \arrow[r, "\Delta_1"] & T_A(V_1) \arrow[r, "T_A(\iota_1)"] \arrow[d, "T_A(\iota)"] & T_AH_A(G) \arrow[d, "1_{T_AH_A(G)}"] \\
\operatorname{Tor}_1^E(H_A(G)/V_2, A) \arrow[r, "\Delta_2"] & T_A(V_2) \arrow[r, "T_A(\iota_2)"] \arrow[d] & T_AH_A(G) \\
 & T_A(V_2/V_1). &
\end{tikzcd}$$

Since V_2/V_1 is torsion-free, $T_A(V_2/V_1)$ cannot be torsion by Theorem 2.1 because A is quasi-fully faithful. Thus, one can choose an element x of $T_A(V_2)$ such that $\mathbf{Z}x \cap \operatorname{im} T_A(\iota) = 0$. Then $m\theta_G T_A(\iota_2)(x) \notin \operatorname{im} \theta_G T_A(\iota_1)$ for all nonzero integers m. Otherwise, there is $y \in T_A(V_1)$ such that $\theta_G T_A(\iota_2)(mx) = \theta_G T_A(\iota_1)(y) = \theta_G T_A(\iota_2 \iota)(y)$. Since $\ker \theta_G$ is torsion, there is a nonzero integer k such that $T_A(\iota_2)(kmx) = T_a(\iota_2)T_A(\iota)(ky)$. Consequently, $kmx - T_A(\iota)(ky) \in \ker T_A(\iota_2) = \operatorname{im} \Delta_2$ which is torsion since A is quasi-flat. There is a nonzero integer l with $lkmx = T_A(\iota)(lky)$ which is not possible in view of the choice of x. Setting $W_i = [\theta_G T_A(\iota_i)](T_A(V_i))$ for $i = 1, 2$ defines A-generated subgroups of G such that $W_1 \subset W_2$ but W_2/W_1 is not torsion as an abelian group. Consequently, $(W_1)_*$ is a proper subgroup of $(W_2)_*$, and $(W_i)_*/S_A((W_i)_*)$ is torsion for $i = 1, 2$.

Therefore, a strictly descending chain of $\mathbf{Q}E$-submodules of $\mathbf{Q}H_A(G)$ which has infinite length induces a strictly descending chain $\{X_n \mid n < \omega\}$ of pure subgroups of G such that $X_n/S_A(X_n)$ is torsion which is not possible by i).

ii) $\Rightarrow$ iii). Since $\mathbf{Q}H_A(G)$ is finitely generated, an exact sequence $\mathbf{Q}E^n \xrightarrow{\pi} \mathbf{Q}H_A(G) \to 0$ exists. But $[\pi(E^n) + H_A(G)]/H_A(G)$ is a finitely generated E-module whose additive group is torsion, so there is a nonzero integer m such that $V = m\pi(E^n) \subseteq H_A(G)$. An application of T_A to the inclusion map $\iota : V \to H_A(G)$ yields the exact sequence $T_A(V) \stackrel{T_A(\iota)}{\to} T_AH_A(G) \to T_A(H_A(G)/V) \to 0$. Then $U = \theta_G T_A(\iota)(T_A(V))$ is a finitely A-generated subgroup of G such that G/U is torsion.

iii) $\Rightarrow$ i). Suppose that it has already been shown that $\mathbf{Q}H_A(G)$ is an Artinian $\mathbf{Q}E$-module. Consider a descending chain $U_0 \supseteq U_1 \supseteq \cdots$ of pure subgroups of G such that $U_n/S_A(U_n)$ is torsion for all $n < \omega$. The submodules $\{V_n = H_A(U_n) \mid n < \omega\}$ of $H_A(G)$ form a descending chain of submodules of $H_A(G)$ such that $H_A(G)/V_n$ is torsion-free. Since $S_A(U_n) = V_nA$, one obtains $V_n = V_{n+1}$ if and only if $S_A(U_n) = S_A(U_{n+1})$. But $U_n = S_A(U_n)_*$ yields $V_n = V_{n+1}$ if and only if $U_n = U_{n+1}$. Because $\mathbf{Q}H_A(G)$ is an Artinian $\mathbf{Q}E$-module, an index m exists such that $\mathbf{Q}V_n = \mathbf{Q}V_m$ for all $m \leq n < \omega$. Then $U_n = U_m$ for all those n since $V_n = \mathbf{Q}V_n \cap H_A(G)$.

It remains to show that $\mathbf{Q}H_A(G)$ is Artinian. Since $\mathbf{Q}E$ is a right Artinian ring, it suffices to establish that $\mathbf{Q}H_A(G)$ is finitely generated. Choose a finitely A-generated subgroup H of G such that G/U is torsion, and consider the induced exact sequence $0 \to U \to G \stackrel{\pi}{\to} G/U \to 0$. It induces the exact sequence $0 \to H_A(U) \to H_A(G) \stackrel{H_A(\pi)}{\to} M \to 0$ where $M = \operatorname{im} H_A(\pi)$ is a submodule of $H_A(G/U)$. The latter yields the top row of the commutative diagram

$$\begin{array}{ccccccccc} & & T_AH_A(U) & \longrightarrow & T_AH_A(G) & \xrightarrow{T_AH_A(\pi)} & T_A(M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta_U} & & \downarrow{\scriptstyle \theta_G} & & \downarrow{\scriptstyle \theta} & & \\ 0 & \longrightarrow & U & \longrightarrow & G & \xrightarrow{\pi} & G/U & \longrightarrow & 0 \end{array}$$

Since $S_A(U) = U$, the Snake-Lemma gives the exact sequence $\ker\theta_G \to \ker\theta \to 0$ whose first term is a torsion group because G is $\mathbf{Q}A$-solvable. Therefore, $T_A(M)$ is torsion. In view of the fact that A is quasi-fully faithful, M is torsion as an abelian group by Theorem 2.1. Then $\mathbf{Q}H_A(G) = \mathbf{Q}H_A(U)$.

By [**7**, Corollary 3.2], U is a $\mathbf{Q}A$-solvable abelian group. Since U is finitely A-generated, an exact sequence $0 \to W \to A^n \to U \to 0$ exists for some $n < \omega$. By Theorem 2.1, this sequence is quasi-A-balanced and induces an exact sequence $0 \to H_A(W) \to H_A(A^n) \to H_A(U) \to T \to 0$ in which T is a right E-module whose additive group is torsion. An application of the exact function $\mathbf{Q}\otimes_{\mathbf{Z}}$ shows that $\mathbf{Q}H_A(U)$ is a finitely generated $\mathbf{Q}E$-module.

b) Suppose that $\overline{G} = G/R_A(G)$ is a $\mathbf{Q}A$-solvable group of finite A-rank. Let $U = \ker\sigma$. Since $R_A(G/U) = 0$, one can find an index set I and a homomorphism $\sigma : G \to A^I$ with $\ker\sigma = U$. Suppose that $\ker(\pi_J\sigma) \neq U$ whenever J is a finite subset of I. There is a sequence $\{i_n \mid n < \omega\}$ of indices in I such that $U_n = \cap_{j=0}^{n} \ker\pi_{i_j}$ contains U_{n+1} as a proper subgroup. Since $\overline{U_n} = U_n/R_A(G)$ is the kernel of a map from $\overline{G}$ into A^{n+1}, one obtains $\overline{U_n}/S_A(\overline{U_n})$ is torsion because A is quasi-flat and $\overline{G}$ is $\mathbf{Q}A$-solvable. But this contradicts the fact that $\overline{G}$ has finite A-rank. Therefore, there is a finite subset J of I such that $\ker(\pi_J\sigma) = U$.

Conversely, there is a homomorphism $\sigma : G \to A^I$ for some index-set I such that $R_A(G) = \ker\sigma$. Choose a finite subset J of I such that $R_A(G) = \ker\pi_J\sigma$. Therefore, $\overline{G}$ is isomorphic to a subgroup of A^n for some $n < \omega$ and is $\mathbf{Q}A$-solvable by [**7**, Corollary 3.2]. Moreover, $\mathbf{Q}H_A(\overline{G})$ is a finitely generated $\mathbf{Q}E$-module since it is isomorphic to a submodule of the finitely generated module $\mathbf{Q}H_A(A^n)$ and $\mathbf{Q}E$ is right Artinian. $\square$

Corollary 3.3. *Let A be a quasi-flat torsion-free abelian group such that $\mathbf{Q}E$ has the* ACC *for right and left annihilators. For every homomorphism $\sigma : A \to A^I$, there is a finite subset J of I such that* $\ker\sigma = \ker\pi_J\sigma$.

Proof. If G is chosen to be A in the proof of part b) of Theorem 3.2, then the arguments used there construct a strictly descending chain of right annihilators in $\mathbf{Q}E$ which gives rise to a strictly ascending chain of left annihilators, which is impossible. Now continue as in the proof of part b) of Theorem 3.2 observing that the fact that A is quasi-fully faithful is not used in the arguments which are relevant for the proof of this corollary. $\square$

Corollary 3.4. *Let A be a torsion-free abelian group which is quasi-flat.*

a) *The following conditions are equivalent if $\mathbf{Q}E$ is right and left Noetherian*:

i) *Every right ideal of $\mathbf{Q}E$ is the right annihilator of a finite subset of $\mathbf{Q}E$.*

ii) *A is a quasi-fully faithful group such that every pure subgroup U of A for which $U/S_A(U)$ is torsion satisfies $R_A(A/U) = 0$.*

b) *Suppose that $\mathbf{Q}E$ is a right Artinian ring such that $l_E r_E(I)/I$ is torsion for all left ideals I of E. If P is isomorphic to an A-projective group of finite A-rank, then an exact sequence $0 \to P \xrightarrow{\alpha} G$ of torsion-free groups quasi-splits if and only if $R_A(G) \cap \alpha(P) = 0$.*

Proof. a) i) $\Rightarrow$ ii). Observe that A is quasi-fully faithful by Corollary 2.3. If U is a pure subgroup of A such that $U/S_A(U)$ is torsion, then $H_A(U)$ is a pure right ideal of E for which $\alpha_1, \dots, \alpha_t \in E$ with $\mathbf{Q}H_A(U) = r_{\mathbf{Q}E}(\alpha_1, \dots, \alpha_t)$. Define a map $\lambda : A \to A^t$ by $\lambda(a) = (\alpha_1(a), \dots, \alpha_t(a))$. To show ii), it is enough to verify that U is the kernel of λ.

Consider $u \in U$. Since $U/S_A(U)$ is torsion, one can find a nonzero integer k, maps $\phi_1, \dots, \phi_s \in H_A(U)$ and $a_1, \dots, a_s \in A$ such that $ku = \Sigma_{j=1}^s \phi_j(a_j)$. Then $k\alpha_i(u) = \Sigma_{j=1}^s \alpha_i \phi_j(a_j) = 0$, and $U \subseteq \ker \lambda$. If $\beta \in H_A(\ker \lambda)$, then $\alpha_i \beta = 0$ yields $\beta \in r_{\mathbf{Q}E}(\alpha_1, \dots, \alpha_t) \cap E = H_A(U)$. Therefore, $S_A(\ker \lambda) \subseteq U$. The quasi-flatness of A guarantees that the group $\ker / S_A(\ker \lambda)$ is torsion [**7**, Theorem 3.1], and so $U = \ker \lambda$.

ii) $\Rightarrow$ i). Let I be a right ideal of E, and set $U = (I_* A)_*$. Because of c), $R_A(A/U) = 0$. By Corollary 3.3, there is a finite set T such that A/U is isomorphic to a subgroup of A^T. Consequently, there are $\beta_1, \dots, \beta_n \in E$ with the property that $U = \cap_{j=1}^n \ker \beta_j$. If $\alpha \in I_*$, then $\alpha(A) \subseteq U$, and so $\beta_i \alpha = 0$ for $i = 1, \dots, n$. Thus, $I_* \subseteq r_{\mathbf{Q}E}(\beta_1, \dots, \beta_n)$. Conversely, if $\phi \in r_{\mathbf{Q}E}(\beta_1, \dots, \beta_n) \cap E$, then $\phi(A) \subseteq \cap_{i=1}^n \ker \beta_i = U$ and $\phi \in H_A(U) = I_*$ since A is quasi-flat. Therefore, $I_* = r_{\mathbf{Q}E}(\beta_1, \dots, \beta_n) \cap E$. Hence, $\mathbf{Q}I = \mathbf{Q}I_* = r_{\mathbf{Q}E}(\beta_1, \dots, \beta_n)$.

b) Let I be a proper left ideal of $\mathbf{Q}E$ such that $r_{\mathbf{Q}E}(I) = 0$. Then $r_E(I \cap E) = 0$ and $E = l_E r_E(I \cap E) = I \cap E$ since $I \cap E$ is pure in E. But this is not possible since I is proper. By [**3**, Theorem 2.3], every exact sequence $0 \to P \to F$ such that F is a quasi-summand of an A-projective group of finite A-rank quasi-splits.

Now consider an exact sequence $0 \to P \xrightarrow{\alpha} G$ in which $\alpha(P) \cap R_A(G) = 0$. Since $G/R_A(G)$ is isomorphic to a subgroup of A^I for some index-set I, one may assume that $G = A^* I$. By Theorem 3.2, there is a finite subset J of I such that $\pi_{J\alpha}$ is a monomorphism where $\pi_J : A^I \to A^J$ is the projection with kernel $A^{I\setminus J}$. By the result of the first paragraph, $\pi_J\alpha$ quasi-splits; and so does α. □

The main result of this section gives two characterizations of the quasi-flat torsion-free groups of finite rank whose quasi-endomorphism ring is quasi-Frobenius. To formulate one of these easier, call a sequence $0 \to B \xrightarrow{\alpha} G$ *quasi-A-cobalanced* if, for every $\phi \in H_A(B)$, there is a nonzero integer m and a map $\psi \in \mathrm{Hom}\,(G, A)$ with $\psi|_B = m\phi$. Moreover, if X is a subset of an abelian group A, then $X^* = \{\alpha \in E \mid \alpha(X) = 0\}$, while $S^* = \{a \in A \mid Sa = 0\}$ if S is a subset of E [**10**].

Theorem 3.5. *The following are equivalent for a quasi-flat torsion-free abelian group A such that $\mathbf{Q}E$ is right and left Noetherian.*

a) i) *Every A-generated subgroup of A is $\mathbf{Q}A$-solvable.*

ii) *$\mathbf{Q}E$ is a quasi-Frobenius ring.*

b) i) *A is quasi-fully faithful.*

ii) *If U is a pure subgroup of A^n for some $n < \omega$, then $U/S_A(U)$ is torsion if and only if $R_A(A/U) = 0$.*

iii) *If I is a pure left ideal of E, then $I = X^*$ for some subset X of A.*

c) i) *A is quasi-flat.*

ii) *An exact sequence $0 \to B \xrightarrow{\alpha} G$ such that $B/R_A(B)$ is $\mathbf{Q}A$-solvable and has finite A-rank is quasi-A-cobalanced if and only if $\alpha(R_A(B)) = \alpha(B) \cap R_A(G)$.*

Proof. a) $\Rightarrow$ b) and c). In order to show c) i), it suffices to verify that $\mathrm{Tor}\,_1^E(E/I, A)$ is torsion for all right ideals I of E. Since $\mathbf{Q}E$ is a quasi-Frobenius ring, $\mathbf{Q}I$ is the right annihilator of a finite subset $\{\alpha_1, \dots, \alpha_n\}$ of $\mathbf{Q}E$. The fact that $\mathbf{Q}E/E$ is torsion allows to assume $\{\alpha_1, \dots, \alpha_n\} \subseteq E$. Observe that $I \subseteq r_E(\alpha_1, \dots, \alpha_n)$ and $r_e(\alpha_1, \dots, \alpha_n)/I$ is torsion. Define a map $\lambda : A \to A^n$ by $\lambda(a) = (\alpha_1(a), \dots, \alpha_n(a))$ whose kernel is denoted by K. Then $H_A(K) = r_E(\alpha_1, \dots, \alpha_n)$, and the exact sequence $0 \to H_A(K)/I \to E/I \to E/H_A(K) \to 0$ induces $\mathrm{Tor}\,_1^E(H_A(K)/I, A) \to \mathrm{Tor}\,_1^E(E/I, A) \to \mathrm{Tor}\,_1^E(E/H_A(K), A)$ in which the first term is a torsion group. Therefore, it is enough to show that the last term has this property, too. For this we consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Tor}\,_1^E(H_A(A)/H_A(K), A) \arrow[r] & T_A H_A(K) \arrow[r] \arrow[d, "\theta_K"] & T_A H_A(A) \arrow[d, "\theta_A", "\wr"'] \\
 & 0 \arrow[r] & \ker\lambda \arrow[r] & A
\end{tikzcd}$$

which yields $\mathrm{Tor}\,_1^E(H_A(A)/H_A(K)) \cong \ker\theta_K$. But $\ker\theta_K = \ker\theta_{S_A(K)}$, and the latter is torsion by a).

The first step in the proof of c) ii) is to assume that $G = A^n$ for some $n < \omega$. The exact sequence $0 \to B \stackrel{\alpha}{\to} A^n$ induces an exact sequence $0 \to \mathbf{Q}H_A(B) \stackrel{\mathbf{Q}H_A(\alpha)}{\to} \mathbf{Q}H_A(A^n)$ of right $\mathbf{Q}E$-modules. There is a map $\psi : \mathbf{Q}H_A(A^n) \to \mathbf{Q}E$ with $\psi\mathbf{Q}H_A(\alpha) = \mathbf{Q}H_A(\phi)$ since $\mathbf{Q}E$ is self-injective. Choose a nonzero integer m such that $m\psi(H_A(A^n)) \subseteq E$. Then $m\psi H_A(\alpha) = H_A(m\phi)$. Apply T_A to obtain $T_A(m\psi)T_A H_A(\alpha) = T_A H_A(m\phi)$. Then

$$m\phi\theta_B = \theta_A T_A H_A(m\phi) = \theta_A T_A(m\psi) T_A H_A(\alpha) = \theta_A T_A(m\psi)\theta_A^{-1}\alpha\theta_B.$$

Since $B = S_A(B)_*$ is torsion, $m\phi = \theta_A T_A(m\psi\theta_A^{-1}\alpha)$.

For the general case, consider an exact sequence $0 \to B \stackrel{\alpha}{\to} G$ such that $\overline{B} = B/R_A(B)$ is a $\mathbf{Q}A$-solvable group of finite A-rank and $\alpha(R_A(B)) = \alpha(B) \cap R_A(G)$. This sequence induces the sequence $0 \to \overline{B} \stackrel{\bar\alpha}{\to} \overline{G}$ where $\overline{G} = G/R_A(G)$. Let $\sigma : \overline{G} \to A^I$ be any monomorphism. By Theorem 3.2, I may be chosen to be finite since quasi-Frobenius rings are right and left Artinian. Observe that A is quasi-fully faithful because of the already established equivalence in Corollary 3.4 a). Every map $\phi : B \to A$ induces a map $\bar\phi : \overline{B} \to A$. By the first part of this proof, there are a map $\psi : A^I \to A$ and a nonzero integer m such that $m\bar\phi = \psi\sigma\bar\alpha$. If $\pi : B \to \overline{B}$ is the projection map, then $\bar\alpha\pi = \alpha$ and $\bar\phi\pi = \phi$. Thus, $\psi\sigma\alpha = m\phi$.

Conversely, assume that $0 \to B \stackrel{\alpha}{\to} G$ is quasi-A-cobalanced. Since $\alpha(R_A(B)) \subset R_A(G)$, consider $b \in B$ such that $\alpha(b) \in R_A(G))$. If $\phi \in \mathrm{Hom}\,(B, A)$, then there are a nonzero integer m and a map $\psi \in \mathrm{Hom}\,(G, A)$ such that $m\phi = \psi\alpha$. Then $m\phi(b) = \psi\alpha(b) = 0$, and $b \in R_A(B)$.

To verify b) it remains to show ii) and iii) because of Corollary 3.4. For the former, consider a pure subgroup U of A^n for some $n < \omega$. If $R_A(A/U) = 0$, then there is a map $\alpha : A^n \to A^I$ for some index-set I with $\ker\alpha = U$. Since $\mathbf{Q}E$ is right and left Noetherian, I may be chosen to be finite by Corollary 3.3. Hence, U is the kernel of a homomorphism between the A-solvable groups A^n and A^I. By [**7**, Theorem 3.1], $U/S_A(U)$ is torsion since A is quasi-flat.

Conversely, assume that $U/S_A(U)$ is torsion and observe that the inclusion $U \subseteq A^n$ induces an exact sequence $0 \to H_A(U) \to H_A(A^n) \to M \to 0$ where M is submodule of $H_A(A^n/U)$. Since U is pure in A^n, the additive group of M is torsion-free. Furthermore, $\mathbf{Q}M$ is isomorphic to a submodule of $\mathbf{Q}E^m$ for some $m < \omega$ since finitely generated modules over a quasi-Frobenius ring are reflexive. If this embedding is denoted by α, then one can find a nonzero integer k such that $k\alpha(A) \subseteq E^m$. The monomorphism $k\alpha$ induces the exact sequence $0 \to \operatorname{Tor}_1^E(E^m/k\alpha(M)) \xrightarrow{\Delta} T_A(M) \xrightarrow{T_A(k\alpha)} T_A(E^m)$. Since A is quasi-flat, and $T_A(E^m)$ is torsion-free, $tT_A(M) = \operatorname{im}\Delta$, and $T_A(M)/tT_A(M)$ is isomorphic to a subgroup of A^m. Consider the induced diagram

$$\begin{tikzcd}
 & T_AH_A(U) \arrow[r] \arrow[d, "\theta_U"] & T_AH_A(A^n) \arrow[r] \arrow[d, "\theta_{A^n}", "\wr"'] & T_A(M) \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & U \arrow[r] & A^n \arrow[r] & A^n/U \arrow[r] & 0
\end{tikzcd}$$

in which the induced map θ is an epimorphism whose kernel is torsion by the Snake-Lemma since $U/S_A(U)$ is torsion. But the fact that A^n/U is torsion-free yields $\ker\theta = tT_A(M)$. Thus $A^n/U \cong T_A(M)/tT_A(M)$ is isomorphic to a subgroup of A^m.

For the proof of iii), consider a pure left ideal I of E and choose $\beta_1, \dots, \beta_s \in E$ such that $\mathbf{Q}I = l_{\mathbf{Q}E}(\beta_1, \dots, \beta_s)$. In particular, $I\beta_i = 0$ for $i = 1, \dots, n$. If $\gamma \in E$ satisfies $\gamma(\beta_a(A) + \cdots + \beta_s(A)) = 0$, then $\gamma \in \mathbf{Q}I \cap E = I$ by the $\mathbf{Z}$-purity of I in E.

b) $\Rightarrow$ a). Observe that the kernel of every map $\alpha : A^n \to A$ satisfies $R_A(A^n/\ker\alpha) = 0$. Thus, $\ker\alpha/S_A(\ker\alpha)$ is torsion by b ii) and A is quasi-flat by [**7**, Corollary 3.1]. An application of [**7**, Theorem 3.1] yields that the class of torsion-free $\mathbf{Q}A$-solvable groups is closed with respect to A-generated subgroups. This shows a) i).

In order to establish a) ii), by part a) of Corollary 3.4, it remains to show that every left ideal I of $\mathbf{Q}E$ is the left annihilator of some finite subset of $\mathbf{Q}E$. Let $J = E \cap I$ and choose $\mathbf{Q}E$-generators $\alpha_1, \dots, \alpha_t \in J$ of I. Define a map $\lambda : A \to A^t$ by $\lambda(a) = (\alpha_1(a), \dots, \alpha_t(a))$. For $\alpha \in J$, find a nonzero integer m and $r_1, \dots, r_t \in e$ with $m\alpha = \Sigma_{i=1}^t r_i \alpha_i$. Therefore, $\alpha(\ker \lambda) = 0$ and $\ker \lambda \subseteq J^* \subseteq \ker(\alpha_1) \cap \cdots \cap \ker(\alpha_t) = \ker \lambda$. Since J is the annihilator of some subset of A, one has $J = J^{**}$ by [**10**], and so $J = (\ker \lambda)^*$. However, the quasi-flatness of A guarantees that $\ker \lambda / S_A(\ker \lambda)$ is torsion. Thus, $J = V^*$ where $V = S_A(\ker \lambda)$. Choose $\beta_1, \dots, \beta_s \in H_A(V)$ which generate $\mathbf{Q}H_A(V)$ as a right $\mathbf{Q}E$-module. This is possible since $\mathbf{Q}H_A(V) \subseteq \mathbf{Q}E$ and the latter is a right Noetherian ring. Then $W = \Sigma_{j=1}^s \beta_i E$ is a finitely generated E-submodule of $H_A(V)$ such that $H_A(V)/W$ is torsion as an abelian group. It remains to show that I is the left annihilator of $\beta_1, \dots, \beta_s$ in $\mathbf{Q}E$.

If $\alpha \in I$, then $m\alpha \in J$ for some nonzero integer m, and $m\alpha(V) = 0$. But this is only possible if $m\alpha\beta_i = 0$ for all $i = 1, \dots, s$. Conversely, if $\beta \in \mathbf{Q}E$ satisfies $\beta\beta_i = 0$ for all i, then there is a nonzero integer k_1 with $k_1\beta \in e$. Consequently, $k_1\beta \in l_E(W) = l_E(H_A(V))$, and so $k_1\beta \in V^* = J$. This shows $\beta \in I$.

c) $\Rightarrow$ a). By [**7**, Theorem 3.1], it remains to show ii). Let I be a right ideal of $\mathbf{Q}E$ and $\phi : I \to \mathbf{Q}E$ a $\mathbf{Q}E$-module morphism. Choose a finitely generated right ideal J of E with $I = \mathbf{Q}J$. This is possible in view of the fact that $\mathbf{Q}E$ is right Noetherian. Since A is quasi-flat, the first term in the sequence $0 \to \mathrm{Tor}_1^E(E/J, A) \xrightarrow{\Delta} T_A(J) \xrightarrow{\iota} T_A(E)$ is torsion where $\iota : J \to E$ denotes the inclusion map. Because J is finitely generated, there is a nonzero integer m such that $m\phi(J) \subseteq E$. Thus $m\phi$ induces a map $T_A(m\phi) : T_A(J) \to T_A(E)$. Since $\mathrm{Tor}_1^E(E/J, A)$ is torsion and $T_A(E)$ is torsion-free, $\mathrm{im}\,\Delta \subseteq \ker T_A(m\phi)$. Consequently, $T_A(m\phi)$ induces a map $\overline{T_A(m\phi)} : T_A(J)/\mathrm{im}\,\Delta \to T_A(E)$. By c) one can find a nonzero integer l and a map $\lambda : T_A(E) \to T_A(E)$ such that $l\overline{T_A(m\phi)} = \lambda\bar{\iota}$ where $\bar{\iota} : T_A(J)/\mathrm{im}\,\Delta \to T_A(E)$ is the monomorphism induced by $T_A(\iota)$. If π denotes the projection of $T_A(J)$ onto $T_A(J)/\mathrm{im}\,\Delta$, then $\iota = \bar{\iota}\pi$ and $T_A(m\phi) = \overline{T_A(m\phi)}\pi$. Consequently, the E-module map $\delta = \phi_E^{-1} H_A(\lambda)\phi_E : E \to E$ has the property

$$\begin{aligned}\delta_\iota &= \phi_E^{-1} H_A(\lambda) H_A T_A(\iota)\phi_J = \phi_E^{-1} H_A(\lambda\bar{\iota}\pi) = \phi_E^{-1} H_A(l\overline{T_A(m\phi)}\pi)\phi_J \\ &= \phi_E^{-1} H_A(lT_A(m\phi))\phi_J = l[\phi_E^{-1} H_A T_A(m\phi))\phi_J] = l[m\phi].\end{aligned}$$

But δ induces a $\mathbf{Q}E$-module morphism $\tilde{\delta} : \mathbf{Q}E \to \mathbf{Q}E$ such that $\tilde{\delta} \mid E = \delta$. Observe that $\tilde{\delta} \mid J = l[m\phi] \mid J$ implies $\tilde{\delta} \mid I = lm\phi$. Since multiplication by lm is a central automorphism of $\mathbf{Q}E$, the map $(ml)^{-1}\tilde{\delta}$ is a $\mathbf{Q}E$-module morphism extending ϕ. □

Corollary 3.6. *The following are equivalent for a torsion-free group of finite rank*:

a) i) *A-generated subgroups of A are $\mathbf{Q}A$-solvable.*

ii) *$\mathbf{Q}E$ is a self-injective ring.*

b) i) *A is quasi-flat.*

ii) *An exact sequence $0 \to B \xrightarrow{\alpha} G$ of torsion-free abelian groups of finite rank such that $B/R_A(B)$ is $\mathbf{Q}A$-solvable is quasi-A-cobalanced if and only if $\alpha(R_A(B)) = \alpha(B) \cap R_A(G)$.* □

While every quasi-A-cobalanced sequence $0 \to B \xrightarrow{\alpha} G$ always satisfies $\alpha(R_A(B)) = \alpha(B) \cap R_A(G)$, the converse may fail if $B/R_A(B)$ is not $\mathbf{Q}A$-solvable of finite A-rank as the next result shows. Therefore, the finiteness condition on the A-rank of $B/R_A(B)$ cannot be removed from part c) of Theorem 3.5.

For an abelian group G, the symbol G^* denotes the left E-module $\mathrm{Hom}\,(G, A)$ while $M^* = \mathrm{Hom}_E(M, A)$ whenever M is a left E-module. There is a natural map $\psi_G : G \to G^{**}$ which is defined by $[\psi_G(g)](\phi) = \phi(g)$ for all $g \in G$ and $\phi \in G^*$. If A is slender, ψ_P is an isomorphism if P is an A-projective group of nonmeasurable cardinality.

Proposition 3.7. *Let A be a slender torsion-free abelian group of nonmeasurable cardinality. An exact sequence $0 \to \oplus_\omega A \to G$ exists with $R_A(G) = 0$ which is not quasi-A-cobalanced.*

Proof. Consider the right E-module $\mathbf{Q}E$ which is the union of an ascending chain $\{U_n\}_{n<\omega}$ given by $U_n = \{(1/n!)E \mid 0 < n < \omega\}$. Since each of these submodules is isomorphic to E, the module $\mathbf{Q}E$ has projective dimension 1.

We consider the free module $F = \oplus_{i=1}^{\infty} E$ with basis $\{e_i \mid 0 < i < \omega\}$ and define epimorphism $\pi : F \to \mathbf{Q}E$ by $\pi(e_i) = (1/i!)1_A$ for

$i = 1, 2, \ldots$. If $x = e_1 r_1 + \cdots + e_n r_n \in \ker \pi$ with $r_1, \ldots, r_n \in E$, then $\Sigma_{i=1}^{n}(1/i!)r_i = 0$. Therefore, $r_n = -\Sigma_{i=1}^{n-1}(n!/i!)r_i$, and $x = \Sigma_{i=1}^{n-1} r_i(e_i - (n!/i!)e_n)$. Since $\pi(e_i - (n!/i!)e_n) = 0$. Therefore, $r_n = -\Sigma_{i=1}^{n-1}(n!/i!)r_i$ and $x = \Sigma_{i=1}^{n-1} r_i(e_i - (n!/i!)e_n)$. Since $\pi(e_i - (n!/i!)e_n) = 0$, one obtains that $P = \ker \pi$ is countably generated. But $\mathbf{Q}E$ has projective dimension 1, so P is projective, and there exists a P' with $P \oplus P' \cong \oplus_\omega E$. Consequently, an exact sequence $0 \to P_1 \xrightarrow{\sigma} P_2 \to \mathbf{Q}E \to 0$ such that $P_1 \cong \oplus_\omega A$ and P_2 is projective. Applying T_A to this sequence induces the exact sequence $\operatorname{Tor}_1^E(\mathbf{Q}E, A) \xrightarrow{\Delta} T_A(P_1) \xrightarrow{T_A(\sigma)} T_A(P_2) \to \mathbf{Q}A \to 0$. Since $T_A(P_1)$ is A-projective, it is reduced; and $\operatorname{im} D = 0$ because $\operatorname{Tor}_1^E(\mathbf{Q}E, A)$ is divisible. In particular, $T_A(\sigma)$ is a monomorphism. Identifying $\oplus_\omega A$ with $T_A(P_1)$ and setting $G = T_A(P_2)$, one obtains the exact sequence $0 \to \oplus_\omega A \xrightarrow{\alpha} G \to \mathbf{Q}A \to 0$ in which $\alpha = T_A(\sigma)$.

Assume that this sequence is quasi-A-cobalanced. It induces the exact sequence $0 = (\mathbf{Q}A)^* \to G \xrightarrow{\alpha^*} (\oplus_\omega A)^* - \operatorname{Ext}(\mathbf{Q}A, A)$. Since $\operatorname{Ext}(\mathbf{Q}A, A)$ is a torsion-free divisible group, and $(\oplus_\omega A)^*/\operatorname{im}\alpha^*$ is torsion, α^* is onto. Thus α^* is an isomorphism, and a commutative diagram exists

$$\begin{tikzcd}
0 \arrow[r] & (\oplus_\omega A)^{**} \arrow[r, "\alpha^{**}"] & G^{**} \arrow[r] & 0 & \\
0 \arrow[r] & \oplus_\omega A \arrow[r, "\alpha"] \arrow[u, "\psi\oplus_\omega A", "\wr"'] & G \arrow[r] \arrow[u, "\psi_G", "\wr"'] & \mathbf{Q}A \arrow[r] & 0
\end{tikzcd}$$

with exact rows in which $\psi_{\oplus_\omega A}$ and ψ_G are isomorphisms since $\oplus_\omega A$ and G are A-projective groups of nonmeasurable cardinality. It yields $\mathbf{Q}A = 0$, which is not possible. $\square$

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