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COMPACT ORBITS OF SMOOTH KILLING VECTOR FIELDS ON RIEMANNIAN MANIFOLDS

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ABSTRACT. Orbits of a family $\mathcal{F}$ of smooth vector fields on a manifold M partition M into connected, immersed submanifolds, not necessarily of the same dimension. Each orbit N is partitioned further into zero-time orbits (sets reachable from a point in zero total time). Compact orbits of a family of smooth, Killing vector fields on a Riemannian manifold M are studied in this paper. It is shown that zero-time orbits form a Riemannian foliation on N; in particular, the distance between the leaves, i.e., the zero-time orbits, is locally constant. Furthermore, zero-time orbits are isometric to each other and are either dense submanifolds of N or constitute fibers of a locally trivial fibration over the circle S^1. Since reachable sets of a family $\mathcal{F}$ of vector fields are translates of zero-time orbits along flows generated by vector fields from $\mathcal{F}$, analogous conclusions hold for a foliation of a compact orbit by reachable sets.

1. Introduction. Let $\mathcal{F}$ denote a family of smooth vector fields on a smooth manifold M. The orbit N of $\mathcal{F}$ through $x \in M$ is the set of points that can be reached from x by piecewise smooth integral curves of vector fields from $\mathcal{F}$. Orbits of $\mathcal{F}$ form a partition of M into connected, immersed submanifolds, not necessarily of the same dimension, in other words, they form a singular foliation on M [**1, 4, 5, 6, 7**]. The zero-time orbit N^0 through x is the set of points reachable from x by moving along piecewise smooth integral curves of $\mathcal{F}$ either forward or backward (moving backward is the same as moving along an integral curve for some negative time) so that the total time is zero, see [**9**], where zero-time orbits are first introduced and used. All zero-time orbits belonging to the same orbit (are of the same dimension and hence) define a regular foliation on that orbit. A vector field X is called Killing if its (local) flow $\exp(tX)$ consists of (local) isometries of M. In this paper orbits of a family of smooth, Killing vector fields on

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a Riemannian manifold (M, γ) are studied. It is shown that zero-time orbits foliate a compact orbit in a special way; the distance between them is locally constant (in the language of foliations, they define a Riemannian foliation on N). Furthermore, only two possibilities can occur: either all zero-time orbits are dense in N or they constitute fibers of a locally trivial fibration $N \to S^1$.

From the proof of the theorem, it will follow that the zero-time orbits inside a single compact orbit are isometric to each other. Since the sets reachable in some time from a point x are translates of zero-time orbits along flows of vector fields from $\mathcal{F}$, the properties of the foliation of N by its zero-time orbits hold for a foliation of N by accessible sets (at positive times).

Statements of the main result and its corollaries appear in Section 2. In Section 3 the languages of orbit theory and foliations are brought closer, and Section 4 specializes in the case of Killing vector fields and Riemannian foliations. The proof of the theorem is presented in Section 5.

An effort has been made to keep notation as simple as possible. The symbol M will always denote a manifold, N an orbit of a family of vector fields $\mathcal{F}$ and N^0 a zero-time orbit. Upper-case calligraphic letters $\mathcal{F}, \mathcal{K}$, etc., are used for families (ideals, Lie algebras) of vector fields, and $\mathcal{F}(x), \mathcal{K}(x)$, etc., denote their evaluations at x. A point x appears as a subscript in the notation for tangent spaces and orbits, e.g., T_xN and N_x. The same symbols are used for families of vector fields and corresponding foliations since the context will keep the meaning clear.

2. Result. Let $\mathcal{F}$ denote a family of smooth vector fields on a smooth manifold M. The orbit N_x of $\mathcal{F}$ through the point $x \in M$ is the set

$$N_x = \{\exp(t_k X_k) \circ \cdots \circ \exp(t_1 X_1)(x)\},$$

where $k \geq 1$, $X_i \in \mathcal{F}$ and $t_i \in \mathbf{R}$ are such that the flows are well defined, $i = 1, \ldots, k$.

Let $\mathcal{G}_{\mathcal{F}}$ be the pseudogroup of local diffeomorphisms generated by one-parameter subgroups $\exp(tX)$ whose infinitesimal generators belong to $\mathcal{F}$. The set N_x is the orbit of $\mathcal{G}_{\mathcal{F}}$ through x. It is a smooth, connected (immersed) submanifold of M. A zero-time orbit N_x^0 of $\mathcal{F}$

through $x \in M$ is a smooth submanifold of M given by

$$N_x^0 = \{\exp(t_k X_k) \circ \cdots \circ \exp(t_1 X_1)(X) \mid t_1 + \cdots + t_k = 0\} \subseteq N_x,$$

where t_i and X_i, $i = 1, \ldots, k$, are as above.

The aim of this paper is to prove the following theorem:

Theorem. *Let N denote a compact orbit of a family $\mathcal{F}$ of smooth Killing vector fields on a smooth Riemannian manifold M. Zero-time orbits of $\mathcal{F}$ define a Riemannian foliation $(N, \mathcal{N}^0)$ on N. One of the following holds*:

(i) *All zero-time orbits coincide with N.*

(ii) *Zero-time orbits are dense in N.*

(iii) *Zero-time orbits are fibers of a locally trivial fibration $\pi : N \to S^1$.*

In Section 4 it is shown that this result holds with a weaker assumption on $\mathcal{F}$; namely, it is enough to assume that the flow of $\mathcal{F}$ preserves the transversal component of the metric.

If $\mathcal{F}$ contains only one vector field, then any compact orbit of $\mathcal{F}$ must be diffeomorphic to S^1. In that case, zero-time orbits are points, and thus the alternative (iii) of the theorem holds (the locally trivial fibration being the identity map).

From the proof of the theorem one can immediately draw the following consequence, see also [**8**, p. 351]:

Corollary. *Let M and $\mathcal{F}$ be as in the theorem, and let N_1^0 and N_2^0 be two zero-time orbits belonging to the same compact orbit of $\mathcal{F}$. Then there is an isometry $\phi : M \to M$ such that $\phi(N_1^0) = N_2^0$.*

The set of points reachable from x at time T, $T \geq 0$, is the set

$$\mathcal{A}_{\mathcal{F}}(x, T) = \{\exp(t_k X_k) \circ \cdots \circ \exp(t_1 X_1)(x) \mid t_1 + \cdots + t_k = T\} \subseteq N_x,$$

where t_i and X_i, $i = 1, \ldots, k$, are as above.

For any $T \geq 0$, $p \in \mathcal{A}_{\mathcal{F}}(x, T)$ and $X \in \mathcal{F}$, $\exp(-TX)(p) \in N_x^0$; thus, $\mathcal{A}_{\mathcal{F}}(x, T) = (\exp(TX))N_x^0$. Since the flow $\exp(TX)$ is an isometry, one concludes that $\mathcal{A}_{\mathcal{F}}(x,T)$ is isometric to N_x^0. From the corollary, we conclude that there is an isometry between any two reachable sets $\mathcal{A}_{\mathcal{F}}(x_1,T_1)$ and $\mathcal{A}_{\mathcal{F}}(x_2,T_2)$.

Clearly, the above theorem applies to the foliation of a compact orbit by its reachable sets.

Example. Let $\mathcal{F}$ be any smooth family of Killing vector fields on the sphere S^2, for example, rotation vector fields, such that $\mathcal{F}$ is reachable, i.e., there is only one orbit, $N = S^2$. For topological reasons, zero-time orbits cannot fiber over S^1. Anyway, if $N^0 \neq N$, then (the proof will show) that there must exist everywhere nonzero transversal direction. Since this is impossible on S^2, it follows that zero-time orbits coincide with the orbit N.

3. Orbits and foliations. Let $\mathcal{F}$ be a family of smooth vector fields on a smooth manifold M, assumed to be everywhere defined, i.e., union of the domains of elements in $\mathcal{F}$ is M. Denote by $\Delta_{\mathcal{F}}$ the distribution on M spanned by $\mathcal{F}$. Recall that $\mathcal{G}_{\mathcal{F}}$ is the pseudogroup of local diffeomorphisms generated by one-parameter subgroups $\exp(tX)$ whose infinitesimal generators belong to $\mathcal{F}$.

A distribution $\Delta_{\mathcal{F}}$ is called $\mathcal{G}_{\mathcal{F}}$-invariant if for each $g \in \mathcal{G}_{\mathcal{F}}$ the differential dg maps $\Delta_{\mathcal{F}}(x) \subseteq T_xM$ into $\Delta_{\mathcal{F}}(g(x)) \subseteq T_{g(x)}M$. Let $P_{\mathcal{F}}$ be the smallest $\mathcal{G}_{\mathcal{F}}$-invariant distribution that contains $\mathcal{F}$.

It can be shown, see [**6**], that the distribution $P_{\mathcal{F}}$ is generated by the pullbacks $\{dg \circ X \circ g^{-1}\}$, where $X \in \mathcal{F}$ and $g \in \mathcal{G}_{\mathcal{F}}$. It is integrable, and Theorem 4.1 in [**6**] implies that the orbits of $\mathcal{F}$ coincide with maximal integral submanifolds of $P_{\mathcal{F}}$. Denote by $\Delta_{\mathcal{F}}^0$ the distribution spanned by the differences $\{dg \circ X \circ g^{-1} - dh \circ Y \circ h^{-1}\}$, where $X, Y \in \mathcal{F}$ and $g, h \in \mathcal{G}_{\mathcal{F}}$. $\Delta_{\mathcal{F}}^0$ is $\mathcal{G}_{\mathcal{F}}$-invariant, and hence its dimension is constant on orbits of $\mathcal{F}$, and of codimension zero or one in $P_{\mathcal{F}}$. Theorem 1.3 in [**2**] implies that $\Delta_{\mathcal{F}}^0$ is integrable, and its maximal integral submanifolds are precisely the zero-time orbits of $\mathcal{F}$.

If $\mathcal{F}$ is a family of analytic vector fields, then Proposition 1.4 in [**2**], see also [**9**], provides a useful way of computing the distributions involved: $P_{\mathcal{F}}$ equals $\mathrm{Lie}\,(\mathcal{F})$, which is the smallest Lie algebra generated by $\mathcal{F}$, and $\Delta_{\mathcal{F}}^0 = \{\sum \alpha_i X_i + X'\}$, where $X_i \in \mathcal{F}$, $\sum \alpha_i = 0$ and $X' \in \mathcal{F}'$ ($\mathcal{F}'$ is the derived algebra of $\mathcal{F}$).

A foliation $(M, \mathcal{F})$ of codimension q, $q \leq n$, on an n-dimensional manifold M is a partition $\{\mathcal{F}_\alpha\}$ of M into connected, not necessarily imbedded, submanifolds, called leaves of $\mathcal{F}$, which satisfy the following requirement: for every point there is an open neighborhood U and a diffeomorphism $\phi : U \to \mathbf{R}^n = \mathbf{R}^{n-q} \times \mathbf{R}^q$ given by $\phi = (x_1, \ldots, x_{n-q}, y_1, \ldots, y_q)$, so that for each leaf $\mathcal{F}_\alpha$ the connected components of $U \cap \mathcal{F}_\alpha$ are given by the equations $y_1 = \text{constant}, \ldots, y_q = \text{constant}$. From the definition it follows that, locally, every foliation is given by the submersion of U onto the local quotient manifold $\pi(U)$, where $\pi : M \to M/\mathcal{F}$ is the projection.

From the above it follows that the distribution $\Delta^0_{\mathcal{F}}$, restricted to an orbit N of $\mathcal{F}$, defines a foliation, $N, \mathcal{N}^0 = \Delta^0_{\mathcal{F}}|_N$, of N by codimension zero (trivial foliation) or codimension one leaves.

4. Killing vector fields and Riemannian foliations. Let (M, γ) be a Riemannian manifold and $(M, \mathcal{F})$ a foliation on M. The tangent space T_xM splits as the direct sum $T_xM = T_xN \oplus T_xQ$ of the space T_xN tangent to the leaf N of $\mathcal{F}$ through x and its orthogonal complement T_xQ. The metric $\gamma = \gamma_N \oplus \gamma_Q$ decomposes into its tangential and transversal components. A foliation $\mathcal{F}$ is called Riemannian if $L_X\gamma_Q = 0$, where L_X denotes the Lie derivative of the metric in the direction of the vector field X tangent to the leaves of $\mathcal{F}$ (given by $L_X\gamma_Q(U, V) = X(\gamma_Q(U, V)) - \gamma_Q([X, U], V) - \gamma_Q(U, [X, V])$, where U and V are vector fields on M). Interpreting the vector fields tangent to the leaves of $\mathcal{F}$ as defining the motion on M, the requirement that the foliation be Riemannian means that the transversal component of the metric is a constant of motion. Equivalently, see [**3**], the foliation is Riemannian if and only if the distance between its leaves is locally constant.

Example. A Riemannian submersion is a smooth submersion $f : M \to P$ between Riemannian manifolds M and P such that the differential $df(x) : T_xM \to T_{f(x)}P$ is an isometry between the horizontal subspace H_x (= orthogonal complement of $df(x)^{-1}(0)$ in T_xM) and $T_{f(x)}P$ for all $x \in M$. The collection $\{f^{-1}(y) \mid y \in P\}$ defines a Riemannian foliation on M of codimension equal to the dimension of P.

Let a Lie group G act on a Riemannian manifold M by isometries. If all orbits of G have the same dimension, then they define a Riemannian foliation on M. In the example at the end of this paper, $M = T^2$, with the standard metric, and $G = \mathcal{G}_{\mathcal{F}}$ acts by isometries (one-parameter groups generated by $a(\partial/\partial\theta_1)$ and $b(\partial/\partial\theta_2)$ are translations). Consequently, the foliation $\Delta^0_{\mathcal{F}}$ on T^2 is a Riemannian foliation.

A vector field X on a Riemannian manifold (M, γ) is called a Killing vector field if its (local) one-parameter group of diffeomorphisms consists of (local) isometries of M. In other words, X is Killing if and only if $L_X\gamma = 0$. The importance of Killing vector fields is that their orbits define a Riemannian foliation, since $L_X\gamma = 0$ implies $L_X\gamma_Q = 0$ (to be precise, such a foliation is called a singular Riemannian foliation since the dimension of its leaves does not have to be constant, see [**5**]; however, in the case under consideration, i.e., the foliation of an orbit by zero-time orbits, all leaves will have the same dimension). Rotation fields on the sphere S^n equipped with the standard metric are Killing vector fields. Constant vector fields on $\mathbf{R}^n$ are Killing vector fields, and they project to Killing vector fields on the flat torus $T^n = \mathbf{R}^n/\mathbf{Z}^n$. Orbits of those fields define Riemannian foliations on $S^n, \mathbf{R}^n$ and T^n, respectively. Complete, Killing vector fields form a Lie subalgebra $\mathcal{K}$ of the algebra $\mathcal{X}(M)$ of all vector fields on M.

5. Proof of the theorem. Let $\mathcal{F}$ be a family of smooth Killing vector fields on a smooth Riemannian manifold (M, γ), and denote by (N, γ) any of its compact orbits (the metric on N is the induced metric from M). The tangent space to N at x is given by

$$T_xN = P_{\mathcal{F}}(x) = \{dg \circ X \circ g^{-1}(x) \mid X \in \mathcal{F}, g \in \mathcal{G}_{\mathcal{F}}\} \supseteq \mathcal{F}(x).$$

We can view vector fields in $\mathcal{F} \subseteq \mathcal{X}(M)$ as vector fields restricted to N and will denote this (restricted) family again by $\mathcal{F}$. Orbit N is assumed to be compact, and, consequently, exponential maps are defined for all times. Since the pullback by a diffeomorphism of a Killing vector field is again a Killing vector field, the zero-time distribution $\Delta^0_{\mathcal{F}}$ is spanned by Killing vector fields, and the foliation $(N, \mathcal{N}^0 = \Delta^0_{\mathcal{F}}|_N)$ of N by its zero-time orbits is a Riemannian foliation whose leaves are of codimension zero or one in N.

Assume that the zero-time orbits N^0 of N do not coincide with N. At every point there is a decomposition $T_xN = T_xN^0 \oplus T_xQ$ into leaf zero-orbit directions and a transversal direction. Let $T \in \mathcal{F}$ be such that $T(x) \notin \Delta^0_{\mathcal{F}}(x)$ for some x. If, for some y, $T(y) \in \Delta^0_{\mathcal{F}}(y)$, then, by the $\mathcal{G}_{\mathcal{F}}$-invariance of $\Delta^0_{\mathcal{F}}$, it follows that $T(x) \in \Delta^0_{\mathcal{F}}(x)$. In other words, the above decomposition can be written as

$$T_xN = \mathcal{N}^0(x) \oplus \mathbf{R}T(x),$$

globally on N, where $\mathbf{R}T(x)$ denotes the linear span of T (a vector field $T \notin \Delta^0_{\mathcal{F}}$ which is nonzero at one point is nonzero everywhere on N; otherwise, the zero-time distribution $\Delta^0_{\mathcal{F}}(x)$ would equal T_xN for all x and $N^0 = N$, contrary to the assumption).

From $[\Delta^0_{\mathcal{F}}, T] = [P_{\mathcal{F}} - P_{\mathcal{F}}, T] = [P_{\mathcal{F}}, T] - [P_{\mathcal{F}}, T] \subseteq P_{\mathcal{F}} - P_{\mathcal{F}} = \Delta^0_{\mathcal{F}}$, it follows that the flow of T preserves zero-time orbits. Consequently, the group $\mathcal{A}(\mathcal{N}^0)$ of automorphisms of the foliation $\mathcal{N}^0$, i.e., diffeomorphisms of N that map zero-time orbits to zero-time orbits, contains flows of all vector fields in $\mathcal{F}$, and therefore has open orbits. Since N is connected, $\mathcal{A}(\mathcal{N}^0)$ acts transitively on N.

Consider the subfamily of the algebra of vector fields on N given by

$$\mathcal{N}^0_b = \{X \in \mathcal{X}(N) \mid L_X f = Xf = 0\} \supseteq \mathcal{N}^0$$

where f is a constant of motion for $\mathcal{N}^0$, i.e., $L_X f = Xf = 0$ for all X in $\mathcal{N}^0$. Let $\phi : N \to N$ be an automorphism of $\mathcal{N}^0$. If f is a constant of motion for $\mathcal{N}^0$, then $\phi \circ f$ is again a constant of motion for $\mathcal{N}^0$. Moreover, $d\phi$ pulls back vector fields from $\mathcal{N}^0_b$ to $\mathcal{N}^0_b$, and therefore $\mathcal{N}^0_b(\phi(x)) = d\phi(\mathcal{N}^0_b(x))$. Since the automorphism group acts transitively, the dimension of $\mathcal{N}^0_b(x)$ is constant on N, and hence $\mathcal{N}^0_b$ defines a distribution, which will also be denoted by $\mathcal{N}^0_b$. Since $\mathcal{N}^0_b = \{X \mid df(X) = 0\}$ for all constants of motion, the distribution $\mathcal{N}^0_b$ (is integrable and hence) defines a foliation of codimension $q_b \leq 1$ on N. If it is of codimension one, the flow of the vector field T is transversal to its leaves (such a foliation is called transversally parallelizable).

From Chapter 4 in [**3**] it follows that the orbits N^0_b of $\mathcal{N}^0_b$ are compact submanifolds of N; they are the closures of orbits of $\mathcal{N}^0$. Moreover, the orbits N^0_b constitute fibers of the locally trivial fibration $\phi : N \to N/\mathcal{N}^0_b$, where $N/\mathcal{N}^0_b$ is a smooth manifold of dimension equal to q_b.

The foliation $(N, \mathcal{N}^0)$ induces a foliation $(N^0_b, \mathcal{N}^0)$ on N^0_b, by zero-time orbits. Since $\mathcal{A}(\mathcal{N}^0)$ is transitive, all orbits N^0_b are isomorphic as foliations. If the dimension of $N/\mathcal{N}_b^0$ is zero, it follows that $\mathcal{N}_b^0$ has only one leaf, which is the closure of leaves of $\mathcal{N}^0$. So in this case all leaves of $\mathcal{N}^0$ are dense in N. If the dimension of $N/\mathcal{N}_b^0$ is one, then π is a fibration over a compact, connected, one-dimensional manifold, i.e., π is a locally trivial fibration over S^1. This completes the proof of the theorem.

Example. Identify the two-dimensional torus T^2 with the product $S^1 \times S^1$ so that the point of T^2 corresponds to (θ_1, θ_2) determined up to a multiple of 2π. Consider the family $\mathcal{F} = \{a(\partial/\partial\theta_1), b(\partial/\partial\theta_2)\}$, where a and b are nonzero constants. Since the vector fields $a(\partial/\partial\theta_1)$ and $b(\partial/\partial\theta_2)$ commute, it follows that every element of $\mathcal{G}_{\mathcal{F}}$ is of the form $\exp(t_1 a(\partial/\partial\theta_1)) \exp(t_2 b(\partial/\partial\theta_2))$, for some t_1 and t_2. Clearly, $\Delta_{\mathcal{F}}(x)$ coincides with the tangent space $T_x T^2$ and, consequently, $\mathcal{F}$ has only one orbit, namely, T^2, i.e., $\mathcal{F}$ is reachable. The distribution $\Delta_{\mathcal{F}}^0$ is spanned by $\{a(\partial/\partial\theta_1) - b(\partial/\partial\theta_2)\}$, and its orbits (the zero-time orbits) foliate the (compact) orbit T^2 of $\mathcal{F}$ either by one-dimensional manifolds diffeomorphic to $\mathbf{R}$, if a/b is irrational, or by leaves diffeomorphic to S^1, if a/b is rational. In the latter case, the theorem proved in this paper implies that the foliation is actually a fibration over S^1. This example also shows that a compact orbit can have noncompact zero-time orbits.

Example. Let $M = S^1 \times S^1 \times S^1$ with coordinates θ_1, θ_2 and θ_3 (modulo 2π), and let $\mathcal{F} = \{(\partial/\partial\theta_1), (\partial/\partial\theta_2), (\partial/\partial\theta_3)\}$. The orbit N of each point is equal to M. For $t_1, t_2 \in \mathbf{R}$, define $X(t_1, t_2) = t_1(\partial/\partial\theta_1) + t_2(\partial/\partial\theta_2) - (t_1 + t_2)(\partial/\partial\theta_3)$. The zero-time orbit of a point $x \in N$ is $N_x^0 = \{\exp X(t_1, t_2) \mid t_1, t_2 \in \mathbf{R}\}$. Clearly zero-time orbits do not give a locally trivial fibration $\pi : N \to S^1$, nor is $N_x^0 = S^1$. If t_1/t_2 is irrational, then $(t_1 + t_2)/t_1$ and $(t_1 + t_2)/t_2$ are also irrational, and so $X(t_1, t_2)$ gives a dense flow on N; i.e., the zero-time orbits are dense in N.

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