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NORMALITY OF COMPLEX CONTACT MANIFOLDS

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ABSTRACT. Complex contact metric manifolds are studied. Normality is defined for these manifolds and equivalent conditions are given in terms of ∇G and ∇H. GH-sectional curvature and $\mathcal{H}$-homothetic deformations are defined. Examples of normal complex contact metric manifolds with constant GH-sectional curvature c are given for $c \geq -3$.

1. Introduction. The theory of complex contact manifolds started with the papers of Kobayashi [**12**] and Boothby [**4**], [**5**] in late 1950's and early 1960's, shortly after the celebrated Boothby-Wang fibration in real contact geometry [**6**]. It did not receive as much attention as the theory of real contact geometry. In 1965, Wolf studied homogeneous complex contact manifolds [**17**]. Recently, more examples are appearing in the literature, especially twistor spaces over quaternionic Kähler manifolds (e.g., [**13**], [**14**], [**15**], [**16**], [**18**]). Other examples include the odd dimensional complex projective spaces [**9**] and the complex Heisenberg group [**1**].

In the 1970's and early 1980's there was a development of the Riemannian theory of complex contact manifolds by Ishihara and Konishi [**8**], [**9**], [**10**]. However, their notion of normality as it appears in [**9**] seems too strong since it does not include the complex Heisenberg group and it forces the structure to be Kähler. In this paper we introduce a slightly different notion of normality which includes the complex Heisenberg group.

In Section 2 we give the necessary definitions and some basic facts about complex contact metric manifolds. In Section 3 we define normality and give the theorem which states the necessary and sufficient conditions, in terms of the covariant derivatives of the structure tensors, for a complex contact metric manifold to be normal. We discuss some curvature properties of normal complex contact metric manifolds in Section 4.

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In Section 5, following the corresponding theory of real contact geometry, we define the GH-sectional curvature for normal complex contact metric manifolds and we classify those with constant GH-sectional curvature $+1$. We give examples of normal complex contact metric manifolds with constant GH-sectional curvature -3 and $+1$ in Section 6. Then, in Section 7, we define $\mathcal{H}$-homothetic deformations and show that they preserve normality. Using $\mathcal{H}$-homothetic deformations, we get examples of normal complex contact metric manifolds with constant GH-sectional curvature c for every $c > -3$. Here we note that Ishihara-Konishi's notion of normality is not preserved under $\mathcal{H}$-homothetic deformations.

2. Basic definitions.

Definition 2.1. Let M be a complex manifold with $\dim_{\mathbf{C}} M = 2n+1$, and let J denote the complex structure on M. M is a complex contact manifold if an open covering $\mathcal{U} = \{\mathcal{O}_\alpha\}$ of M exists, such that

1) on each $\mathcal{O}_\alpha$ there is a holomorphic 1-form ω_α with $\omega_\alpha \wedge (d\omega_\alpha)^n \neq 0$ everywhere, and

2) if $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \varnothing$, then there is a nonvanishing holomorphic function $\lambda_{\alpha\beta}$ in $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ such that

$$\omega_\alpha = \lambda_{\alpha\beta}\omega_\beta \quad \text{in } \mathcal{O}_\alpha \cap \mathcal{O}_\beta.$$

On each $\mathcal{O}_\alpha$ we define $\mathcal{H}_\alpha = \{X \in T\mathcal{O}_\alpha \mid \omega_\alpha(X) = 0\}$. Since $\lambda_{\alpha\beta}$'s are nonvanishing, $\mathcal{H}_\alpha = \mathcal{H}_\beta$ on $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$. So $\mathcal{H} = \cup\mathcal{H}_\alpha$ is a well-defined, holomorphic, nonintegrable subbundle on M, called the *horizontal subbundle.*

Definition 2.2. Let M be a complex manifold with $\dim_{\mathbf{C}} M = 2n+1$, complex structure J and Hermitian metric g. M is called a complex almost contact metric manifold if an open covering $\mathcal{U} = \{\mathcal{O}_\alpha\}$ of M exists such that

1) in each $\mathcal{O}_\alpha$ there are 1-forms u_α and $v_\alpha = u_\alpha J$, (1,1) tensors G_α

and $H_\alpha = G_\alpha J$, unit vector fields U_α and $V_\alpha = -JU_\alpha$ such that

$$\begin{aligned} H_\alpha^2 = G_\alpha^2 &= -\mathrm{Id} + u_\alpha \otimes U_\alpha + v_\alpha \otimes V_\alpha \\ g(G_\alpha X, Y) &= -g(X, G_\alpha Y) \\ g(U_\alpha, X) &= u_\alpha(X) \\ G_\alpha J &= -JG_\alpha \\ G_\alpha U_\alpha &= 0 \\ u_\alpha(U_\alpha) &= 1, \end{aligned}$$

2) if $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \varnothing$, then there are functions a, b on $\mathcal{O}_\alpha \cap \mathcal{O}_\beta$ such that

$$\begin{aligned} u_\beta &= au_\alpha - bv_\alpha \\ v_\beta &= bu_\alpha + av_\alpha \\ G_\beta &= aG_\alpha - bH_\alpha \\ H_\beta &= bG_\alpha + aH_\alpha \\ a^2 + b^2 &= 1. \end{aligned}$$

As a result of this definition, on a complex almost contact metric manifold M, the following identities hold (cf. [**9**]):

$$\begin{aligned} H_\alpha G_\alpha = -G_\alpha H_\alpha &= J + u_\alpha \otimes V_\alpha - v_\alpha \otimes U_\alpha \\ JH_\alpha = -H_\alpha J &= G_\alpha \\ g(H_\alpha X, Y) &= -g(X, H_\alpha Y) \\ G_\alpha V_\alpha = H_\alpha U_\alpha &= H_\alpha V_\alpha = 0 \\ u_\alpha G_\alpha = v_\alpha G_\alpha &= u_\alpha H_\alpha = v_\alpha H_\alpha = 0 \\ JV_\alpha &= U_\alpha, g(U_\alpha, V_\alpha) = 0. \end{aligned}$$

From now on, we will suppress the subscripts if $\mathcal{O}_\alpha$ is understood.

Let $(M, \{\omega\})$ be a complex contact manifold. We can find a nonvanishing, complex-valued function multiple π of ω such that on $\mathcal{O} \cap \mathcal{O}'$, $\pi = h\pi'$ with

$$h : \mathcal{O} \cap \mathcal{O}' \longrightarrow \mathbf{S}^1.$$

Let $\pi = u - iv$. Then $v = uJ$ since ω is holomorphic. Locally we can define a vector field U by $du(U, X) = 0$ for all X in $\mathcal{H}$ and $u(U) = 1$,

$v(U) = 0$. Then we have a global subbundle $\mathcal{V}$ locally spanned by U and $V = -JU$ with $TM = \mathcal{H} \oplus \mathcal{V}$. We call $\mathcal{V}$ the *vertical subbundle* of the contact structure. Here we note that we can find a local (1,1) tensor G such that $(u, v, U, V, G, H = GJ, g)$ form a complex almost contact metric structure on M (cf. [**10**]).

Definition 2.3. Let $(M, \{\omega\})$ be a complex contact manifold with the complex structure J and Hermitian metric g. We call (M, u, v, U, V, g) a complex contact metric manifold if

1) there is a local (1,1) tensor G such that $(u, v, U, V, G, H = GJ, g)$ is a complex almost contact metric structure on M and

2) $g(X, GY) = du(X, Y)$ and $g(X, HY) = dv(X, Y)$ for all X, Y in $\mathcal{H}$.

In his thesis [**7**], Foreman shows the existence of complex contact metric structures on complex contact manifolds.

We will assume that the subbundle $\mathcal{V}$ is integrable. Since every known example of a complex contact manifold has an integrable vertical subbundle, this is a reasonable assumption for our work. From now on, we will work with a complex contact metric manifold M with structure tensors (u, v, U, V, G, H, g) and complex structure J.

Define 2-forms $\hat{G}$ and $\hat{H}$ on M by

$$\hat{G}(X, Y) = g(X, GY), \hat{H}(X, Y) = g(X, HY).$$

Then for horizontal vector fields X, Y,

$$\hat{G}(X, Y) = du(X, Y), \quad \hat{H}(X, Y) = dv(X, Y).$$

In general, we have

$$\hat{G} = du - \sigma \wedge v, \tag{1}$$

$$\hat{H} = dv + \sigma \wedge u, \tag{2}$$

where $\sigma(X) = g(\nabla_X U, V)$ (cf. [**7**]).

In real contact geometry, there is a symmetric operator $h = (1/2)\mathcal{L}_\xi \phi$, where ξ is the characteristic vector field and ϕ is the structure tensor of the real contact metric structure. Here $\mathcal{L}$ denotes the Lie differentiation. In particular, on a real contact metric manifold, we have

$$\nabla_X \xi = -\phi X - \phi h X,$$

cf. [**3**].

Similarly, we define symmetric operators $h_U, h_V : TM \to \mathcal{H}$ as follows:

$$h_U = \frac{1}{2}\,\mathrm{sym}\,(\mathcal{L}_U G) \circ p$$
$$h_V = \frac{1}{2}\,\mathrm{sym}\,(\mathcal{L}_V H) \circ p$$

where 'sym' denotes the symmetrization and $p : TM \to \mathcal{H}$ is the projection map. Then we have

$$h_U G = -G h_U, \qquad h_V H = -H h_V,$$
$$h_U(U) = h_U(V) = h_V(U) = h_V(V) = 0,$$

and

$$\nabla_X U = -GX - G h_U X + \sigma(X) V, \tag{3}$$
$$\nabla_X V = -HX - H h_V X - \sigma(X) U, \tag{4}$$

where ∇ is the Levi-Civita connection of g (cf. [**7**]). Hence,

$$\begin{aligned} \nabla_U U &= \sigma(U) V, & \nabla_V U &= \sigma(V) V, \\ \nabla_U V &= -\sigma(U) U, & \nabla_V V &= -\sigma(V) U. \end{aligned} \tag{5}$$

It can easily be seen by a direct computation that

$$(\nabla_X \hat{G})(Y,Z) + (\nabla_Y \hat{G})(Z,X) + (\nabla_Z \hat{G})(X,Y) = 3d\hat{G}(X,Y,Z),$$

and

$$(\nabla_X \hat{H})(Y,Z) + (\nabla_Y \hat{H})(Z,X) + (\nabla_Z \hat{H})(X,Y) = 3d\hat{H}(X,Y,Z).$$

Then, using equations (1) and (2) we get

$$\begin{aligned} &(\nabla_X \hat{G})(Y,Z) + (\nabla_Y \hat{G})(Z,X) + (\nabla_Z \hat{G})(X,Y) \\ &\quad = -v(X)\Omega(Y,Z) - v(Y)\Omega(Z,X) - v(Z)\Omega(X,Y) \\ &\qquad + \sigma(X) g(Y,HZ) + \sigma(Y) g(Z,HX) + \sigma(Z) g(X,HY), \end{aligned} \tag{6}$$

and

$$(7)\quad \begin{aligned}&(\nabla_X \hat{H})(Y,Z) + (\nabla_Y \hat{H})(Z,X) + (\nabla_Z \hat{H})(X,Y) \\ &\qquad = u(X)\Omega(Y,Z) + u(Y)\Omega(Z,X) + u(Z)\Omega(X,Y) \\ &\qquad\quad - \sigma(X)g(Y,GZ) - \sigma(Y)g(Z,GX) - \sigma(Z)g(X,GY),\end{aligned}$$

where $\Omega = d\sigma$.

Lemma 2.4. *$\nabla_U G = \sigma(U)H$ and $\nabla_V H = -\sigma(V)G$.*

Proof. By equations (6) and (3) we get

$$(8)\quad (\nabla_U \hat{G})(X,Y) = v(X)\Omega(U,Y) + v(Y)\Omega(X,U) + \sigma(U)g(X,HY).$$

If X and Y are horizontal, then

$$(\nabla_U \hat{G})(X,Y) = \sigma(U)g(X,HY).$$

On the other hand, by (5)

$$(\nabla_U \hat{G})(U,Y) = -g(\nabla_U U, GY) = 0,$$

and

$$(\nabla_U \hat{G})(V,Y) = -g(\nabla_U V, GY) = 0.$$

So $(\nabla_U G)Y = \sigma(U)HY$ for any Y.

Similarly, using (7) and (4), we get

$$(9)\quad (\nabla_V \hat{H})(X,Y) = u(X)\Omega(Y,V) + u(Y)\Omega(V,X) - \sigma(V)g(X,GY).$$

Again, by (5), $(\nabla_V \hat{H})(U,Y) = (\nabla_V \hat{H})(V,Y) = 0$. So $(\nabla_V H)Y = -\sigma(V)GY$. □

Now, if we use Lemma 2.4 in equations (8) and (9), we get

$$(10)\quad \Omega(U,X) = v(X)\Omega(U,V),$$

and

$$(11)\quad \Omega(V,X) = -u(X)\Omega(U,V).$$

3. Normality on complex contact metric manifolds. Let M be a complex contact metric manifold. Ishihara and Konishi [**9**] defined (1,2) tensors S and T on a complex almost contact manifold as follows:

$$\begin{aligned}S(X,Y) = {} & [G,G](X,Y) + 2v(Y)HX - 2v(X)HY + 2g(X,GY)U \\ & - 2g(X,HY)V - \sigma(GX)HY + \sigma(GY)HX + \sigma(X)GHY \\ & - \sigma(Y)GHX\end{aligned}$$

$$\begin{aligned}T(X,Y) = {} & [H,H](X,Y) + 2u(Y)GX - 2u(X)GY + 2g(X,HY)V \\ & - 2g(X,GY)U + \sigma(HX)GY - \sigma(HY)GX - \sigma(X)HGY \\ & + \sigma(Y)HGX\end{aligned}$$

where

$$[G,G](X,Y) = (\nabla_{GX}G)Y - (\nabla_{GY}G)X - G(\nabla_X G)Y + G(\nabla_Y G)X$$

is the Nijenhuis torsion of G. In [**9**], they introduced a notion of normality which is the vanishing of the two tensors S and T. One of their results is that if M is normal, then it is Kähler. This result suggests that Ishihara-Konishi's notion of normality is too strong. Here we will give a somewhat weaker definition.

Definition 3.1. A complex contact metric manifold M is normal if

1) $S(X,Y) = T(X,Y) = 0$ for all X, Y in $\mathcal{H}$, and

2) $S(U,X) = T(V,X) = 0$ for all X.

In real contact geometry, normality implies the vanishing of the operator h. The following proposition is the analogous result for complex contact geometry.

Proposition 3.2. *If M is normal, then $h_U = h_V = 0$.*

Proof. Since M is normal,

$$S(GX,U) = 0.$$

By (5), $G(\nabla_V U) = G(\nabla_U V) = G(\nabla_U U) = 0$ and $u(\nabla_U GX) = v(\nabla_U GX) = 0$. Also, by (3) $u(\nabla_{GX}U) = 0$, and $v(\nabla_{GX}U) = \sigma(GX)$. Hence, by Lemma 2.4, $S(GX,U) = 2h_U X$. Therefore, $h_U = 0$.

Similarly, using $T(HX,V)=0$ and Lemma 2.4, we get $h_V=0$. $\square$

By the above proposition, on a normal contact metric manifold, we have

$$\nabla_X U = -GX + \sigma(X)V \tag{12}$$

and

$$\nabla_X V = -HX - \sigma(X)U. \tag{13}$$

In the next proposition, we give necessary and sufficient conditions, in terms of ∇G and ∇H, for M to be normal. Again, compare with the condition for a real contact metric manifold to be normal.

Proposition 3.3. *Let M be a complex contact metric manifold. M is normal if and only if*

(I)

$$\begin{aligned} g((\nabla_X G)Y,Z) = {} & \sigma(X)g(HY,Z)+v(X)\Omega(GZ,GY)-2v(X)g(HGY,Z)\\ & - u(Y)g(X,Z) - v(Y)g(JX,Z) + u(Z)g(X,Y)\\ & - v(Z)g(X,JY) \end{aligned}$$

and

(II)

$$\begin{aligned} g((\nabla_X H)Y,Z) = {} & -\sigma(X)g(GY,Z)+u(X)\Omega(HZ,HY)-2u(X)g(HGY,Z)\\ & + u(Y)g(JX,Z) - v(Y)g(X,Z) + u(Z)g(X,JY)\\ & + v(Z)g(X,Y). \end{aligned}$$

Proof. Suppose that M is normal. For arbitrary vector fields X and Y, we can write

$$X = X' + u(X)U + v(X)V, \qquad Y = Y' + u(Y)U + v(Y)V$$

where X' and Y' are in $\mathcal{H}$. Then $GX = GX'$, $GY = GY'$ and

$$\begin{aligned}S(X,Y) = S(X',Y') &+ 4v(Y)HX - 4v(X)HY - u(X)G(\nabla_U G)Y \\ &- v(X)G(\nabla_V G)Y + u(Y)G(\nabla_U G)X + v(Y)G(\nabla_V G)X \\ &+ u(X)\sigma(U)GHY + v(X)\sigma(V)GHY \\ &- u(Y)\sigma(U)GHX - v(Y)\sigma(V)GHX.\end{aligned}$$

From (6) and (11) we get

$$\begin{aligned}(\nabla_V \hat{G})(X,Y) = 2g(X,GHY) &+ 2u\wedge v(X,Y)\Omega(U,V) \\ &- \Omega(X,Y) + \sigma(V)g(X,HY).\end{aligned} \tag{14}$$

Now, using equation (14), Lemma 2.4 and the fact that $S(X',Y') = 0$ for any vector field Z, we have

$$\begin{aligned}g(S(X,Y),Z) = 2v(Y)g(HX,Z) &- 2v(X)g(HY,Z) \\ &- v(X)\Omega(GZ,Y) + v(Y)\Omega(GZ,X).\end{aligned} \tag{15}$$

If we take $Y = V$ and GX instead of X in (15), we get

$$g(S(GX,V),Z) = 2g(HGX,Z) + \Omega(GZ,GX). \tag{16}$$

On the other hand, by (3) and (4), $u(\nabla_V GX) = v(\nabla_V GX) = 0$. When we substitute these in $S(GX,V)$, we get

$$S(GX,V) = 4HGX + (\nabla_V G)X - \sigma(V)HX.$$

Hence,

$$g(S(GX,V),Z) = 2g(HGX,Z) - 2u\wedge v(X,Z)\Omega(U,V) + \Omega(X,Z).$$

Combining with (16), we get

$$\Omega(GZ,GX) = \Omega(X,Z) - 2u\wedge v(X,Z)\Omega(U,V). \tag{17}$$

Applying the above process to $T(X,Y)$, we get

$$\begin{aligned}g(T(X,Y),Z) = 2u(Y)g(GX,Z) &- 2u(X)g(GY,Z) \\ &+ u(X)\Omega(HZ,Y) - u(Y)\Omega(HZ,X)\end{aligned} \tag{18}$$

and

$$\Omega(HZ, HX) = \Omega(X, Z) - 2u \wedge v(X, Z)\Omega(U, V). \tag{19}$$

Combining (17) with (19) gives

$$\Omega(GZ, GX) = \Omega(HZ, HX). \tag{20}$$

Equation (20) implies

$$\Omega(G^2 Z, G^2 X) = \Omega(HGZ, HGX).$$

If we compute the lefthand side and the righthand side separately using (10) and (11), we get

$$\Omega(G^2 Z, G^2 X) = \Omega(Z, X) + (u(X)v(Z) - v(X)u(Z))\Omega(U, V),$$

and

$$\Omega(HGZ, HGX) = \Omega(JZ, JX) + (u(X)v(Z) - u(Z)v(X))\Omega(U, V).$$

Therefore,

$$\Omega(Z, X) = \Omega(JZ, JX). \tag{21}$$

Replacing X with GX in (17), we get

$$\begin{aligned}\Omega(GX, Z) &= \Omega(GZ, G^2 X)\\ &= -\Omega(GZ, X) + u(X)\Omega(GZ, U) + v(X)\Omega(GZ, V).\end{aligned}$$

Equations (10) and (11) imply $\Omega(GZ, U) = \Omega(GZ, V) = 0$. Hence,

$$\Omega(GX, Z) = \Omega(X, GZ). \tag{22}$$

Similarly, replacing X with HX in (19), we get

$$\Omega(HX, Z) = \Omega(X, HZ). \tag{23}$$

Finally, replacing X with JX in (21), we get

$$\Omega(JX, Z) = -\Omega(X, JZ). \tag{24}$$

We now want to compute $S(X,Y)$ in a different way. First, we can rewrite $G(\nabla_X G)Y$ as

$$G(\nabla_X G)Y = -u(Y)GX - v(Y)HX - (\nabla_X G)GY + g(X,GY)U + g(X,HY)V. \tag{25}$$

Now let us substitute (25) in $S(X,Y)$ to get

$$\begin{aligned} S(X,Y) = {} & (\nabla_{GX}G)Y - (\nabla_{GY}G)X + (\nabla_X G)GY - (\nabla_Y G)GX \\ & + u(Y)GX + 3v(Y)HX - u(X)GY - 3v(X)HY - 4g(X,HY)V \\ & - \sigma(GX)HY + \sigma(GY)HX + \sigma(X)GHY - \sigma(Y)GHX. \end{aligned}$$

Taking the inner product with Z and using equations (6), (22) and (25) gives

$$\begin{aligned} g(S(X,Y),Z) = {} & 2g((\nabla_Z G)Y,GX) + 2v(Z)\Omega(X,GY) - v(Y)\Omega(X,GZ) \\ & + v(X)\Omega(Y,GZ) + 2\sigma(Z)g(Y,HGX) + 2u(Y)g(GX,Z) \\ & + 4v(Y)g(HX,Z) - 2v(X)g(HY,Z) - 4v(Z)g(X,HY). \end{aligned}$$

If we combine the above equation with equation (15), we get

$$\begin{aligned} & 2g((\nabla_Z G)Y,GX) + 2v(Z)\Omega(X,GY) + 2\sigma(Z)g(Y,HGX) \\ & \quad + 2u(Y)g(GX,Z) + 2v(Y)g(HX,Z) - 4v(Z)g(X,HY) = 0. \end{aligned}$$

In order to get the equation we want, we replace X with GX which gives

$$\begin{aligned} & 2g((\nabla_Z G)X,Y) + 2v(Z)\Omega(GX,GY) + 2\sigma(Z)g(X,HY) \\ & \quad - 2u(Y)g(X,Z) - 2v(Y)g(X,JZ) - 2v(Y)u(Z)v(X) \\ & \quad + 4v(Z)g(X,GHY) + 2u(X)g(Z,Y) - 2v(X)g(Z,JY) \\ & \quad + 2v(X)v(Y)u(Z) = 0. \end{aligned}$$

Now equation (I) follows.

Applying the same process to $T(X,Y)$, we can easily see that equation (II) also holds.

Conversely, suppose that formulas (I) and (II) hold. To show that M is normal, first let us check $S(X,U)$. Since formula (I) holds,

$$\begin{aligned} g(S(X,U),Y) &= g((\nabla_U G)GY,X)+g((\nabla_{GX}G)U,Y)+g((\nabla_X G)U,GY) \\ &\quad - \sigma(U)g(GHX,Y) \\ &= \sigma(U)g(HGY,X) - g(GX,Y) - g(X,GY) \\ &\quad - \sigma(U)g(GHX,Y) \\ &= 0. \end{aligned}$$

Therefore, $S(X,U)=0$. Similarly, $T(X,V)=0$.

Now let X and Y be two vector fields in $\mathcal{H}$. Making use of the fact that $u(X)=v(X)=u(Y)=v(Y)=0$ and applying formula (I), we get

$$\begin{aligned} g(S(X,Y),Z) &= g((\nabla_{GX}G)Y,Z) + g((\nabla_{GY}G)Z,X) + g((\nabla_X G)Y,GZ) \\ &\quad + g((\nabla_Y G)GZ,X)+2u(Z)g(X,GY)-2v(Z)g(X,HY) \\ &\quad -\sigma(GX)g(HY,Z)+\sigma(GY)g(HX,Z)+\sigma(X)g(GHY,Z) \\ &\quad - \sigma(Y)g(GHX,Z) \\ &= \sigma(GX)g(HY,Z) + u(Z)g(GX,Y) - v(Z)g(GX,JY) \\ &\quad + \sigma(GY)g(HZ,X) - u(Z)g(GY,X) - v(Z)g(JGY,X) \\ &\quad +\sigma(X)g(HY,GZ)+\sigma(Y)g(HGZ,X)+2u(Z)g(X,GY) \\ &\quad -2v(Z)g(X,HY)-\sigma(GX)g(HY,Z)+\sigma(GY)g(HX,Z) \\ &\quad + \sigma(X)g(GHY,Z) - \sigma(Y)g(GHX,Z) \\ &= 0. \end{aligned}$$

Therefore, $S(X,Y)=0$.

In a similar way, we can also show that $T(X,Y)=0$. Therefore, M is normal. $\square$

At the moment, normality appears to be a local notion since the tensors S and T were defined locally. Our next step is to show that normality is, in fact, a global notion. Towards this end, let us define a third tensor W as follows:

$$\begin{aligned}W(X,Y) = [G,H](X,Y) + \frac{1}{2}\big(&\sigma(GX)GY - \sigma(HX)HY\\ &- \sigma(GY)GX + \sigma(HY)HX\big)\\ &- u(Y)HX - V(Y)GX + u(X)HY + v(X)GY\\ &+ 2g(X,GY)V + 2g(X,HY)U\end{aligned}$$

where $[G,H](X,Y) = 1/2([GX,HY] + [HX,GY] - G[HX,Y] - H[GX,Y] - G[X,HY] - H[X,GY])$.

If M is normal, in other words if

$$\begin{aligned}&S(U,X) = T(V,X) = 0 \quad \text{for all } X, \text{ and}\\ &S(X,Y) = T(X,Y) = 0 \quad \text{for all } X \text{ and } Y \text{ in } \mathcal{H},\end{aligned}$$

then equations (I) and (II) hold. Then, using (I) and (II), we get

$$\begin{aligned}&g([G,H](X,Y),Z)\\ &\quad = \frac{1}{2}\big(\sigma(HX)g(HY,Z) - \sigma(GX)g(GY,Z)\\ &\qquad - 4u(Z)g(X,HY) - 4v(Z)g(X,GY) + \sigma(GY)g(GX,Z)\\ &\qquad - \sigma(HY)g(HX,Z) + u(X)\Omega(GZ,Y) - v(X)\Omega(HZ,Y)\\ &\qquad + v(Y)\Omega(HZ,X) - u(Y)\Omega(GZ,X)\big).\end{aligned}$$

Hence, for X, Y in $\mathcal{H}$,

$$\begin{aligned}&W(X,Y)\\ &\quad = \frac{1}{2}\big(\sigma(HX)HY - \sigma(GX)GY - 4g(X,HY)U - 4g(X,GY)V\\ &\qquad + \sigma(GY)GX - \sigma(HY)HX + \sigma(GX)GY - \sigma(HX)HY\\ &\qquad - \sigma(GY)GX + \sigma(HY)HX\big) + 2g(X,GY)V + 2g(X,HY)U\\ &\qquad = 0.\end{aligned}$$

We now check the normality condition on an overlap $\mathcal{O} \cap \mathcal{O}'$. On the open set $\mathcal{O}$, we have tensors u, v, G, H, S, T and W. On $\mathcal{O}'$, we have u', v', G', H', S', T'. Since M is a contact metric manifold, there are functions a and b on $\mathcal{O} \cap \mathcal{O}'$ such that

$$\begin{aligned} u' &= au - bv \\ v' &= bu + av \\ G' &= aG - bH \\ H' &= bG + aH \\ a^2 + b^2 &= 1. \end{aligned}$$

Lemma 3.4. *$S' = a^2S + b^2T - 2abW$ and $T' = b^2S + a^2T + 2abW$.*

Proof. First of all $U' = aU - bV$ and $V' = bU + aV$. Using this fact we see that

$$\sigma'(X) = \sigma(X) + bX(a) - aX(b).$$

Note that $aX(a) + bX(b) = 0$ for any X since $a^2 + b^2 = 1$. Also $G'H' = GH$. Now if we compute $S'(X,Y)$ using what we have so far and grouping terms under a^2, b^2 and ab, we get

$$S'(X,Y) = a^2S(X,Y) + b^2T(X,Y) - 2abW(X,Y).$$

Similarly,

$$T'(X,Y) = b^2S(X,Y) + a^2T(X,Y) + 2abW(X,Y). \qquad \square$$

Now assume that $S(X,Y) = T(X,Y) = 0$ for all horizontal X and Y and $S(U,X) = T(V,X) = 0$ for all X. Then, as we checked above, $W(X,Y) = 0$ for all horizontal X and Y. Therefore, $S'(X,Y) = T'(X,Y) = 0$ by the above lemma.

For an arbitrary vector field X, apply the above lemma to $S'(U',X)$ to get

$$\begin{aligned} S'(U',X) = a^2b[&G(\nabla_V G)X + \sigma(V)HGX + G(\nabla_U H)X + H(\nabla_U G)X] \\ &- ab^2[H(\nabla_U H)X + \sigma(U)HGX + G(\nabla_V H)X + H(\nabla_V G)X]. \end{aligned}$$

Now, taking the inner product with Y and using equations (I) and (II) gives

$$g(S'(U',X),Y) = 0.$$

Therefore, $S'(U', X) = 0$.

Similarly, we can show that $T'(V', X) = 0$.

Therefore, normality conditions agree on the overlaps. So the notion of normality is global.

We now give an expression for $\nabla_X J$. Recall that on a complex contact manifold we have $H = GJ = -JG$, $V = -JU$, $U = JV$. Also, using Proposition 3.2, we have

$$(\nabla_X J)U = HX + \sigma(X)U - J(-GX + \sigma(X)V) = 0$$

and

$$(\nabla_X J)V = -GX + \sigma(X)V - J(-HX - \sigma(X)U) = 0.$$

Then we can write

$$(\nabla_X H)GY = (\nabla_X J)Y - J(\nabla_X G)GY.$$

Taking the inner product with Z and applying equations (I) and (II) gives

(III)

$$\begin{aligned} g((\nabla_X J)Y, Z) = u(X)&(\Omega(Z, GY) - 2g(HY, Z)) \\ &+ v(X)(\Omega(Z, HY) + 2g(GY, Z)). \end{aligned}$$

4. Some basic facts on normal complex contact metric manifolds. In this section we will establish some basic formulas for a normal complex contact metric manifold M with structure tensors u, v, U, V, G, H, J, g. First we will consider the curvature of the vertical plane, $g(R(U, V)V, U)$. Using Proposition 3.2,

$$\begin{aligned} R(U, V)V &= \nabla_U(-\sigma(V)U) - \nabla_V(-\sigma(U)U) + \sigma([U, V])U \\ &= -U(\sigma(V))U - \sigma(V)\sigma(U)V + V(\sigma(U))U \\ &\quad + \sigma(U)\sigma(V)V + \sigma([U, V])U \\ &= -2\Omega(U, V)U. \end{aligned}$$

Therefore,

$$g(R(U,V)V,U) = -2\Omega(U,V). \tag{26}$$

Now let X and Y be two horizontal vector fields. Then, using Proposition 3.2,

$$R(X,Y)U = -(\nabla_X G)Y + (\nabla_Y G)X + 2\Omega(X,Y)V - \sigma(Y)HX + \sigma(X)HY.$$

By equation (I) we know that

$$(\nabla_X G)Y = \sigma(X)HY + g(X,Y)U + g(JX,Y)V.$$

If we substitute this in $R(X,Y)U$ we get

$$R(X,Y)U = 2(g(X,JY) + \Omega(X,Y))V. \tag{27}$$

Similarly, using Proposition 3.2, we have

$$R(X,Y)V = -2(g(X,JY) + \Omega(X,Y))U. \tag{28}$$

Now we can compute $R(X,U)U$ for horizontal X, using Proposition 3.2:

$$R(X,U)U = 2\Omega(X,U)V - \sigma(U)HX + (\nabla_U G)X + X.$$

Since X is horizontal, $\Omega(X,U) = 0$ by (10), and $(\nabla_U G)X = \sigma(U)HX$ by Lemma 2.4. Therefore

$$R(X,U)U = X. \tag{29}$$

Similarly,

$$R(X,V)V = X. \tag{30}$$

Again, for a horizontal vector field X we can compute $R(X,U)V$ and $R(X,V)U$ using Proposition 3.2 to get

$$R(X,U)V = \sigma(U)GX + (\nabla_U H)X - JX \tag{31}$$

and

$$R(X,V)U = -\sigma(V)HX + (\nabla_V G)X + JX. \tag{32}$$

Now define a new tensor P_G by

$$P_G(X,Y,Z,W) = g(R(X,Y)GZ,W) + g(R(X,Y)Z,GW)$$

and similarly define tensors P_H and P_J.

Our next step is to get an expression for P_G free of the curvature tensor R. By a direct computation, it is easy to see that we can write

$$P_G(X,Y,Z,W) = -(\nabla_X\nabla_Y\hat{G} - \nabla_Y\nabla_X\hat{G} - \nabla_{[X,Y]}\hat{G})(Z,W).$$

For horizontal vector fields X, Y, Z and W, if we compute the right-hand side of the above equation using (I), we get:

$$\begin{aligned} P_G(X,Y,Z,W) = {} & 2g(HZ,W)\Omega(X,Y) - 2g(HX,Y)\Omega(Z,W) \\ & + 4g(HX,Y)g(JZ,W) + g(GX,Z)g(Y,W) \\ & + g(HX,Z)g(JY,W) - g(GX,W)g(Y,Z) \\ & - g(HX,W)g(JY,Z) - g(GY,Z)g(X,W) \\ & - g(HY,Z)g(JX,W) + g(GY,W)g(X,Z) \\ & + g(HY,W)g(JX,Z). \end{aligned} \tag{33}$$

In the same way, we can show that

$$\begin{aligned} P_H(X,Y,Z,W) = {} & -2g(GZ,W)\Omega(X,Y) + 2g(GX,Y)\Omega(Z,W) \\ & - 4g(GX,Y)g(JZ,W) + g(HX,Z)g(Y,W) \\ & - g(GX,Z)g(JY,W) - g(HX,W)g(Y,Z) \\ & + g(GX,W)g(JY,Z) - g(HY,Z)g(X,W) \\ & + g(GY,Z)g(JX,W) + g(HY,W)g(X,Z) \\ & - g(GY,W)g(JX,Z). \end{aligned} \tag{34}$$

Since $JX = HGX = -GHX$ for horizontal X,

(35)

$$\begin{aligned} P_J(X,Y,Z,W) &= g(R(X,Y)HGZ,W) - g(R(X,Y)Z,GHW) \\ &= P_H(X,Y,GZ,W) - P_G(X,Y,Z,HW) \\ &= 2g(GX,Y)\Omega(GZ,W) + 2g(HX,Y)\Omega(HZ,W) \\ &\quad + 4g(GX,Y)g(HZ,W) - 4g(HX,Y)g(GZ,W). \end{aligned}$$

Lemma 4.1. *For horizontal vector fields X, Y, Z and W, the curvature tensor satisfies the following equations*:

(i)

$$\begin{aligned} &g(R(GX, GY)GZ, GW) \\ &\quad = g(R(X, Y)Z, W) - 2g(JZ, W)\Omega(X, Y) + 2g(HX, Y)\Omega(GZ, W) \\ &\qquad + 2g(JX, Y)\Omega(Z, W) - 2g(HZ, W)\Omega(GX, Y), \end{aligned}$$

(ii)

$$\begin{aligned} &g(R(HX, HY)HZ, HW) \\ &\quad = g(R(X, Y)Z, W) - 2g(JZ, W)\Omega(X, Y) - 2g(GX, Y)\Omega(HZ, W) \\ &\qquad + 2g(JX, Y)\Omega(Z, W) + 2g(GZ, W)\Omega(HX, Y). \end{aligned}$$

Proof. By the definition of P_G, the lefthand side of (i) is equal to

$$g(R(X, Y)Z, W) + P_G(Z, W, X, GY) + P_G(GX, GY, Z, GW).$$

Equation (33) gives

$$\begin{aligned} &P_G(Z, W, X, GY) + P_G(GX, GY, Z, GW) \\ &\quad = 2g(JX, Y)\Omega(Z, W) - 2g(HZ, W)\Omega(GX, Y) - 2g(JZ, W)\Omega(X, Y) \\ &\qquad + 2g(HX, Y)\Omega(GZ, W). \end{aligned}$$

Therefore equation (i) holds.

Similarly, using the definition of P_H and equation (34) we obtain (ii). □

Lemma 4.2. *The following equations hold for horizontal vector fields X, Y, Z and W*:

(i)

$$\begin{aligned} &g(R(X, GX)Y, GY) \\ &\quad = g(R(X, Y)X, Y) + g(R(X, GY)X, GY) + 4g(JX, Y)\Omega(X, Y) \\ &\qquad - 4g(HX, Y)\Omega(GX, Y) - 2g(GX, Y)^2 - 4g(HX, Y)^2 \\ &\qquad - 2g(X, Y)^2 + 2g(X, X)g(Y, Y) - 4g(JX, Y)^2 \end{aligned}$$

(ii)

$$\begin{aligned}
&g(R(X,HX)Y,HY)\\
&\quad = g(R(X,Y)X,Y) + g(R(X,HY)X,HY) + 4g(JX,Y)\Omega(X,Y)\\
&\qquad + 4g(GX,Y)\Omega(HX,Y) - 2g(HX,Y)^2 - 4g(GX,Y)^2\\
&\qquad - 2g(X,Y)^2 + 2g(X,X)g(Y,Y) - 4g(JX,Y)^2.
\end{aligned}$$

Proof. By Bianchi's first identity,

$$g(R(X,GX)Y,GY) = -g(R(GX,Y)X,GY) - g(R(Y,X)GX,GY).$$

The definition of P_G implies

$$-g(R(GX,Y)X,GY) = g(R(X,GY)X,GY) - P_G(X,GY,X,Y)$$

and

$$-g(R(Y,X)GX,GY) = g(R(X,Y)X,Y) + P_G(X,Y,X,GY).$$

Using equation (33), we get

$$\begin{aligned}
&P_G(X,Y,X,GY) - P_G(X,GY,X,Y)\\
&\quad = 4g(JX,Y)\Omega(X,Y) - 4g(HX,Y)\Omega(GX,Y) - 4g(HX,Y)^2\\
&\qquad - 2g(X,Y)^2 - 4g(JX,Y)^2 - 2g(GX,Y)^2 + 2g(X,X)g(Y,Y)
\end{aligned}$$

which gives equation (i) and equation (ii) is obtained in the same way. □

Lemma 4.3. *If X is a horizontal vector field, then*

$$\begin{aligned}
g(R(X,GX)GX,X) + g(R(X,HX)HX,X) + g(R(X,JX)JX,X)&\\
= -6g(X,X)(\Omega(JX,X) + g(X,X)).&
\end{aligned}$$

Proof. Recall that $GX = -HJX$. Then, by the definition of P_H,

$$g(R(X,GX)GX,X) = g(R(X,GX)JX,GJX) - P_H(X,GX,JX,X).$$

By Lemma 4.2,

$$g(R(X,GX)JX,GJX) \\ = -g(R(X,JX)JX,X) - g(R(X,HX)HX,X) \\ - 4g(X,X)\Omega(JX,X) - 2g(X,X)^2.$$

We can compute $P_H(X,GX,JX,X)$ using equation (34) to get

$$P_H(X,GX,JX,X) = 2g(X,X)\Omega(JX,X) + 4g(X,X)^2.$$

We get the lemma by joining the above equations. □

We can use the definition of P_G and equation (33) to see that the following formulas hold for a horizontal vector field X:

$$g(R(X,HX)JX,GX) = -g(R(X,HX)HX,X) \\ - 2g(X,X)\Omega(JX,X) - 4g(X,X)^2, \tag{36}$$

$$g(R(X,JX)HX,GX) = g(R(X,JX)JX,X) \\ + 2g(X,X)\Omega(JX,X) - 2g(X,X)^2, \tag{37}$$

$$g(R(GX,HX)HX,GX) = g(R(X,JX)JX,X), \tag{38}$$

$$g(R(GX,JX)JX,GX) = g(R(X,HX)HX,X). \tag{39}$$

Similarly, using the definition of P_J and equation (35) we get the following formulas for horizontal vector fields X, Y:

$$g(R(JX,JY)JY,JX) = g(R(X,Y)Y,X), \tag{40}$$

$$g(R(X,Y)JX,JY) \\ = g(R(X,Y)Y,X) + 2g(X,GY)\Omega(X,HY) \\ - 2g(X,HY)\Omega(X,GY) + 4g(X,GY)^2 + 4g(X,HY)^2, \tag{41}$$

$$g(R(Y,JX)JX,Y) = g(R(X,JY)JY,X), \tag{42}$$

(43) $$\begin{aligned} g(R(X,JY)JX,Y) &= g(R(X,JY)JY,X) - 2g(X,HY)\Omega(X,GY) \\ &\quad + 2g(X,GY)\Omega(X,HY) + 4g(X,HY)^2 + 4g(X,GY)^2. \end{aligned}$$

By Bianchi's first identity,

$$g(R(X,JX)JY,Y) = -g(R(JX,JY)X,Y) - g(R(JY,X)JX,Y).$$

Substituting formulas (41) and (43) in the above equation, we get

(44) $$\begin{aligned} &g(R(X,JX)JY,Y) \\ &\quad = g(R(X,Y)Y,X) + g(R(X,JY)JY,X) + 4(g(X,GY)\Omega(X,HY) \\ &\qquad - g(X,HY)\Omega(X,GY) + 2g(X,GY)^2 + 2g(X,HY)^2). \end{aligned}$$

5. ***GH*-sectional curvature.** Let M be a normal complex contact metric manifold with structure tensors u, v, U, V, G, H, J, g. For a horizontal vector field X, the plane section generated by X and $Y = aGX + bHX$, $a^2 + b^2 = 1$, is called a GH-*section* or an $\mathcal{H}$-*holomorphic section*. We define the GH-sectional curvature $\mathcal{GH}_{a,b}(X)$ as the curvature of a GH-section:

$$\mathcal{GH}_{a,b}(X) = K(X, aGX + bHX)$$

where $K(X,Y)$ is the curvature of the plane section generated by X and Y.

Lemma 5.1. *$\mathcal{GH}_{a,b}(X)$ is independent of the choice of the numbers a and b if and only if $K(X,GX) = K(X,HX)$ and $g(R(X,GX)HX,X) = 0$.*

Proof. We can write the GH-sectional curvature as

$$\begin{aligned} &\mathcal{GH}_{a,b}(X) \\ &\quad = a^2K(X,GX) + b^2K(X,HX) + \frac{2ab}{g(X,X)^2}\, g(R(X,GX)HX,X). \end{aligned}$$

If $\mathcal{GH}_{a,b}(X)$ is independent of the choice of a and b, then taking $a = 1$, $b = 0$ gives $\mathcal{GH}_{a,b}(X) = K(X,GX)$ and taking $a = 0$,

$b = 1$ gives $\mathcal{GH}_{a,b}(X) = K(X, HX)$. So $K(X, GX) = K(X, HX)$ and $g(R(X, GX)HX, X) = 0$.

Conversely, if $K(X, GX) = K(X, HX) = K$ and $g(R(X, GX)HX, X) = 0$, then $\mathcal{GH}_{a,b}(X) = K$ and hence $\mathcal{GH}_{a,b}(X)$ is independent of the choice of a and b. □

From now on, we will assume that $\mathcal{GH}_{a,b}(X)$ is independent of the choice of a and b and denote it by $\mathcal{GH}(X)$.

As the next step, we want to write holomorphic curvature in terms of GH-sectional curvature. In order to do this, we are going to use the formulas from Section 4.

Proposition 5.2. *For a horizontal vector field X,*

$$K(X, JX) = \frac{1}{2}\left(\mathcal{GH}(X + GX) + \mathcal{GH}(X - GX)\right) + 3.$$

Proof. Since $\mathcal{GH}(X)$ is independent of the choice of a and b, we can choose $a = 0$, $b = 1$. Then $\mathcal{GH}(X) = K(X, HX)$. So $\mathcal{GH}(X + GX) = K(X + GX, HX + JX)$ and $\mathcal{GH}(X - GX) = K(X - GX, HX - JX)$. By direct computation we get

$$\begin{aligned} &g(R(X{+}GX, HX + JX)HX + JX, X + GX) \\ &\qquad = g(R(X, HX)HX, X) + g(R(X, JX)JX, X) \\ &\qquad\quad + g(R(GX, HX)HX, GX) + g(R(GX, JX)JX, GX) \\ &\qquad\quad + 2[g(R(X, HX)HX, GX) + g(R(X, HX)JX, X) \\ &\qquad\quad + g(R(X, HX)JX, GX) + g(R(X, JX)HX, GX) \\ &\qquad\quad + g(R(X, JX)JX, GX) + g(R(GX, HX)JX, GX)] \end{aligned}$$

and

$$\begin{aligned} &g(R(X{-}GX, HX - JX)HX - JX, X - GX) \\ &\qquad = g(R(X, HX)HX, X) + g(R(X, JX)JX, X) \\ &\qquad\quad + g(R(GX, HX)HX, GX) + g(R(GX, JX)JX, GX) \\ &\qquad\quad + 2[-g(R(X, HX)HX, GX) - g(R(X, HX)JX, X) \\ &\qquad\quad + g(R(X, HX)JX, GX) + g(R(X, JX)HX, GX) \\ &\qquad\quad - g(R(X, JX)JX, GX) - g(R(GX, HX)JX, GX)]. \end{aligned}$$

If we add the two equations above, we get

$$\begin{aligned}\mathcal{GH}(X+GX)&+\mathcal{GH}(X-GX)\\ &=\frac{1}{2g(X,X)^2}\,[g(R(X,HX)HX,X)+g(R(X,JX)JX,X)\\ &\quad+g(R(GX,HX)HX,GX)+g(R(GX,JX)JX,GX)\\ &\quad+2[g(R(X,HX)JX,GX)+g(R(X,JX)HX,GX)]].\end{aligned}$$

Now, using formulas (36)–(39), we have

$$\mathcal{GH}(X+GX)+\mathcal{GH}(X-GX)=2K(X,JX)-6.$$

Therefore

$$K(X,JX)=\frac{1}{2}\,(\mathcal{GH}(X+GX)+\mathcal{GH}(X-GX))+3. \qquad \square$$

We now want to work with the assumption that the GH-sectional curvature is independent of the choice of the GH-section at each point. Let $\mathcal{GH}(X)=c$ where c does not depend on X. Then by the previous proposition

$$K(X,JX)=c+3.$$

Next we give an expression for the sectional curvature in terms of the holomorphic curvature.

Lemma 5.3. *For horizontal vector fields X and Y, we have*

$$\begin{aligned}g(R(X,Y)Y,X)=\frac{1}{32}\,&[3Q(X+JY)+3Q(X-JY)-Q(X+Y)\\ &-Q(X-Y)-4Q(X)-4Q(Y)]\\ +\frac{3}{2}\,&[g(X,HY)\Omega(X,GY)-g(X,GY)\Omega(X,HY)\\ &-2g(X,GY)^2-2g(X,HY)^2],\end{aligned}$$

where $Q(X)=g(R(X,JX)JX,X)$.

Proof. By direct computation

$$\begin{aligned} Q(X+JY) = {} & g(R(X,JX)JX,X) + g(R(Y,JY)JY,Y) \\ & + g(R(JX,JY)JY,JX) + g(R(X,Y)Y,X) \\ & + 2[g(R(X,JX)JX,JY) - g(R(X,JX)Y,X) \\ & \quad - g(R(X,JX)Y,JY) - g(R(X,Y)JX,JY) \\ & \quad + g(R(X,Y)Y,JY) - g(R(JY,JX)Y,JY)] \end{aligned}$$

and

$$\begin{aligned} Q(X-JY) = {} & g(R(X,JX)JX,X) + g(R(Y,JY)JY,Y) \\ & + g(R(JX,JY)JY,JX) + g(R(X,Y)Y,X) \\ & + 2[-g(R(X,JX)JX,JY) + g(R(X,JX)Y,X) \\ & \quad - g(R(X,JX)Y,JY) - g(R(X,Y)JX,JY) \\ & \quad - g(R(X,Y)Y,JY) + g(R(JY,JX)Y,JY)]. \end{aligned}$$

By combining the two equations above, we get

$$\begin{aligned} & Q(X+JY)+Q(X-JY) \\ & \quad = 2[g(R(X,JX)JX,X) + g(R(Y,JY)JY,Y) \\ & \qquad + g(R(JX,JY)JY,JX) + g(R(X,Y)Y,X)] \\ & \quad - 4[g(R(X,JX)Y,JY) + g(R(X,Y)JX,JY)]. \end{aligned}$$

Using the formulas (40), (41) and (44), we have

$$\begin{aligned} & Q(X+JY)+Q(X-JY) \\ & \quad = 2[g(R(X,JX)JX,X) + g(R(Y,JY)JY,Y)] \\ & \qquad + 4[3g(R(X,Y)Y,X) + g(R(X,JY)JY,X)] \\ & \qquad + 24[g(X,GY)\Omega(X,HY) - g(X,HY)\Omega(X,GY) \\ & \qquad\quad + 2g(X,GY)^2 + 2g(X,HY)^2]. \end{aligned}$$

Doing the same calculations for $Q(X+Y)+Q(X-Y)$ and using the formulas (42), (43) and (44), we get

$$\begin{aligned} & Q(X+Y)+Q(X-Y) \\ & \quad = 2[g(R(X,JX)JX,X) + g(R(Y,JY)JY,Y)] \\ & \qquad + 4[3g(R(X,JY)JY,X) + g(R(X,Y)Y,X)] \\ & \qquad + 24[g(X,GY)\Omega(X,HY) - g(X,HY)\Omega(X,GY) \\ & \qquad\quad + 2g(X,GY)^2 + 2g(X,HY)^2]. \end{aligned}$$

Finally, combining what we have so far,

$$\begin{aligned}3Q(X+JY)+3Q(X-JY)-Q(X+Y)-Q(X-Y)-4Q(X)-4Q(Y)\\ =32g(R(X,Y)Y,X)+48[g(X,GY)\Omega(X,HY)\\ -g(X,HY)\Omega(X,GY)+2g(X,GY)^2+2g(X,HY)^2],\end{aligned}$$

giving us the desired result. □

Since $K(X,JX)=c+3$ does not depend on X, from the above lemma we get

(45)

$$\begin{aligned}g(R(X,Y)Y,X)=\frac{c+3}{4}&[g(X,X)g(Y,Y)-g(X,Y)^2+3g(X,JY)^2]\\ +\frac{3}{2}&[g(X,HY)\Omega(X,GY)-g(X,GY)\Omega(X,HY)\\ &-2g(X,GY)^2-2g(X,HY)^2],\end{aligned}$$

for horizontal X and Y.

Now let X and Y be two arbitrary vector fields. We can write

$$X=Z+u(X)U+v(X)V,\qquad Y=W+u(Y)U+v(Y)V$$

where Z and W are in $\mathcal{H}$. Then, using the formulas (26)–(32) and (45), we have

$$\begin{aligned}g(R(X,Y)Y,X)&\\ =g(R(Z,W)W,Z)&-2(u(X)u(Y)+v(X)v(Y))g(Z,W)\\ &+(u(Y)^2+v(Y)^2)g(Z,Z)+(u(X)^2+v(X)^2)g(W,W)\\ &-12u\wedge v(X,Y)g(Z,JW)-12u\wedge v(X,Y)\Omega(Z,W)\\ &-8(u\wedge v(X,Y))^2\Omega(U,V)\\ =g(R(Z,W)W,Z)&-2(u(X)u(Y)+v(X)v(Y))g(X,Y)\\ &+(u(Y)^2+v(Y)^2)g(X,X)+(u(X)^2+v(X)^2)g(Y,Y)\\ &-12u\wedge v(X,Y)g(X,JY)-12u\wedge v(X,Y)\Omega(X,Y)\\ &+16(u\wedge v(X,Y))^2(1+\Omega(U,V))\end{aligned}\tag{46}$$

$$\begin{aligned}
&= \frac{c-1}{2}\,[u(X)u(Y) + v(X)v(Y)]g(X,Y) \\
&\quad - \frac{c-1}{4}\,[(u(Y)^2+v(Y)^2)g(X,X)+(u(X)^2+v(X)^2)g(Y,Y)] \\
&\quad - 3(c+7)u \wedge v(X,Y)g(X,JY) \\
&\quad + \frac{c+3}{4}\,[g(X,X)g(Y,Y) + 3g(X,JY)^2 - g(X,Y)^2] \\
&\quad + \frac{3}{2}\,[g(X,HY)\Omega(X,GY) - g(X,GY)\Omega(X,HY) \\
&\quad - 2g(X,GY)^2 - 2g(X,HY)^2] + 4(c+7)(u \wedge v(X,Y))^2 \\
&\quad - 12u \wedge v(X,Y)\Omega(X,Y) + 16(u \wedge v(X,Y))^2\Omega(U,V).
\end{aligned} \tag{47}$$

In order to simplify the above equation somewhat, we need to examine the term $\Omega(X,Y)$. Since $\mathcal{GH}(X) = c+3$ does not depend on X,

$$g(R(X,GX)GX,X) = g(R(X,HX)HX,X) = cg(X,X)^2$$

and

$$g(R(X,JX)JX,X) = (c+3)g(X,X)^2.$$

Substituting these in Lemma 4.3, we get

$$\Omega(JX,X) = -\frac{c+3}{2}\,g(X,X) \tag{48}$$

for horizontal X.

In order to compute $\Omega(JX,X)$ for an arbitrary vector field X, we can apply formula (48) to the horizontal component of X to get

$$\begin{aligned}
\Omega(JX,X) = -\frac{c+3}{2}\,g(X,X) + \frac{c+3}{2}\,(u(X)^2 + v(X)^2) \\
+ (u(X)^2 + v(X)^2)\Omega(U,V).
\end{aligned} \tag{49}$$

Replacing X with $JX+Y$ in (49), we have

$$\Omega(X,Y) = \frac{c+3}{2}\,g(JX,Y) + u \wedge v(X,Y)(c+3+2\Omega(U,V)). \tag{50}$$

Now if we substitute (50) in (47) we get a somewhat simpler expression for the sectional curvature as

$$\begin{aligned} g(R(X,Y)Y,X) & \\ &= \frac{c-1}{2}\,[u(X)u(Y)+v(X)v(Y)]g(X,Y) \\ &\quad - \frac{c-1}{4}\,[(u(Y)^2+v(Y)^2)g(X,X)+(u(X)^2+v(X)^2)g(Y,Y)] \\ &\quad + 3(c-1)u\wedge v(X,Y)g(X,JY) \\ &\quad + \frac{c+3}{4}\,[g(X,X)g(Y,Y)+3g(X,JY)^2-g(X,Y)^2] \\ &\quad + 3\frac{c-1}{4}\,[g(X,GY)^2+g(X,HY)^2] \\ &\quad - 8(u\wedge v(X,Y))^2(c+1+\Omega(U,V). \end{aligned} \tag{51}$$

Now to get an expression for the curvature tensor, we will use the following identity of [**2**]:

$$\begin{aligned} 6g(R(X,Y)Z,W) = \frac{\partial^2}{\partial s\partial t}\,(B(X+sW,Y+tz) \\ - B(X+sZ,Y+tW))|_{s=0,t=0}, \end{aligned}$$

where $B(X,Y)=g(R(X,Y)Y,X)$.

If we compute the righthand side of the above identity using (51), we get the following expression for the curvature tensor:

$$\begin{aligned} R(X,Y)Z = \frac{c+3}{4}\,[g(Y,Z)X - g(X,Z)Y + g(Z,JY)JX \\ + g(X,JZ)JY + 2g(X,JY)JZ] \\ + \frac{c-1}{4}\big[(u(X)u(Z)+v(X)v(Z))Y-(u(Y)u(Z)+v(Y)v(Z))X \\ + 4u\wedge v(X,Y)JZ + 2u\wedge v(X,Z)JY + 2u\wedge v(Z,Y)JX \\ + 2g(X,GY)GZ + g(X,GZ)GY + g(Z,GY)GX \\ + 2g(X,HY)HZ + g(X,HZ)HY + g(Z,HY)HX \\ + [u(Y)g(X,Z) - u(X)g(Y,Z) + v(X)g(Z,JY) \\ + v(Y)g(X,JZ) + 2v(Z)g(X,JY)]U \end{aligned}$$

$$
\begin{aligned}
&+ [v(Y)g(X,Z) - v(X)g(Y,Z) - u(X)g(Z,JY) \\
&- u(Y)G(X,JZ) - 2u(Z)g(X,JY)]V\big] \\
&- \frac{4}{3}\,(c+1+\Omega(U,V))[(v(X)u\wedge v(Z,Y) + v(Y)u\wedge v(X,Z) \\
&+2v(Z)u\wedge v(X,Y))U - (u(X)u\wedge v(Z,Y) + u(Y)u\wedge v(X,Z) \\
&+ 2u(Z)u\wedge v(X,Y))V]. \qquad (52)
\end{aligned}
$$

Now we are ready to prove the following proposition.

Proposition 5.4. *Let M be a normal complex contact metric manifold with complex dimension greater than or equal to 5. If the GH-sectional curvature is independent of the choice of the GH-section at each point, then it is constant on M.*

Proof. Suppose that the complex dimension of M is $2n+1$. If the GH-sectional curvature is independent of the choice of the GH-section at each point, then the curvature tensor has the form (52). Let us choose a local orthonormal basis of the form

$$\{X_i, GX_i, HX_i, JX_i, U, V \mid 1 \le i \le n\}.$$

Then the Ricci tensor has the form

$$
\begin{aligned}
\rho(X,Y) &= \sum_{i=1}^{n} [g(R(X_i,X)Y,X_i) + g(R(GX_i,X)Y,GX_i) \\
&\quad + g(R(HX_i,X)Y,HX_i) + g(R(JX_i,X)Y,JX_i)] \\
&\quad + g(R(U,X)Y,U) + g(R(V,X)Y,V) \\
&= ((n+2)c + 3n + 2)g(X,Y) + (-(n+2)c + n - 2 \\
&\quad - 2\Omega(U,V))(u(X)u(Y) + v(X)v(Y)).
\end{aligned}
$$

The scalar curvature τ has the form

$$
\begin{aligned}
\tau &= \sum_{i=1}^{n} [\rho(X_i,X_i) + \rho(GX_i,GX_i) + \rho(HX_i,HX_i) \\
&\quad + \rho(JX_i,JX_i)] + \rho(U,U) + \rho(V,V) \\
&= 2(n+2)(2n-1)c + 4n(3n+4) - 4\Omega(U,V).
\end{aligned}
$$

Since $\Omega = d\sigma$, $d\Omega = 0$. In particular, $d\Omega(U, V, X) = 0$, which implies

$$X\Omega(U, V) = u(X)U\Omega(U, V) + v(X)V\Omega(U, V).$$

By Bianchi's identity,

$$2\Big[\sum_{i=1}^{n}((\nabla_{X_i}\rho)(X, X_i) + (\nabla_{GX_i}\rho)(X, GX_i) + (\nabla_{HX_i}\rho)(X, HX_i) + (\nabla_{JX_i}\rho)(X, JX_i)) + (\nabla_U\rho)(X, U) + (\nabla_V\rho)(X, V)\Big] - \nabla_X\tau = 0.$$

Substituting the expressions for $\rho(X, Y)$ and τ, the above equation gives

$$2(1-n)X(c) - (u(X)U(c) + v(X)V(c)) = 0.$$

If we let $X = U$, we get $U(c) = 0$, and if we let $X = V$, we get $V(c) = 0$. Therefore, $X(c) = 0$ if n is different from 1. So c is constant on M when $n > 1$. □

Definition 5.5. A normal complex contact metric manifold M with constant GH-sectional curvature is called a complex contact space form.

The following theorem is an easy consequence of Proposition 5.2 and Lemma 5.3.

Theorem 5.6. *Let M be a normal complex contact metric manifold. Then M has constant GH-sectional curvature c if and only if, for horizontal X, the holomorphic sectional curvature of the plane generated by X and JX is $c + 3$.*

This theorem gives rise to a natural question: is it possible for a normal complex contact metric manifold to have constant holomorphic sectional curvature? We answer this question by the following proposition.

Proposition 5.7. *Let M be a normal complex contact metric manifold. If M has constant holomorphic sectional curvature c, then $c = 4$ and M is Kähler.*

Proof. For an arbitrary unit vector field X, let $X = Z + u(X)U + v(X)V$, where Z is horizontal. If we take $Y = JX$, $W = JZ$ in equation (46), we get

$$\begin{aligned} g(R(X, JX)JX, X) \\ &= g(R(Z, JZ)JZ, Z) + 6(u(X)^2 + v(X)^2)\Omega(X, JX) \\ &\quad - 4(u(X)^2 + v(X)^2) + 4(u(X)^2 + v(X)^2)^2(1+\Omega(U, V)). \end{aligned} \tag{53}$$

Since M has constant holomorphic curvature c,

$$g(R(X, JX)JX, X) = g(R(U, V)V, U) = c,$$

and

$$g(R(Z, JZ)JZ, Z) = g(Z, Z)^2 c.$$

Theorem 5.6 implies that $\mathcal{GH}(X) = c - 3$. Also, by formula (50)

$$\Omega(X, Y) = \frac{c}{2}\, g(JX, Y) + u \wedge v(X, Y)(c + 2\Omega(U, V)).$$

Since $g(R(U, V)V, U) = -2\Omega(U, V)$, $\Omega(U, V) = -(c/2)$. Therefore, $\Omega(X, Y) = (c/2)g(JX, Y)$, and hence $\Omega(X, JX) = (c/2)$. Since X is unit, $g(Z, Z) = 1 - u(X)^2 - v(X)^2$. Substituting these back into (53), we get

$$(c - 4)(u(X)^2 + v(X)^2)(1 - u(X)^2 - v(X)^2) = 0.$$

We can choose X so that $u(X) \neq 0$, $v(X) \neq 0$ and $u(X)^2 + v(X)^2 \neq 1$. Then we must have $c = 4$. In this case $\mathcal{GH}(X) = 1$ and $\Omega(U, V) = -2$.

Since M is normal, by equation (III)

$$\begin{aligned} g((\nabla_X J)Y, Z) &= u(X)\Omega(Z, GY) + v(X)\Omega(Z, HY) - 2u(X)g(HY, Z) \\ &\quad + 2v(X)g(GY, Z) \\ &= 2u(X)g(JZ, GY) + 2v(X)g(JZ, HY) - 2u(X)g(HY, Z) \\ &\quad + 2v(X)g(GY, Z) \\ &= 0. \end{aligned}$$

Hence, M is Kähler. $\square$

Theorem 5.8. *Let M be a normal complex contact metric manifold with constant GH-sectional curvature* 1 *and* $\Omega(U,V) = -2$. *Then M has constant holomorphic sectional curvature* 4 *and it is Kähler. If, in addition, M is complete and simply connected, then M is isometric to* $\mathbf{CP}^{2n+1}$ *with the Fubini-Study metric of constant holomorphic curvature* 4.

Proof. Since $\mathcal{GH}(X) = 1$, $g(R(X,JX)JX,X) = 4g(X,X)^2$ for a horizontal vector field X by Theorem 5.6. Substituting $c = 1$ and $\Omega(U,V) = -2$ in (50), we get $\Omega(X,Y) = 2g(JX,Y)$. For an arbitrary unit vector field X, let $X = Z + u(X)U + v(X)V$, where Z is horizontal. Then $g(Z,Z) = 1 - u(X)^2 - v(X)^2$. Now, from (53) it follows that

$$\begin{aligned} g(R(X,JX)JX,X) &= 4(1-u(X)^2 - v(X)^2)^2 - 4(u(X)^2 + v(X)^2) \\ &\quad + 12(u(X)^2 + v(X)^2) - 4(u(X)^2 + v(X)^2)^2 \\ &= 4. \end{aligned}$$

Hence M has constant holomorphic curvature 4, and by Proposition 5.7, M is Kähler. □

6. Examples of normal complex contact metric manifolds. Our first example of a normal complex contact metric manifold is the complex Heisenberg group. The complex Heisenberg group is the closed subgroup $\mathbf{H_C}$ of $\mathrm{GL}(3,\mathbf{C})$ given by

$$\left\{ \begin{pmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{pmatrix} \mid b_{12}, b_{13}, b_{23} \in \mathbf{C} \right\}.$$

Blair defined the following complex contact metric structure on $\mathbf{H_C}$ in [**1**]. See also [**11**]. Let z_1, z_2, z_3 be the coordinates on $\mathbf{H_C} \simeq \mathbf{C}^3$, defined by $z_1(B) = b_{23}$, $z_2(B) = b_{12}$, $z_3(B) = b_{13}$ for B in $\mathbf{H_C}$. Then the Hermitian metric (matrix)

$$g = \frac{1}{8} \left(\begin{array}{ccc|ccc} & & & 1+|z_2|^2 & 0 & -z_2 \\ & 0 & & 0 & 1 & 0 \\ & & & -\bar{z}_2 & 0 & 1 \\ \hline 1+|z_2|^2 & 0 & -\bar{z}_2 & & & \\ 0 & 1 & 0 & & 0 & \\ -z_2 & 0 & 1 & & & \end{array} \right)$$

is a left invariant metric on $\mathbf{H_C}$. Define a holomorphic 1-form $\theta = (dz_3 - z_2 dz_1)/2$ and set $\theta = u - iv$ and $4(\partial/\partial z_3) = U + iV$.

Also define a (1-1) tensor

$$G = \left(\begin{array}{ccc|ccc} & & & 0 & 1 & 0 \\ & 0 & & -1 & 0 & 0 \\ & & & 0 & z_2 & 0 \\ \hline 0 & 1 & 0 & & & \\ -1 & 0 & 0 & & 0 & \\ 0 & \bar{z}_2 & 1 & & & \end{array}\right).$$

Then $(u, v, U, V, G, H = GJ, g)$ is a complex contact metric structure on $\mathbf{H_C}$. Blair also computed the covariant derivatives of G and H as

$$\begin{aligned}(\nabla_X G)Y = g(X,Y)U - u(Y)X - g(X,JY)V \\ - v(Y)JX + 2v(X)GHY\end{aligned}$$

and

$$\begin{aligned}(\nabla_X H)Y = g(X,Y)V - v(Y)X - g(X,JY)U \\ + u(Y)JX - 2u(X)GHY.\end{aligned}$$

In [**1**], the following are also listed:

$$\begin{aligned} g(\nabla_X U, V) &= 0, \\ \nabla_X U &= -GX, \\ \nabla_X V &= -HX. \end{aligned}$$

As a consequence of the first equality, we see that σ is identically zero. Therefore, by Proposition 3.3 this structure on $\mathbf{H_C}$ is normal.

The Hermitian connection of g is also given in [**1**]. So we can establish the following curvature identities easily:

$$\begin{aligned} g(R(X,GX)GX,X) &= g(R(X,HX)HX,X) \\ &= -3g(X,X)^2, \\ g(R(X,GX)HX,X) &= 0. \end{aligned}$$

Therefore, $\mathbf{H_C}$ has constant GH-sectional curvature -3.

Our second example is the odd-dimensional complex projective space $\mathbf{CP}^{2n+1}$ with the standard Fubini-Study metric g of constant holomorphic curvature 4. It is established in [**8**] that $(\mathbf{CP}^{2n+1}(4), g)$ admits a normal complex contact metric structure via the Hopf fibering

$$\pi : \mathbf{S}^{4n+3} \longrightarrow \mathbf{CP}^{2n+1}.$$

Since this structure has constant holomorphic curvature 4, $(\mathbf{CP}^{2n+1}(4), g)$ has constant GH-sectional curvature 1 by Theorem 5.6.

7. $\mathcal{H}$-homothetic deformations. The odd-dimensional complex projective space with the Fubini-Study metric is an example of a normal complex contact metric manifold with constant GH-sectional curvature 1. To get other examples with constant GH-sectional curvature, we need to study the $\mathcal{H}$-homothetic deformations.

Let M be a normal complex contact metric manifold with structure tensors (u, v, U, V, G, H, g). For a positive constant α, we define new tensors by $\tilde{u} = \alpha u$, $\tilde{v} = \alpha v$, $\tilde{U} = U/\alpha$, $\tilde{V} = V/\alpha$, $\tilde{G} = G$, $\tilde{H} = H$, $\tilde{g} = \alpha g + \alpha(\alpha - 1)(u \otimes u + v \otimes v)$. This change of structure is called an *$\mathcal{H}$-homothetic deformation.*

Proposition 7.1. *If (u, v, U, V, G, H, g) is a normal complex contact metric structure on (M, J), then $(\tilde{u}, \tilde{v}, \tilde{U}, \tilde{V}, \tilde{G}, \tilde{H}, \tilde{g})$ is also a normal complex contact metric structure on (M, J).*

Proof. Clearly, $\tilde{\omega} = \alpha\omega$ is a complex contact structure on M. Also, $\tilde{\mathcal{H}} = \mathcal{H}$, $d\tilde{u}(\tilde{U}, X) = du(U, X) = 0$ for all X in $\mathcal{H}$, $\tilde{u}(\tilde{U}) = u(U) = 1$ and $\tilde{v}(\tilde{U}) = 0$. We can easily check the first condition of Definition 2.2 by noting that

$$\begin{aligned}
\tilde{G}^2 &= -Id + \tilde{u} \otimes \tilde{U} + \tilde{v} \otimes \tilde{V}, \\
\tilde{g}(\tilde{G}X, Y) &= -\tilde{g}(X, \tilde{G}Y), \\
\tilde{g}(\tilde{U}, X) &= \tilde{u}(X), \\
\tilde{G}J &= GJ = -JG = -J\tilde{G}, \\
\tilde{G}\tilde{U} &= G\tilde{U} = \frac{1}{\alpha}\, GU = 0.
\end{aligned}$$

If $\mathcal{O}\cap\mathcal{O}' \neq \varnothing$, then there are functions a and b on $\mathcal{O}\cap\mathcal{O}'$ which satisfy the second condition of Definition 2.2. Then

$$\begin{gathered}\tilde{u}' = \alpha u' = \alpha(au - bv) = a\tilde{u} - b\tilde{v}\\ \tilde{v}' = \alpha v' = \alpha(bu + av) = b\tilde{u} + a\tilde{v}\\ \tilde{G}' = G' = aG - bH = a\tilde{G} - b\tilde{H}\\ \tilde{H}' = H' = bG + aH = b\tilde{G} + a\tilde{H}\\ a^2 + b^2 = 1.\end{gathered}$$

Therefore the first condition of Definition 2.3 is satisfied.

For horizontal X and Y, $d\tilde{u}(X,Y) = \alpha du(X,Y) = \alpha g(X,GY) = \tilde{g}(X,GY)$ and $d\tilde{v}(X,Y) = \alpha dv(X,Y) = \alpha g(X,HY) = \tilde{g}(X,HY)$. So the second condition of Definition 2.3 is also satisfied, and hence $(\tilde{u},\tilde{v},\tilde{U},\tilde{V},\tilde{G},\tilde{H},\tilde{g})$ is a complex contact metric structure on (M,J).

To check for normality, first we need to see how the covariant derivative changes. By a direct computation, we can see that

$$\tilde{\nabla}_X Y = \nabla_X Y + (1-\alpha)[u(Y)GX + v(Y)HX + u(X)GY + v(X)HY]. \tag{54}$$

If we take $Y = U$ in (54), we get

$$\tilde{\nabla}_X U = \nabla_X U + (1-\alpha)GX.$$

Hence

$$\begin{aligned}\tilde{\sigma}(X) &= \tilde{g}(\tilde{\nabla}_X \tilde{U}, \tilde{V})\\ &= \frac{1}{\alpha^2}\,\tilde{g}(\tilde{\nabla}_X U, V)\\ &= \frac{1}{\alpha}\,g(\tilde{\nabla}_X U, V) + \frac{\alpha-1}{\alpha}\,v(\tilde{\nabla}_X U)\\ &= \frac{1}{\alpha}\,g(\nabla_X U, V) + \frac{\alpha-1}{\alpha}\,v(\nabla_X U)\\ &= g(\nabla_X U, V) = \sigma(X).\end{aligned}$$

Thus, $\sigma = \tilde{\sigma}$. Then

$$\tilde{S}(X,Y) = S(X,Y) + 2(\alpha-1)(v(Y)HX - v(X)HY).$$

Similarly, we can show that

$$\tilde{T}(X,Y) = T(X,Y) + 2(\alpha-1)(u(Y)GX - u(X)GY).$$

Thus,

$$\tilde{S}(\tilde{U}, X) = \frac{1}{\alpha}\,\tilde{S}(U, X) = \frac{1}{\alpha}\,S(U, X) = 0,$$

and

$$\tilde{T}(\tilde{V}, X) = \frac{1}{\alpha}\,\tilde{T}(V, X) = \frac{1}{\alpha}\,T(V, X) = 0.$$

If X and Y are horizontal, then

$$\tilde{S}(X, Y) = S(X, Y) = 0,$$

and

$$\tilde{T}(X, Y) = T(X, Y) = 0.$$

Therefore, the deformed structure is also normal. □

Now we want to see what happens to the GH-sectional curvature under an $\mathcal{H}$-homothetic deformation. First we check how the sectional curvature changes.

For horizontal vector fields X and Y,

$$\begin{aligned}
&\tilde{R}(X,Y)Y \\
&\quad= \tilde{\nabla}_X\tilde{\nabla}_Y Y - \tilde{\nabla}_Y\tilde{\nabla}_X Y - \tilde{\nabla}_{[X,Y]}Y \\
&\quad= \tilde{\nabla}_X\nabla_Y Y - \tilde{\nabla}_Y\nabla_X Y - \nabla_{[X,Y]}Y \\
&\qquad - (1-\alpha)(u([X,Y])GY + v([X,Y])HY) \\
&\quad= \nabla_X\nabla_Y Y + (1-\alpha)(u(\nabla_Y Y)GX + v(\nabla_Y Y)HX) - \nabla_Y\nabla_X Y \\
&\qquad - (1-\alpha)(u(\nabla_X Y)GY + v(\nabla_X Y)HY) - \nabla_{[X,Y]}Y \\
&\qquad - (1-\alpha)(u([X,Y])GY + v([X,Y])HY).
\end{aligned}$$

Since X and Y are horizontal and M is normal, we have

$$u(\nabla_X Y) = g(\nabla_X Y, U) = -g(\nabla_X U, Y) = g(GX, Y),$$

and

$$v(\nabla_X Y) = g(\nabla_X Y, V) = -g(\nabla_X V, Y) = g(HX, Y).$$

Hence, $u(\nabla_Y Y) = v(\nabla_Y Y) = 0$, $u([X,Y]) = 2g(GX,Y)$, $v([X,Y]) = 2g(HX,Y)$. Therefore,

$$\tilde{R}(X,Y)Y = R(X,Y)Y + 3(1-\alpha)(g(X,GY)GY + g(X,HY)HY)$$

for X, Y in $\mathcal{H}$. So, for horizontal vector fields X and Y,

$$\begin{aligned}\tilde{g}(\tilde{R}(X,Y)Y,X) = \alpha g(R(X,Y)Y,X) + 3\alpha(1-\alpha)(g(X,GY)^2 \\ + g(X,HY)^2).\end{aligned}$$

Assume that the original structure on M has constant GH-sectional curvature c. Let X be a unit horizontal vector field with respect to the new structure on M. Let $Y = a\tilde{G}X + b\tilde{H}X$ with $a^2 + b^2 = 1$. Then $GY = -aX - bJX$ and $HY = aJX - bX$. Thus,

$$\begin{aligned}&\tilde{g}(\tilde{R}(X,Y)Y,X)\\ &\quad = \alpha g(R(X,Y)Y,X) + 3\alpha(1-\alpha)(g(X,-aX-bJX)^2 + g(X,aJX-bX)^2)\\ &\quad = \alpha c g(X,X)^2 + 3\alpha(1-\alpha)(a^2 g(X,X)^2 + b^2 g(X,X)^2)\\ &\quad = \alpha c\frac{1}{\alpha^2}\,\tilde{g}(X,X)^2 + 3\alpha(1-\alpha)\frac{1}{\alpha^2}\,\tilde{g}(X,X)^2\\ &\quad = \frac{c}{\alpha} + \frac{3(1-\alpha)}{\alpha}\\ &\quad = \frac{c+3}{\alpha} - 3.\end{aligned}$$

Hence the new structure has constant GH-sectional curvature $(c+3)/\alpha - 3$.

Next we want to see how the curvature of the vertical plane changes under an $\mathcal{H}$-homothetic deformation. We know that $\sigma = \tilde{\sigma}$. So $\Omega = \tilde{\Omega}$. Hence

$$\begin{aligned}\tilde{g}(\tilde{R}(\tilde{U},\tilde{V})\tilde{V},\tilde{U}) &= -2\tilde{\Omega}(\tilde{U},\tilde{V})\\ &= -\frac{2}{\alpha^2}\Omega(U,V) = \frac{1}{\alpha^2}\,g(R(U,V)V,U).\end{aligned}$$

In particular, if $c = 1$ and $\Omega(U,V) = -2$, then the new structure has constant GH-sectional curvature $(4/\alpha) - 3$ with $\tilde{\Omega}(\tilde{U},\tilde{V}) = -(2/\alpha^2)$. This observation gives us the following theorem.

Theorem 7.2. *In addition to its standard structure, complex projective space* $\mathbf{CP}^{2n+1}$ *also carries a normal complex contact metric structure with constant GH-sectional curvature* $(4/\alpha) - 3$ *and* $\Omega(U,V) = -(2/\alpha^2)$ *for every* α *greater than* 0.

With this theorem we get examples of normal complex contact metric manifolds with constant GH-sectional curvature $\tilde{c} > -3$. Conversely, as we state in the following theorem, every such manifold is $\mathcal{H}$-homothetic to a normal complex contact metric manifold with constant GH-sectional curvature $c = 1$.

Theorem 7.3. *A normal complex contact metric manifold with metric* $\tilde{g}$ *of constant GH-sectional curvature* $\tilde{c} > -3$ *is* $\mathcal{H}$*-homothetic to a normal complex contact metric manifold with metric* g *of constant GH-sectional curvature* $c = 1$*. Moreover, if* $\Omega(\tilde{U},\tilde{V}) = -(\tilde{c}+3)^2/8$*, then the metric* g *is Kähler and has constant holomorphic curvature* 4.

Proof. Let M be a normal complex contact metric manifold with metric $\tilde{g}$ of constant GH-sectional curvature $\tilde{c} > -3$. Apply an $\mathcal{H}$-homothetic deformation to $(M,\tilde{g})$ with $\alpha = (\tilde{c}+3)/4 > 0$. We know that the new structure is also a normal complex contact metric structure with constant GH-sectional curvature $c = (\tilde{c}+3)/\alpha - 3 = 1$. Moreover, if $\Omega(\tilde{U},\tilde{V}) = -(\tilde{c}+3)^2/8$, then $\Omega(U,V) = (1/\alpha^2)\Omega(\tilde{U},\tilde{V}) = -2$. Then, by Theorem 5.8, (M,g) is Kähler and has constant holomorphic curvature 4. □

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