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DENSE SUBGROUPS AND DIVISIBLE QUOTIENT GROUPS OF LOCALLY COMPACT ABELIAN GROUPS

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Introduction. Recall that an Abelian group A is said to be divisible if $nA = A$ for all natural numbers n. It follows that homomorphic images of divisible groups are again divisible. A topological group is called monothetic if it contains a dense cycle subgroup. It is known that a locally compact monothetic group is either topologically isomoprhic with the discrete group of integers or is compact (for more details see [**1**, p. 25 and Section 5.5]). The following results concerning a compact monothetic group G were proved as part of Lemma 3 of [**6**] and relate the notions of (topological) density and (algebraic) divisibility: (1) if H is a dense subgroup of G, then G/H is divisible, (2) if G is also totally disconnected and H is a subgroup such that g/H is divisible, then H is dense in G. These results suggest the following three properties that an LCA, i.e., locally compact Hausdorff Abelian, group G may or may not possess.

Property D_1. For an arbitrary subgroup H, G/H is divisible implies that H is dense in G.

Property D_2. For an arbitrary subgroup H, H is dense in G implies that G/H is divisible.

Property D_3. An arbitrary subgroup H is dense in G if and only if G/H is divisible.

In this note we study the beautiful interplay between the topological property of denseness of subgroups and the algebraic property of divisibility of quotients in the form of the three properties defined above. Additionally, we shall say that an LCA group G has property

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C if every proper closed subgroup of G is contained in a maximal subgroup. This last property was introduced and studied in some detail in [**4**].

Notation. We shall denote the maximal torsion subgroup and the connected component of the zero element by $t(G)$ and G_0, respectively. $G^\wedge$ stands for the dual of G. The symbols $\cong$ and $\approx$ denote topological isomorphism and group isomorphism, respectively, while the symbols Π and $\oplus$ stand for topological direct product and sum, respectively. T, J_p and $Z(p^\infty)$ stand for the compact multiplicative group of unit circle, the compact group of p-adic integers and discrete quasi-cyclic group (for the last group, see [**2**, Section 3]). We shall often use the fact that an Abelian group is divisible if and only if it contains no maximal subgroup, i.e., a subgroup of prime index (see [**2**, p. 99]). For other results, notation and terminology, we follow [**1**].

1. Property D_1. To start with, we remark that an LCA group G which is not totally disconnected cannot have property D_1, for G has in this case a closed subgroup H such that $G/H \cong T$. Even some totally disconnected subgroups are devoid of this property. The most deficient in this respect are divisible, torsion LCA groups: all of their homomorphic images are divisible but they have no proper dense subgroups at all ([**3**, Corollary 5.4]). In the next theorem, we shall show that property D_1 is equivalent to property C.

Theorem 1.1. *An* LCA *group has property* D_1 *if and only if it has property* C.

Proof. Suppose G has property C and G/H is divisible for some proper subgroup H. Suppose H^- is a proper subgroup G; then G/H^- would be divisible and so it would not have any maximal subgroup. Hence, there would be no maximal subgroup of G containing H^-, contradicting property C. Hence H must be dense in G. Conversely, from the comment above, G must be totally disconnected. Suppose property C does not hold. Then [**4**, Theorem 3(b)] and the second isomorphism theorem imply that there is a proper open subgroup K such that G/K is divisible. But this contradicts D_1. The proof is now complete. □

It follows from this theorem that all the results about property C in [**4**] are also true for property D_1. In particular, our Corollaries 1.2, 1.3 and 1.4 are, except for minor verbal changes, Theorem 3 and Corollaries 8 and 10 of [**4**].

Corollary 1.2. *The following are equivalent for an* LCA *group* G:

(a) G *has property* D_1.

(b) G *is totally disconnected and* G/K *is nondivisible for each open subgroup* K *of* G.

(c) $G^\wedge$ *contains no nonzero closed torsion-free subgroup.*

Corollary 1.3. *A compact Abelian group* G *has property* D_1 *if and only if it is totally disconnected.*

Corollary 1.4. *If an* LCA *group* G *has property* D_1 *and* H *is a closed subgroup, then* H *and* G/H *also have property* D_1.

We next discuss the behavior of property D_1 under direct products.

Corollary 1.5. *An* LCA *group* G *which is a topological direct product of an arbitrary family of* LCA *groups with property* D_1 *has itself property* D_1.

Proof. First let $G = G_1 \oplus G_2$ where both G_1 and G_2 have property D_1. Now G_1 and G_2 both being totally disconnected have bases at zero consisting of compact, open subgroups [**1**, p. 27]. It is easy to see that $G = G_1 \oplus G_2$ also has such a basis and so is totally disconnected. Suppose G does not have property D_1. By Corollary 1.2 there exists a proper open subgroup K such that G/K is divisible. By taking suitable homomorphic images, we can assume that $G/K = Z(p^\infty)$ for some prime p [**2**, 23.1]. Set $A = G_1 \times \{0_2\}$ and $B = \{0_1\} \times G_2$ where 0_i is the identity element of G_i, $i = 1, 2$. Also denote by $\pi_i : G \to G_i$ the natural map ($i = 1, 2$). We easily verify that

$A+K = G_1 \times \Pi_2[K]$. We consider two cases: If $A+K = G$ then $Z(p^\infty) = G/K = (A+K)/K \cong A/(K \cap A)$, since K is open. But since $A \cong G_1$, this contradicts the fact that G_1 has the property D_1 (Corollary 1.2). Next suppose that $A+K$ is a proper subgroup of G. Then $Z(p^\infty) = G/(A+K) \cong (G_1 \times G_2)/(G_1 \times \Pi_2[K]) \cong G_2/\Pi_2[K]$ which contradicts the fact that G_2 has property D_1 (Corollary 1.2). This shows that $G_1 \times G_2$ has property D_1 if each factor does. In the general case, let G_1 be the direct product of all compact groups in the given product for G. Then $G_1^\wedge$, as a direct sum of torsion groups, is itself a torsion group, so the compact group G_1 has property C (Corollary 1.3 above and [**1**, Corollary 3.6]). Let G_2 be the direct product of noncompact factors (necessarily finite in number). Then, by the first case, G_2 has property C and therefore $G = G_1 \oplus G_2$ has property C as well. This completes the proof. □

2. Property D_2. The next theorem characterizes Property D_2.

Theorem 2.1. *An* LCA *group has property D_2 if and only if the subgroup $pG = \{pg : g \in G\}$ is open for each prime p.*

Proof. Let pG be open for each prime p and H dense in G. If G/H is not divisible, then it contains a maximal subgroup of (necessarily prime) index, say q. Then there would be a subgroup K of G containing H of index q in G. Since $qG \subseteq K$, K is open. But then $H^- \subseteq K \neq G$, contradicting the fact that H is dense in G. Hence we conclude that G/H is divisible. Conversely, suppose that pG is nonopen for a prime p. If every subgroup of index p in G were closed, then by applying Exercises 3(a) and 4(a) of Section 3 in [**2**], pG would be a closed, nonopen subgroup of G. Applying Corollary 5.2 of [**3**] to nondiscrete p-bounded LCA group G/pG, we see that there is a proper dense subgroup of G, say H, containing pG. If D_2 holds, then G/H must be divisible, which is impossible because, since $pG \subseteq H$, one would have $\{H\} = p(G/H) = G/H$. So there must be a dense subgroup of index p in G. We have shown that if pG is nonopen, then there is a proper dense subgroup M of G such that G/M is nondivisible. This completes the proof. □

It follows trivially from this theorem that each divisible LCA group, in particular each connected LCA group, has property D_2. We now turn to the class of σ-compact LCA groups where the D_2 property yields interesting consequences.

Following [**5**], we shall say that G is power-open if, for every natural number n the continuous endomorphism $f_n : G \to G$, defined by $f_n(g) = ng$ is an open map, i.e., U open in G implies $f_n(U)$ is open in G. Clearly G is power open implies G has D_2 (see Theorem 2.1). The converse is false. As a counter example, consider the compact group $H = J_p^m$, where m is infinite, and let G be a minimal divisible extension of H, topologized in the usual manner so that H is open. Using Theorem 2.1 we see that G has property D_2. But $f_p(H) = pH$ is of infinite index in H and so cannot be open. Hence G is not power-open. However, we show in the lemma below that for σ-compact groups, the converse is true.

Lemma 2.2. *Let G be a σ-compact* LCA *group with property D_2. Then G is power-open.*

Proof. First, if p is a prime, notice that $f_p : G \to pG$ is an open map by the open mapping theorem [**1**, p. 30, (b)]. As pG is open, $f_p : G \to G$ is also an open map. Let U be any open set in G, and let n be an arbitrary natural member. Let p^m be the highest power of a prime p dividing n. By repeated application of the open map f_p, it is clear that p^mU is open. Doing the same for other prime divisors of n, we conclude that nU is open. Hence, $f_n : G \to G$ is open for each n, and G is consequently power-open.

Clearly, Lemma 2.2 and Theorem 2.1 imply that a σ-compact group is power-open if and only if it is D_2. Following [**5**] again, we shall say that an LCA group G is power-rich if for every natural number n and for every open neighborhood U of zero, $\lambda(nU) > 0$, where λ is a Haar measure on G.

Our next theorem is an immediate consequence of Theorem 2.1, Lemma 2.2 and Theorem 1.1 of [**5**].

Theorem 2.3. *The following are equivalent for a σ-compact* LCA *group G:*

(a) *G has property D_2.*

(b) *G is power-open.*

(c) *G is power-rich.*

Using Proposition 1.1 of [**5**], we can be more specific on the structure of compact groups with property D_2.

Proposition 2.4. *The following are equivalent for a compact Abelian group G:*

(a) *G has property D_2.*

(b) *$G/G_0 \cong \Pi A_p$, where A_p is a topological direct product of finitely many copies of J_p and finitely many cyclic p-groups for each prime p.*

Corollary 2.5. *Any compact monothetic Abelian group has property D_2.*

Proof. Use Propositions 2.4 and 5.5(f) of [**1**].

Since closed subgroups of σ-compact groups are again σ-compact, we deduce the following from Corollary 1.3 of [**5**].

Corollary 2.6. *Let G be a σ-compact* LCA *group with property D_2. Then every closed subgroup containing G_0 has property D_2.*

However the property D_2 is not hereditary for closed subgroups in general. The group T^m, m infinite, has property D_2, but its closed subgroup $(Z(p))^m$ is devoid of it (use Proposition 2.4). Again J_p enjoys property D_2 but if m is infinite, J_p^m is devoid of it. However, if each member in a finite collection of LCA *groups has property D_2, then so does $\oplus G_i$ because $q(\oplus G_i) = \oplus qG_i$ is open for each prime q.*

3. Property D_3. Clearly D_3 is satisfied if both D_1 and D_2 are satisfied. Hence the following theorem follows from Theorems 1.1 and 2.1.

Theorem 3.1. *An* LCA *group G satisfies property D_3 if and only if G has property C and pG is open for each prime p.*

Theorem 3.2. *A compact Abelian group G has property D_3 if and only if $G \cong \Pi A_p$, where A_p is a topological direct product of finitely many copies of J_p and finitely many cyclic p-groups. In particular, every compact monothetic totally disconnected group has property D_3.*

Proof. To see the necessity of condition, notice that G has property D_1 so G_0 is trivial. Since G has property D_2, necessity follows from Proposition 2.4. For the sufficiency, apply Corollary 1.3 and Proposition 2.4. The rest of the corollary follows from Corollaries 1.3 and 2.5. □

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REFERENCES

1. D.L. Armacost, *The structure of locally compact abelian groups*, Marcel Dekker, Inc., New York, 1981.

2. L. Fuchs, *Infinite abelian groups*, Vol. 1, Academic Press, New York, 1970.

3. M.A. Khan, *Chain conditions on subgroups of* LCA *groups*, Pacific J. Math. **86** (1980), 517–534.

4. ———, *When are proper subgroups of* LCA *groups contained in maximal ones?*, Proc. Amer. Math. Soc. **83** (1981), 131–137.

5. ———, *Power-rich and power-deficient* LCA *groups*, Canad. J. Math. **33** (1981), 664–670.

6. ———, *The existence of totally dense subgroups in* LCA *groups*, Pacific J. Math. **112** (1984), 383–390.

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