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CHARACTERIZING A CLASS OF WARFIELD MODULES BY RELATION ARRAYS

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Introduction. In this paper we examine the Warfield modules in a class $\mathcal{H}$ with the property that the torsion submodule is a direct sum of cyclics and the quotient modulo torsion is divisible of arbitrary rank. We give necessary and sufficient conditions about when modules in $\mathcal{H}$ are Warfield if their torsion-free rank is countable and the indicators of torsion-free elements are exclusively of ω-type or exclusively of finite-type. We give two examples to show that the conditions placed on such modules cannot be eliminated. Indeed, we explicitly describe two non-Warfield modules in the class $\mathcal{H}$ of torsion-free rank 2 where the indicators of all torsion-free elements are either of finite-type or of ω-type, respectively, but yet the modules do not satisfy the aforementioned conditions. In addition we prove that a Warfield module is equivalent to a simply presented module if the indicators of torsion-free elements are all of ω-type or all of finite-type. We show that this result is in some sense the best possible by giving an example of a Warfield module in $\mathcal{H}$ which is not simply presented whose torsion-free rank is 2 and contains indicators of both the finite and ω-type. This example complements one given by Warfield of a mixed module of torsion-free rank 1. The proofs of our results rely on the description of these modules by generators and relations, their corresponding relation arrays, and the results established in [**3**], [**4**], [**5**], [**6**].

1. Notation. Let $\mathbf{N}$ denote the set of natural numbers and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Let R denote a *discrete valuation domain*, i.e., a local principal ideal domain with prime p and quotient field $\mathbf{F}$. All modules are understood to be R-modules.

We now recall from [**5**] the definition of a module by generators and relations. Let G be a module in the class $\mathcal{H}$ of torsion-free rank d

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with torsion submodule $\mathbf{t}G = \oplus_{i\in\mathbf{N}} \oplus_{v\in I_i} Rx_i^v$ of isomorphism type $\lambda = (s_i \mid i \in \mathbf{N})$, where $s_i = |I_i|$ and $\mathbf{ann}\, x_i^v = p^i R$ for $i \in \mathbf{N}$, $v \in I_i$. Then the quotient module $G/\mathbf{t}G$ is a vector space over $\mathbf{F}$ of dimension d. Let I be an index set of cardinality d. We call the subset

$$B = \{x_i^v, a_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\} \subset G$$

a *basic generating system of* G if

(1) $\{x_i^v \mid i \in \mathbf{N}, v \in I_i\}$ is a basis of $\mathbf{t}G$ with $\mathbf{ann}\, x_i^v = p^i R$ for all $i \in \mathbf{N}$, $v \in I_i$,

(2) $G/\mathbf{t}G = \oplus_{k\in I}\mathbf{F}\bar{a}_0^k$,

where $\bar{a}_{i-1}^k = a_{i-1}^k + \mathbf{t}G$ and $p\bar{a}_i^k = \bar{a}_{i-1}^k$ for all $i \in \mathbf{N}$, $k \in I$. Note that $G = \langle x_i^v, a_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\rangle$. It follows that for every pair $(i,k) \in \mathbf{N} \times I$ the equation

$$pa_i^k = a_{i-1}^k + \sum_{j\in\mathbf{N}} \sum_{u\in I_j} \alpha_{i-1,j}^{k,u}\, x_j^u \tag{1.1}$$

holds true for some elements $\alpha_{i-1,j}^{k,u}$, $j \in \mathbf{N}$, $u \in I_j$. Moreover, for every fixed pair (k,i) we have $\alpha_{i,j}^{k,u} \in p^j R$ for almost all pairs (j,u). The latter property is called *row finiteness in* j *and* u. A relation array $(\alpha_{i-1,j}^{k,u})$ is called *restricted* if $\alpha_{i-1,j}^{k,u} \in (R \setminus pR) \cup \{0\}$ for all $k \in I$, all $i,j \in \mathbf{N}$ and all $u \in I_j$. The array $(\alpha_{i-1,j}^{k,u})$ is called a *relation array of format* (λ, d). Note that two relation arrays corresponding to different basic generating systems may be different but are of the same format.

Let G be an arbitrary R-module with $g \in G$. As in [**1**, Section 37], let $h^*(g)$ denote the generalized p-height of g in G. Then the *p-indicator of* g is given by

$$\mathbf{H}(g) = (h^*(g), h^*(pg), \dots, h^*(p^n g), \dots).$$

We use the terms gap and equivalence of indicators as in [**1**, Section 103]. If $\mathbf{H}$ and $\mathbf{K}$ are any indicators then we will write $\mathbf{H} \cong \mathbf{K}$ to express their equivalence. A module G of torsion-free rank 1 has a unique equivalence class of indicators, denoted by $\mathbf{H}(G)$, the *indicator* of G.

An indicator is called of *finite-type* if all the entries are natural numbers and there are infinitely many gaps; it is called of *ω-type* if there is an entry $\omega + k$, $k \in \mathbf{N}_0$, with no gaps beyond this entry; and it is called of ∞-*type* if there is an ∞ in the indicator.

Given two strictly increasing functions $i : \mathbf{N}_0 \to \mathbf{N}_0$ and $j : \mathbf{N}_0 \to \mathbf{N}$ where $j(l) - i(l)$ is monotonically increasing and nonnegative, then we obtain an array $(\alpha_{i,j})$ defined by

$$(1.2) \qquad \alpha_{i(l),j(l)} = 1 \quad \text{and} \quad \alpha_{i,j} = 0 \quad \text{otherwise}.$$

We will call such an array *concave*. It is called *strictly concave* if $j(l)-i(l)$ is strictly increasing, and *diagonal* if $j(l) = i(l)$. Using strictly concave relation arrays one can define an indicator $\mathbf{H} = (\beta_0, \beta_1, \dots)$ of finite-type with gaps at $g_0 < g_1 < \cdots$ where $g_l = j(l) - i(l) - 1$ and

$$(1.3) \qquad \beta_n = \begin{cases} j(l) - 1 & \text{for } n = g_l, \\ j(l) - 1 + n - g_l & \text{for } g_l < n < g_{l+1}. \end{cases}$$

Note that this indicator inherits the given functions $i(l)$ and $j(l)$. In this setting we call the function $i(l)$ the *height difference function* and $j(l)$ the *Ulm-Kaplansky exponent function.*

A module is called *strictly reduced* if it has no elements of infinite height, i.e., the first Ulm submodule is zero. Let G be a strictly reduced module in the class $\mathcal{H}$ with a basic generating system $B = \{x_i^v, a_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$ and a corresponding restricted relation array $(\alpha_{i-1,j}^{k,u})$. Observe that for fixed k we may write $(\alpha_{i-1,j}^{k,u})$ as an $\omega \times \omega$-matrix with $|I_j|$-tuples as entries in the jth column, i.e.,

$$(\alpha_{i-1,j}^{k,u}) = \begin{pmatrix} (\alpha_{0,1}^{k,u} \mid u \in I_1) & (\alpha_{0,2}^{k,u} \mid u \in I_2) & \cdots & (\alpha_{0,j}^{k,u} \mid u \in I_j) & \cdots \\ (\alpha_{1,1}^{k,u} \mid u \in I_1) & (\alpha_{1,2}^{k,u} \mid u \in I_2) & \cdots & (\alpha_{i,j}^{k,u} \mid u \in I_j) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\alpha_{i,1}^{k,u} \mid u \in I_1) & (\alpha_{i,2}^{k,u} \mid u \in I_2) & \cdots & (\alpha_{i,j}^{k,u} \mid u \in I_j) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Since the relation array $(\alpha_{i-1,j}^{k,u})$ is row finite in j and in u, this matrix is row finite and has row finite tuples as entries.

Let $k \in I$ be fixed, and let $\mathbf{H}(a_0^k) = \mathbf{H}^k = (\beta_0^k, \beta_1^k, \beta_2^k, \dots)$ be an indicator with finite entries, where $j_k(n) = \beta_{g_n}^k + 1$ is the Ulm-Kaplansky exponent and $i_k(n) = \beta_{g_n}^k - g_n$ the height difference function.

We define an infinite tuple

$$A^k(\alpha) = A(\alpha, \mathbf{H}^k) = (\varrho_j^k \mid j \in \mathbf{N})$$

with entries ϱ_j^k given by

$$\varrho_j^k = \begin{cases} (\alpha_{i_k(n), j_k(n)}^{k,u} \mid u \in I_{j_k(n)}) & \text{if } j = j_k(n) \text{ for some } n \in \mathbf{N}_0, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$A(\alpha) = (A^k(\alpha) \mid k \in I) = (\varrho_j^k)_{k,j}$$

be the $|I| \times \omega$-matrix with rows $A^k(\alpha) = A(\alpha, \mathbf{H}^k) = (\varrho_j^k \mid j \in \mathbf{N})$, $k \in I$. We say that $A(\alpha)$ is the *gap-matrix of* G *relative to* $(\alpha_{i-1,j}^{k,u})$. The choice of notation here is motivated by the correlation between the nth gap of the indicator $\mathbf{H}(a_0^k)$ and the nth gap-tuple of $\mathbf{H}(a_0^k)$.

For a fixed $j \in \mathbf{N}$, let $J_i \subseteq I$ be the subset consisting of all elements k such that the entry ϱ_j^k of $A(\alpha)$ is nonzero, i.e.,

$$J_j = \{k \in I \mid \varrho_j^k \neq 0\} = \{k \in I \mid j = j_k(n) \text{ for some } n \in \mathbf{N}_0\}.$$

If $J_j \neq \varnothing$, then for every $k \in J_j$ the number n satisfying $j_k(n) = j$ is fixed. For each $k \in J_j$ we define a torsion element $t_j^k(\alpha) = t_j(\alpha, \mathbf{H}^k)$ by

$$t_j^k(\alpha) = \sum_{u \in I_j} \alpha_{i_k(n),j}^{k,u} x_j^u,$$

where $j = j_k(n)$ as above. We call $t_j^k(\alpha)$ the *gap-element relative to* a_0^k. Observe that for every $k \in J_j$ the R-module $Rt_j^k(\alpha)$ is a cyclic torsion module with annihilator $p^j R$, since the relation array $(\alpha_{i-1,j}^{k,u})$ is restricted. Furthermore, the set $\{t_j^k(\alpha) \mid k \in J_j\}$ generates a torsion module $\sum_{k \in J_j} Rt_j^k(\alpha) \subseteq \mathbf{t}G$ such that

$$\sum_{j \in \mathbf{N}} \sum_{k \in J_j} R\varrho_j^k / p^j R\varrho_j^k \cong \sum_{j \in \mathbf{N}} \sum_{k \in J_j} Rt_j^k(\alpha). \tag{1.4}$$

An independent set X of torsion-free elements in a module M is called a *basis* if $M/\langle X \rangle$ is torsion. A basis X is called a *decomposition basis* if whenever $\{x_1, \dots, x_n\}$ is a subset of X and $r_1, \dots, r_n$ are elements of R,

$$h^*(r_1x_1 + \cdots + r_nx_n) = \min_i \big(h^*(r_ix_i)\big).$$

We now define a generalization of the concept of decomposition bases to apply to torsion modules. Let G be an R-module, and let $X \subset G$ be an independent subset. We will say the set X is *height independent* if, for all linear combinations $\sum_{x\in X} r_x x$ we have $h^*(\sum_{x\in X} r_x x) = \min_{x\in X}(h^*(r_x x))$. Note that independent sets can fail to be height independent. Let x and y be two independent elements in M of various heights. Then the set $\{x, x+y\}$ is independent but not height independent.

As in [**8**], we call a module *simply presented* if it can be defined in terms of generators and relations in such a way that the only relations are of the form $px = y$ or $px = 0$. Those modules which are direct summands of simply presented modules are called *Warfield modules*.

2. Referenced results. This paper references results of the authors in [**3**], [**4**], [**5**]. To facilitate its readability we state several results from these papers.

Lemma 2.1 [**4**, 2.1]. *Every module in the class $\mathcal{H}$ has a basic generating system with a corresponding restricted relation array.*

Moreover, if a module G in the class $\mathcal{H}$ has a basic generating system $B = \{x_i^v, a_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$ with a corresponding relation array $(\alpha_{i-1,j}^{k,u})$ then there are torsion-free elements $b_{i-1}^k \in a_{i-1}^k + \mathbf{t}G$, $i \in \mathbf{N}$, $k \in I$, such that the set $\{x_i^v, b_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$ is a basic generating system of G with a corresponding restricted relation array that is similar to $(\alpha_{i-1,j}^{k,u})$.

Lemma 2.2 [**4**, 2.2]. *Let G be a module in the class $\mathcal{H}$ with basic generating system $B = \{x_i^v, a_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$ and corresponding restricted relation array $(\alpha_{i-1,j}^{k,u})$, and let $\mathbf{H} = (\beta_0 = 0, \beta_1, \beta_2, \dots)$ be an indicator of finite-type with gaps at $0 = g_0 < g_1 < g_2 < \cdots$. Then $\mathbf{H}(a_0^k) = \mathbf{H}$ for a fixed $k \in I$ if and only if for every $n \in \mathbf{N}_0$ the following hold true.*

(1) $\big(\alpha^{k,u}_{i,j} \mid u \in I_j\big) = 0$ *whenever* $i < \beta_{g_{n+1}} - g_{n+1}$ *and* $j > g_n + 1 + i$;

(2) $\big(\alpha^{k,u}_{\beta_{g_n}-g_n,\beta_{g_n}+1} \mid u \in I_{\beta_{g_n}+1}\big) \neq 0$.

The above establishes a one-to-one correspondence between the set of strictly concave relation arrays and the set of indicators of finite-type. The tuple $(\alpha^{k,u}_{i(l),j(l)} \mid u \in I_{j(l)})$ is called the lth *gap-tuple of* α *corresponding to* $\mathbf{H}$.

Let $\mathcal{H}^1$ be the subclass of $\mathcal{H}$ consisting of the modules in $\mathcal{H}$ of torsion-free rank one.

Corollary 2.3 [**4**, 3.6]. *A module in the class* $\mathcal{H}$ *is simply presented if and only if it is isomorphic to a direct sum of modules in the class* $\mathcal{H}^1$.

Corollary 2.4 [**4**, 3.7]. *A module in the class* $\mathcal{H}$ *is Warfield if and only if it is a direct sum of a countably generated Warfield module in the class* $\mathcal{H}$*, a direct sum of modules in the class* $\mathcal{H}^1$ *and a direct sum of cyclics.*

Lemma 2.5 [**3**, 3.6]. *Let* G *be a module in the class* $\mathcal{H}^1$ *with an indicator of finite-type and* $\{b_{i-1}, x_i^v \mid i \in \mathbf{N}, v \in I_i\}$ *a basic generating system with relation array* β. *Then for each* $l \in \mathbf{N}_0$ *and all sufficiently large* k *we have*

$$h^*(p^l b_0) = h^*\bigg(\sum_{j\in\mathbf{N}}\sum_{u\in I_j}\sum_{s=0}^{k-1}\beta^u_{s,j}p^{s+l}x^u_j\bigg).$$

Theorem 2.6 [**3**, 4.4]. *For a reduced module* G *in the class* $\mathcal{H}^1$ *the following are equivalent*:

(1) $p^\omega G \neq 0$.

(2) *The indicator of* G *is of* ω*-type.*

(3) *There is an element* $g \in G\backslash\{0\}$ *with* $h^*(g) = \omega$.

(4) $p^\omega G \neq 0$ *is cyclic.*

3. Warfield modules.

Lemma 3.1. *Let G be a module in the class $\mathcal{H}$ having a decomposition basis X. Then there is a basic generating system $B = \{x_i^v, a_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$ of G with a corresponding restricted relation array such that $\{a_0^k \mid k \in I\} = X$.*

Proof. The decomposition basis $X = \{a_0^k \mid k \in I\}$ is a maximal independent set of torsion-free elements. Hence $\{x_i^v, a_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$ is a basic generating system where $a_i^k \in p^{-i}(a_0^k + \mathbf{t}G)$ is any representative. By Lemma 2.1 one can choose suitable representatives modulo $\mathbf{t}G$ within the cosets $p^{-i}(a_0^k + \mathbf{t}G)$ to get a corresponding relation array that is restricted. $\square$

We begin our consideration of the Warfield modules by examining those whose zeroth Ulm factor is torsion. Recall that a torsion-free module is called *completely decomposable* if it is a direct sum of rank one modules. Furthermore, a submodule G of A is said to be *pure*, if an equation $nx = g$ is solvable in G, whenever it is solvable in the whole module A. If A is a torsion-free module and S a subset, then the intersection of all pure submodules containing S is the minimal pure submodule that contains S; this intersection is called the *pure hull of S in A*, cf. [**1**, Section 26].

Proposition 3.2. *A module G of countable torsion-free rank, whose zeroth Ulm factor is torsion and whose torsion submodule is a direct sum of cyclics, is Warfield if and only if its first Ulm submodule is completely decomposable. In particular, if G is reduced, it is Warfield if and only if its first Ulm submodule is free.*

Proof. Let G be a module of countable torsion-free rank, whose zeroth Ulm factor is torsion and whose torsion submodule is a direct sum of cyclics. Assume that $p^\omega G$ is completely decomposable with decomposition basis $\{a^k \mid k \in I\}$. Then by [**1**, Section 37], we obtain

$$h^*_{p^\omega G}\left(\sum_{k\in I} r_k a^k\right) = \min_{k\in I} \left(h^*_{p^\omega G}(r_k a^k)\right)$$

for every linear combination $\sum_{k\in I} r_k a^k$. Since $h^*_G(g) = h^*_{p^\omega G}(g) + \omega$ for every $g \in p^\omega G$, cf. [**1**, Section 79], we obtain

$$\begin{aligned} h^*_G\biggl(\sum_{k\in I} r_k a^k\biggr) &= h^*_{p^\omega G}\biggl(\sum_{k\in I} r_k a^k\biggr) + \omega \\ &= \min_{k\in I} \bigl(h^*_{p^\omega G}(r_k a^k)\bigr) + \omega \\ &= \min_{k\in I} \bigl(h^*_G(r_k a^k)\bigr). \end{aligned}$$

Furthermore, since $G/p^\omega G \cong (G/\langle a^k \mid k \in I\rangle)/(p^\omega G/\langle a^k \mid k \in I\rangle)$ is torsion by assumption, and since $p^\omega G/\langle a^k \mid k \in I\rangle$ is torsion because $\{a^k \mid k \in I\}$ is a decomposition basis of $p^\omega G$, the quotient $G/\langle a^k \mid k \in I\rangle$ is torsion, too. Hence the set $\{a^k \mid k \in I\}$ is a decomposition basis of G. Since G has countable torsion-free rank, and since $\mathbf{t}G$ is a direct sum of cyclics, we may write $G = G' \oplus T$, where T is a direct summand of $\mathbf{t}G$ and G' is a countably generated module which has the decomposition basis $\{a^k \mid k \in I\}$. By [**7**, Theorem 12], it follows that G' is Warfield and hence so is G. If G is reduced, and if its first Ulm submodule $p^\omega G$ is free, then $p^\omega G$ is in particular completely decomposable. Thus, G is Warfield by the above.

Conversely, let G be Warfield. Then G has a decomposition basis $\{a^k \mid k \in I\}$. Since $G/p^\omega G$ is torsion, we may assume that $a^k \in p^\omega G$, $k \in I$. Then we obtain

$$\begin{aligned} \min_{k\in I} \bigl(h^*_{p^\omega G}(r_k a^k)\bigr) + \omega &= \min_{k\in I} \bigl(h^*_G(r_k a^k)\bigr) \\ &= h^*_G\biggl(\sum_{k\in I} r_k a^k\biggr) \\ &= h^*_{p^\omega G}\biggl(\sum_{k\in I} r_k a^k\biggr) + \omega \end{aligned}$$

for every linear combination $\sum_{k\in I} r_k a^k$. Since $p^\omega G/\langle a^k \mid k \in I\rangle$ is torsion, we deduce that the set $\{a^k \mid k \in I\}$ is a decomposition basis of $p^\omega G$. Therefore, $p^\omega G$ is completely decomposable by its torsion-freeness. In particular we have $p^\omega G = \oplus_{k\in I} \langle a^k\rangle_*^{p^\omega G}$ where $\langle a^k\rangle_*^{p^\omega G}$ is the pure hull of a^k in $p^\omega G$. In particular, if G is reduced, all of these pure hulls are cyclic, and $p^\omega G$ is free. □

Example. As in [**1**, Section 3], let $\mathbf{Z}_p$ be the localization of the integers at p, and let $\hat{\mathbf{Z}}_p$ be the ring of p-adic integers.

For all indecomposable torsion-free $\mathbf{Z}_p$-modules H of rank 2 there is a non-Warfield $\mathbf{Z}_p$-module G such that

(1) $G/p^\omega G$ is torsion,

(2) $\mathbf{t}G$ is a direct sum of cyclics,

(3) $p^\omega G \cong H$.

It is enough to construct such a local group G where $\mathbf{t}G \cong G/p^\omega G \cong \oplus_{i=1}^\infty \mathbf{Z}_p x_i$ with $\mathbf{ann}\, x_i = p^i \mathbf{Z}_p$. Let $\pi = \sum_{i=0}^\infty \pi_i p^i \in \hat{\mathbf{Z}}_p$ be the standard expansion of π, i.e., $\pi_i \in \mathbf{Z}$, where $0 \le \pi_i < p$. In particular, let π be a unit in $\hat{\mathbf{Z}}_p$, i.e., $\pi_0 \ne 0$. Let G be a $\mathbf{Z}_p$-module in $\mathcal{H}$ with a basic generating system $\{x_i, a_{i-1}^k \mid i \in \mathbf{N}, k = 1, 2\}$ and corresponding relation array $\alpha = (\alpha^1, \alpha^2)$ defined by

$$a_{i-1,j}^1 = \begin{cases} 1 & \text{if } i-1 = j \\ 0 & \text{otherwise} \end{cases},$$

$$\alpha_{i-1,j}^2 = \begin{cases} \pi_{i-j-2} & \text{if } i-j-2 \ge 0 \\ 0 & \text{otherwise} \end{cases}, \quad i, j \in \mathbf{N}.$$

Recall that in general we have the relations

$$\begin{aligned} p^i a_{i+n} - a_n &= \sum_{j=1}^\infty \left(\sum_{s=n}^{i+n-1} \alpha_{s,j} p^{s-n} \right) x_j \\ &= \sum_{s=n}^{i+n-1} p^{s-n} \sum_{j=1}^\infty \alpha_{s,j} x_j, \quad i, n \in \mathbf{N}_0. \end{aligned}$$

These relations yield the following

$$a_n^1 = p^i a_{i+n}^1 - \sum_{\substack{s=n \\ s \ge 2}}^{i+n-1} p^{s-n} x_{s-1},$$

$$a_n^2 = p^i a_{i+n}^2 - \sum_{s=n}^{i+n-1} p^{s-n} \sum_{j=1}^{s-1} \pi_{s-j-1} x_j, \quad i, n \in \mathbf{N}_0,$$

with the understanding that empty sums are 0. In particular, for $n = 0$, we obtain $a_0^k = p^i a_i^k$ for $k = 1, 2$ and all $i \in \mathbf{N}_0$. Therefore, $h^*(a_0^k) \ge \omega$,

$k = 1, 2$. Hence $\langle a_0^1, a_0^2\rangle \subseteq p^\omega G$. For every $n \in \mathbf{N}_0$ consider the element $g_n = a_n^2 - \sum_{l=1}^{n} \pi_{n-l} a_l^1$. First we show that $g_n \in p^\omega G$ for all n. The relations above enable us to write

$$\begin{aligned} g_n &= a_n^2 - \sum_{l=1}^{n} \pi_{n-l} a_l^1 \\ &= p^i \left(a_{i+n}^2 - \sum_{i=1}^{n} \pi_{n-l} a_{i+l}^1 \right) - \sum_{s=n}^{i+n-1} p^{s-n} \sum_{j=1}^{s-1} \pi_{s-j-1} x_j \\ &\quad + \sum_{l=2}^{n} \pi_{n-l} \sum_{s=l}^{i+l-1} p^{s-l} x_{s-1} \end{aligned}$$

for all $i, n \in \mathbf{N}_0$. We will show that for every pair $(i, n) \in \mathbf{N}_0 \times \mathbf{N}_0$ the sum of the last two terms is in $p^i G$. This implies $g_n \in p^\omega G$ for all $n \in \mathbf{N}_0$. Let $\sum_{l=2}^{n} \pi_{n-l} \sum_{s=l}^{i+l-1} p^{s-l} x_{s-1} = \sum_{m=1}^{i+n-2} \lambda_m x_m$. Then

$$\begin{aligned} \lambda_m &= \sum_{l=2}^{\min\{n,m+1\}} p^{m-l+1} \pi_{n-l} \\ &= p^{m-1} \pi_{n-2} + p^{m-2} \pi_{n-3} + \cdots \\ &\quad + p^{m-\min\{n,m+1\}+1} \pi_{n-\min\{n,m+1\}}. \end{aligned}$$

Similarly, let $\sum_{s=n}^{i+n-1} p^{s-n} \sum_{j=1}^{s-1} \pi_{s-j-1} x_j = \sum_{m=1}^{i+n-2} \mu_m x_m$, then

$$\begin{aligned} \mu_m &= \sum_{s=\max\{n,m+1\}}^{i+n-1} p^{s-n} \pi_{s-m-1} \\ &= p^{i-1} \pi_{i+n-m-2} + p^{i-2} \pi_{i+n-m-3} + \cdots \\ &\quad + p^{\max\{n,m+1\}-n} \pi_{\max\{n,m+1\}-m-1}. \end{aligned}$$

For $i - m \geq 0$ we have $(\lambda_m - \mu_m) x_m = 0$. If $i < m$, then

$$\begin{aligned} (\lambda_m - \mu_m) x_m &= \big(p^{m-1} \pi_{n-2} + p^{m-2} \pi_{n-3} + \cdots + p^i \pi_{i+n-m-1} \big) x_m \\ &= p^i \big(p^{m-i-1} \pi_{n-2} + p^{m-i-2} \pi_{n-3} + \cdots + \pi_{i+n-m-1} \big) x_m. \end{aligned}$$

Hence

$$\sum_{l=2}^{n} \pi_{n-l} \sum_{s=l}^{i+l-1} p^{s-l} x_{s-1} - \sum_{s=n}^{i+n-1} p^{s-n} \sum_{j=1}^{s-1} \pi_{s-j-1} x_j = p^i \biggl(\sum_{m=i+1}^{n+i-2} \biggl(\sum_{k=0}^{m-i-1} p^k \pi_{n-k-2} \biggr) x_m \biggr),$$

as necessary for $g_n \in p^\omega G$. Finally, since

$$\begin{aligned} p^n g_n &= p^n a_n^2 - \sum_{l=1}^{n} \pi_{n-l} p^n a_l^1 = a_0^2 - \sum_{l=1}^{n} \pi_{n-l} p^{n-l} a_0^1 \\ &= a_0^2 - \biggl(\sum_{l=0}^{n-1} \pi_l p^l \biggr) a_0^1, \quad n \in \mathbf{N}_0, \end{aligned}$$

the pure hull of $\langle a_0^1, a_0^2 \rangle$ in $p^\omega G$ equals

$$\langle a_0^1, a_0^2 \rangle_*^{p^\omega G} = \langle a_0^1, a_0^2, p^{-\infty}(a_0^2 - \pi a_0^1) \rangle_{\mathbf{Z}_p}, \tag{3.1}$$

cf. [**1**, Section 88, Example 5]. If $\pi \in \hat{\mathbf{Z}}_p \setminus \mathbf{Z}_p$, then $p^\omega G$ is a local Pontryagin group, i.e., $p^\omega G$ is homogeneous and strongly indecomposable, and therefore G is reduced. Moreover, G is non-Warfield.

Every indecomposable torsion-free $\mathbf{Z}_p$-module H of rank 2, or local Pontryagin group, can be given in the form (3.1) where π is a p-adic integer which is not rational, cf. [**1**, Section 88, Example 5]. Hence all such H can be realized as $p^\omega G$.

If $\pi \in \mathbf{Z}_p$, then G is not reduced, and the divisible part of G equals the pure hull of $a_0^2 - \pi a_0^1$ in $p^\omega G$. Then $p^\omega G = R a_0^1 \oplus \mathbf{F}(a_0^2 - \pi a_0^1) \cong R \oplus \mathbf{F}$ is completely decomposable, and G is Warfield.

The proof of the analog of Proposition 3.2 to modules with indicators of finite-type is much more extensive. We begin with a remark and a technical lemma on computing heights.

Remark 3.3. Let T be a reduced torsion module, and let $X \subset T[p] \setminus \{0\}$ be a subset such that for all linear combinations $\sum_{x \in X} r_x x$ we have

$$h^*\biggl(\sum_{x \in X} r_x x \biggr) = \min_{x \in X} \bigl(h^*(r_x x) \bigr). \tag{3.2}$$

Then the set X is independent and in particular height independent. Otherwise there is a nontrivial sum $\sum r_x x = 0$, contradicting (3.2), since T is reduced.

For every $x \in X$, let $y_x \in T$ be such that $x \in \langle y_x \rangle$. Clearly, $\{y_x \mid x \in X\}$ is independent too.

Lemma 3.4. *Let G be a strictly reduced module in the class $\mathcal{H}$ with a basic generating system $B = \{x_i^v, a_{i-1}^k \mid k \in I, i \in \mathbf{N}, v \in I_i\}$ and a corresponding restricted relation array $(\alpha_{i-1,j}^{k,u})$. Let $i_k : \mathbf{N}_0 \to \mathbf{N}_0$ be the height difference and $j_k : \mathbf{N}_0 \to \mathbf{N}$ the Ulm-Kaplansky exponent function corresponding to $\mathbf{H}(a_0^k)$, $k \in I$. Then for every finite sum $\sum_k r_k a_0^k$ there is a sufficiently large i such that*

$$h^*\bigg(\sum_k r_k a_0^k\bigg) = h^*\bigg(\sum_k r_k \sum_{j > h^*(r_k a_0^k)} \sum_{u \in I_j} \sum_{s=i_k(n_k)}^{i-1} p^s \alpha_{s,j}^{k,u} x_j^u\bigg), \tag{3.3}$$

whenever the number n_k satisfies $j_k(n_k - 1) < h^*(r_k a_0^k) < j_k(n_k)$.

Proof. Let $\sum_k r_k a_0^k$ be a finite sum, and for every k let n_k satisfy

$$j_k(n_k - 1) < h^*(r_k a_0^k) < j_k(n_k). \tag{3.4}$$

Considering $\sum_k r_k a_0^k$ as a torsion-free generator of a modified basic generating system, we infer by Lemma 2.5 the existence of some $i \in \mathbf{N}$ such that

$$h^*\bigg(\sum_k r_k a_0^k\bigg) = h^*\bigg(\sum_{j \in \mathbf{N}} \sum_{u \in I_j} \sum_{s=0}^{i-1} p^s \bigg(\sum_k r_k \alpha_{s,j}^{k,u}\bigg) x_j^u\bigg). \tag{3.5}$$

For each k we have $r_k \in p^{m(k)}R \setminus p^{m(k)+1}R$ for some $m(k) \in \mathbf{N}_0$. Then $m(k)$ satisfies $h^*(r_k a_0^k) - m(k) = i_k(n_k)$ by the definition of the height difference function i_k. Since $h^*(r_k a_0^k) = h^*(p^{m(k)} a_0^k)$ is an entry in the indicator $\mathbf{H}(a_0^k)$ between $h^*(p^{j_k(n_k-1) - i_k(n_k-1)} a_0^k)$ and $h^*(p^{j_k(n_k) - i_k(n_k) - 1} a_0^k)$ by (3.4) and by (1.3) we conclude that

$$j_k(n_k - 1) - i_k(n_k - 1) \le m(k) < j_k(n_k) - i_k(n_k). \tag{3.6}$$

It follows that $\alpha_{s,u}^{k,u} p^{s+m(k)} x_j^u = 0$ if $j \leq j_k(n_k - 1) - i_k(n_k - 1) + s$ by (3.6) or if $s < i_k(n_k)$ and $j > j_k(n_k-1) - i_k(n_k-1) + s$ by Lemma 2.2. Since p^{m_k} divides r_k we have

$$\sum_k r_k \sum_{j \leq i_k(n_k)+m(k)} \sum_{u \in I_j} \sum_{s=i_k(n_k)}^{i-1} p^s \alpha_{s,j}^{k,u} x_j^u = 0.$$

Therefore, we may write (3.5) as

$$h^*\bigg(\sum_k r_k a_0^k\bigg) = h^*\bigg(\sum_k r_k \sum_{j > i_k(n_k)+m(k)} \sum_{u \in I_j} \sum_{s=i_k(n_k)}^{i-1} p^s \alpha_{s,j}^{k,u} x_j^u\bigg),$$

and the proof is complete. □

We now prove our result on when a module in $\mathcal{H}$ with indicators of finite-type is Warfield.

Proposition 3.5. *A strictly reduced module G in the class $\mathcal{H}$ of countable torsion-free rank is Warfield if and only if it has a basic generating system $B = \{x_i^v, a_{i-1}^k \mid k \in I, i \in \mathbf{N}, v \in I_i\}$ with a corresponding restricted relation array $(\alpha_{i-1,j}^{k,u})$ satisfying one of the following equivalent conditions*:

(1) *The set $\{a_0^k \mid k \in I\}$ is a decomposition basis of G.*

(2) *The sequence of the elements $p^{j-1} t_j^k(\alpha)$, $k \in J_j$, are height independent for every $j \in \mathbf{N}$ satisfying $J_j \neq \varnothing$.*

(3) *The sequence of gap-elements $t_j^k(\alpha)$, $j \in \mathbf{N}$, $k \in J_j$, are independent.*

(4) *The sequence of tuples ϱ_j^k, $k \in J_j$, are independent for every $j \in \mathbf{N}$ satisfying $J_j \neq \varnothing$.*

Indeed, conditions (1)–(4) *are equivalent irrespective of the torsion-free rank.*

Proof. We begin by proving that conditions (1) through (4) are equivalent. For each $k \in I$, let $i_k : \mathbf{N}_0 \to \mathbf{N}_0$ and $j_k : \mathbf{N}_0 \to \mathbf{N}$ be the height difference and Ulm-Kaplansky exponent functions corresponding to $\mathbf{H}(a_0^k)$, respectively. Recall that, since G is strictly reduced, all indicators $\mathbf{H}(a_0^k)$ are of finite type, i.e., $g_n = j(n) - i(n) - 1$ is strictly increasing.

Properties (3) and (4) are clearly equivalent since the ϱ_j^k precisely reflect the coefficients of the elements $t_j^k(\alpha)$.

(1) $\Longrightarrow$ (2). Let $\tilde{j} \in \mathbf{N}$ satisfying $J_{\tilde{j}} \neq \varnothing$ be fixed. Then the gap-elements $t_{\tilde{j}}^k(\alpha) = \sum_{u \in I_{\tilde{j}}} \alpha^{k,u}_{i_k(n_k),\tilde{j}} x^u_{\tilde{j}}$ are uniquely determined for all $k \in J_{\tilde{j}}$. Note that $\tilde{j} = j_k(n_k)$ and that $\tilde{j} - i_k(n_k) - 1 \geq 0$. From Lemma 2.2 we have $\alpha^{k,u}_{i_k(n_k),j} = 0$, whenever $j > \tilde{j}$. Now let $\sum_k r_k \, p^{\tilde{j}-1} \, t^k_{\tilde{j}}(\alpha)$, $r_k \in R \setminus pR$ be a finite sum, where $k \in J_{\tilde{j}}$. Then

$$
\begin{aligned}
f &= \sum_k r_k \, p^{\tilde{j}-i_k(n_k)-1} \sum_{j > \tilde{j}-1} \sum_{u \in I_{\tilde{j}}} \sum_{s=i_k(n_k)}^{i-1} p^s \alpha^{k,u}_{s,j} x^u_j \\
&= \sum_k r_k \, p^{\tilde{j}-1} \, t^k_{\tilde{j}}(\alpha) \\
&\quad + \sum_k r_k \, p^{\tilde{j}-i_k(n_k)-1} \sum_{j > \tilde{j}-1} \sum_{u \in I_{\tilde{j}}} \sum_{s=i_k(n_k)+1}^{i-1} p^s \alpha^{k,u}_{s,j} x^u_j .
\end{aligned}
\tag{3.7}
$$

We have

$$h^*(p^{\tilde{j}-i_k(n_k)-1} a_0^k) = \beta_{\tilde{j}-i_k(n_k)-1} = \beta^k_{g_{n_k}} = j_k(n_k) - 1 = \tilde{j} - 1.$$

Thus, by Lemma 3.4,

$$h^*(f) = h^*\bigg(\sum_k p^{\tilde{j}-i_k(n_k)-1} r_k a_0^k \bigg).$$

Since $\{a_0^k \mid k \in I\}$ is a decomposition basis, we have $h^*(f) = \tilde{j} - 1$. The height of the second sum in (3.7) is bigger than $\tilde{j} - 1$ because the relation array α is restricted. Consequently,

$$h^*\bigg(\sum_k r_k \, p^{\tilde{j}-1} \, t^k_{\tilde{j}}(\alpha) \bigg) = \tilde{j} - 1,$$

and by Remark 3.3 property (2) holds.

(2) $\Longrightarrow$ (3). By Remark 3.3 the $t_j^k(\alpha)$, $k \in J_j$, are independent for every fixed j. Since $t_j^k(\alpha) \in \oplus_{u\in I_j} Rx_j^u$ property (3) follows.

(3) $\Longrightarrow$ (1). Since the set $\{a_0^k \mid k \in I\}$ is a maximal independent set of torsion-free elements, we need to show that $h^*(\sum_k r_k a_0^k) = \min_k(h^*(r_k a_0^k))$. Without loss of generality we may assume that $h^*(r_k a_0^k) = \sigma$ for all k. For each k we have $j_k(n_k - 1) < \sigma < j_k(n_k)$ for some n_k. By Lemma 3.4 and by (3) we obtain

$$\begin{aligned} h^*\bigg(\sum_k r_k a_0^k\bigg) &= h^*\bigg(\sum_k r_k \sum_{j>h^*(r_k a_0^k)} \sum_{u\in I_j} \sum_{s=i_k(n_k)}^{i-1} p^s \alpha_{s,j}^{k,u} x_j^u\bigg) \\ &= h^*\bigg(\sum_k r_k \sum_{j>\sigma} \sum_{u\in I_j} \alpha_{i_k(n_k),j}^{k,u}\, p^{i_k(n_k)} x_j^u\bigg), \end{aligned} \tag{3.8}$$

where the last equality follows from the independence of the gap-elements $t_{j_k(n_k)}^k(\alpha)$ and the omission of elements of bigger height. By Lemma 2.2 we know that $\alpha_{i_k(n_k),j}^{k,u} = 0$ for all $j > j_k(n_k)$. Therefore, we may write (3.8) as

$$\begin{aligned} h^*\bigg(\sum_k r_k a_0^k\bigg) &= h^*\bigg(\bigg[\sum_k r_k\, p^{i_k(n_k)} t_{j_k(n_k)}^k(\alpha)\bigg] \\ &\qquad + \bigg[\sum_k r_k\, p^{i_k(n_k)} \sum_{j>\sigma}^{j_k(n_k)-1} \sum_{u\in I_j} \alpha_{i_k(n_k),j}^{k,u} x_j^u\bigg]\bigg) \\ &= h^*\bigg(\sum_k r_k\, p^{i_k(n_k)} t_{j_k(n_k)}^k(\alpha)\bigg), \end{aligned} \tag{3.9}$$

since the second sum has no bearing on the height by (3) and the fact that α is restricted. Again, by the independence of the gap-elements, we have

$$h^*\bigg(\sum_k r_k a_0^k\bigg) = \min_k\, \big(r_k\, p^{i_k(n_k)} t_{j_k(n_k)}^k(\alpha)\big) = \min_k\, \big(h^*(r_k a_0^k)\big),$$

where the last equation follows from the definition of i_k, i.e., $h^*(r_k a_0^k) - m(k) = i_k(n_k)$ with $r_k \in p^{m(k)} R \setminus p^{m(k)+1} R$. Thus the set $\{a_0^k \mid k \in I\}$ is a decomposition basis of G.

We now show that these conditions are equivalent to a module being Warfield. If G is Warfield, then by [**7**, Theorem 11] it has a decomposition basis. An application of Lemma 3.1 completes one direction of the proof. Conversely, assume that one of the conditions (1)–(4) above holds. Without loss of generality it suffices to show that G is Warfield if the first condition is satisfied. Let $B = \{x_i^v, a_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$ be a basic generating system of G with a corresponding restricted relation array $(\alpha_{i-1,j}^{k,u})$ such that the set $\{a_0^k \mid k \in I\}$ is a decomposition basis. Since $|I| \leq \aleph_0$, and since the relation array $(\alpha_{i-1,j}^{k,u})$ is row finite in j and u, we may write $G = G' \oplus T$, where T is a direct summand of $\mathbf{t}G$ and G' is a countably generated module which has the decomposition basis $\{a_0^k \mid k \in I\}$. By [**7**, Theorem 12] it follows that G' is Warfield and hence so is G. □

We now give an example showing the necessity of conditions (1)–(4) in Proposition 3.5. Our example is a torsion-free rank 2 module in $\mathcal{H}$ which is non-Warfield yet all torsion-free elements have indicators of finite-type.

Example 3.6. Let G be a mixed module in the class $\mathcal{H}$ with a basic generating system $B = \{x_i, a_{i-1}^k \mid i \in \mathbf{N}, k = 1, 2\}$ and a corresponding relation array $(\alpha_{i-1,j}^k)$ defined by

$$\alpha_{i,4i+1}^1 = 1, \alpha_{i,2i+1}^2 = 1 \quad \text{and} \quad \alpha_{i,j}^k = 0,\ k = 1, 2, \quad \text{otherwise.}$$

We show that G is not Warfield in several steps. First we calculate the indicator of any linear combination $ra_0^1 + sa_0^2 \neq 0$. Then we show that for a fixed pair of linear combinations of the form above there is a gap torsion element relative to both. Finally we apply Proposition 3.5 to deduce that G is not Warfield since the sequence of gap-elements are not independent.

Let $ra_0^1 + sa_0^2 \neq 0$ be a linear combination. If either $r = 0$ or $s = 0$, then the indicator $\mathbf{H}(ra_0^1 + sa_0^2)$ equals either $\mathbf{H}(sa_0^1)$ or $\mathbf{H}(ra_0^2)$ and there is a sufficiently large $J \in \mathbf{N}$ depending on the p-divisibility of r and s such that for every $j \geq J$ the torsion generator x_{4j+1} is a gap-element relative to $\mathbf{H}(ra_0^1 + sa_0^2)$. Now assume that both r and s are both nonzero. Then integers $m(r), m(s) \in \mathbf{N}_0$ exist such that $r \in p^{m(r)}R \backslash p^{m(r)+1}R$ and $s \in p^{m(s)}R \backslash p^{m(s)+1}R$. Considering $ra_0^1 + sa_0^2$

as a torsion-free generator of a modified basic generating system, by Lemma 2.5 we conclude that for every $n \in \mathbf{N}_0$ some $i(n) \in \mathbf{N}$ exists such that

$$h^*\big(p^n(ra_0^1 + sa_0^2)\big) = h^*\Bigg(\sum_{\nu=0}^{i(n)-1} p^{n+\nu}\big((rx_{4\nu+1}) + (sx_{2\nu+1})\big)\Bigg).$$

From this we infer that for fixed $n \in \mathbf{N}$ the smallest $\nu \in \{0, \dots, i(n)-1\}$ satisfying at least one of the conditions

(1) $m(r) + n + \nu \leq 4\nu \Longleftrightarrow m(r) + n \leq 3\nu$

(2) $m(s) + n + \nu \leq 2\nu \Longleftrightarrow m(s) + n \leq \nu$

is relevant for the height of $p^n(ra_0^1 + sa_0^2)$. Define two functions $\nu_r : \mathbf{N}_0 \to \mathbf{N}_0$ by $\nu_r(n) = \min\{z \in \mathbf{N} \mid 3z \geq m(r) + n\}$, and $\nu_s : \mathbf{N}_0 \to \mathbf{N}_0$ by $\nu_s(n) = m(s)+n$. Then we obtain $h^*(p^n(ra_0^1+sa_0^2)) = \min\{m(r) + n + \nu_r(n), m(s) + n + \nu_s(n)\}$ if $m(r) + n + \nu_r(n) \neq m(s)+n+\nu_s(n)$. Observe that the function $\nu_s(n) - \nu_r(n)$ is increasing. Hence for all sufficiently large n satisfying $m(r) + n \equiv 0 \pmod 3$ we obtain $n + m(r) = 3\nu_r(n)$ but $\nu_s(n) > \nu_r(n) + m(r) - m(s)$. By the above this yields $h^*(p^n(ra_0^1 + sa_0^1)) = m(r) + n + \nu_r(n) = 4\nu_r(n)$. The definition of the functions ν_r and ν_s implies that all elements x_{4j+1}, where $j \geq \nu_r(n)$, are gap-elements relative to $\mathbf{H}(ra_0^1 + sa_0^2)$.

Altogether, we conclude that for every linear combination $ra_0^1+sa_0^2 \neq 0$ a natural number J exists depending on r and s such that the elements x_{4j+1}, where $j \geq J$, are gap-elements relative to $\mathbf{H}(ra_0^1 + sa_0^2)$. Since every indicator that is realized in G is equivalent to the indicator of some linear combination of a_0^1 and a_0^2, therefore any two pairs of indicators of torsion-free elements of G have common gap-elements and hence are not independent. Thus the module G is not Warfield by Proposition 3.5.

Our goal is to characterize the Warfield modules in $\mathcal{H}$ which are either strictly reduced or whose zeroth Ulm factor is torsion. These turn out to be familiar objects, namely, the simply presented modules in $\mathcal{H}$, or equivalently, the direct sums of modules of torsion-free rank 1, cf. Corollary 2.3. We begin by recalling some definitions from [**2**] and [**8**].

If M is a module with decomposition basis X then, for any equivalence class $\overline{\mathbf{E}}$ of indicators, let $g_X(\overline{\mathbf{E}}, M)$ be the number of elements $x \in X$ such that $\mathbf{H}(x) \in \overline{\mathbf{E}}$. Warfield proved that the $g_X(\overline{\mathbf{E}}, M)$ are independent of the decomposition basis X, cf. [**8**], and we can write $g_X(\overline{\mathbf{E}}, M) = g(\overline{\mathbf{E}}, M)$. As in [**2**], we will call the invariant $g(\overline{\mathbf{E}}, M)$ the *Warfield invariant of* M *at* $\overline{\mathbf{E}}$. In [**8**, Theorem 5.3] it was shown that two Warfield modules are isomorphic if and only if they have equal Warfield invariants and equal Ulm-Kaplansky invariants.

Theorem 3.7. *For a module* G *in the class* $\mathcal{H}$ *which is either strictly reduced or has a torsion zeroth Ulm factor, the following are equivalent*:

(1) G *is Warfield*.

(2) G *is simply presented*.

(3) G *is a direct sum of modules of torsion-free rank* 1.

Proof. The statement (2) $\Longrightarrow$ (1) is obvious, while (3) $\Longrightarrow$ (2) follows from Corollary 2.3. It still remains to prove that (1) $\Longrightarrow$ (3).

Let G be a Warfield module in the class $\mathcal{H}$ that either has a torsion zeroth Ulm factor or is strictly reduced. By Corollary 2.4 we may assume that G is countably generated. We will construct a module H such that the following hold true.

(1) H is a direct sum of modules of torsion-free rank 1,

(2) $g(\overline{\mathbf{E}}, H) = g(\overline{\mathbf{E}}, G)$ for every equivalent class $\overline{\mathbf{E}}$ of indicators,

(3) $f_\sigma(H) = f_\sigma(G)$ for every ordinal σ, and $f_\infty(H) = f_\infty(G)$.

If this is done an application of Warfield's theorem [**8**, Theorem 5.3] to G and H will conclude the proof.

Since G is Warfield, by [**7**, Theorem 11] and Lemma 3.1 we may assume that there is a basic generating system $B = \{x_i^v, a_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$ of G with a corresponding restricted relation array $(\alpha_{i-1,j}^{k,u})$ such that the set $\{a_0^k \mid k \in I\}$ is a decomposition basis of G. We distinguish two cases, when G has a torsion zeroth Ulm factor or when G is strictly reduced.

First assume that G has a torsion zeroth Ulm factor. We may assume that G is reduced. Since $G/p^\omega G$ is torsion and since G is reduced, the indicators $\mathbf{H}(a_0^k)$, $k \in I$, are all of ω-type. Hence, we obtain

$$g(\overline{\mathbf{E}}, G) = \begin{cases} |I| & \text{if } \overline{\mathbf{E}} \text{ is the equivalence class of the} \\ & \text{indicators of } \omega\text{-type} \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Let $J = \{j \in \mathbf{N} \mid I_j \neq \varnothing\}$. Since $G/\mathbf{t}G$ is divisible and since G is assumed to be reduced, we infer that $|J| = \aleph_0$. Hence we have $|I \times \mathbf{N}| = |I| \cdot |\mathbf{N}| = \aleph_0 = |J|$, and there is an injection $j : I \times \mathbf{N} \to J$. By the definition of the set J and since $j(k,l) \in J$ for every pair $(k,l) \in I \times \mathbf{N}$, there is some $u(k,l) \in I_{j(k,l)}$ for every pair $(k,l) \in I \times \mathbf{N}$. Now let $(\beta_{i-1,j}^{k,u})$ be a relation array defined by

$$\beta_{j(k,l),j(k,l)}^{k,u(k,l)} = 1 \quad \text{and} \quad \beta_{i-1,j}^{k,u} = 0 \quad \text{otherwise.}$$

Furthermore, let H be a module in the class $\mathcal{H}$ with a basic generating system $C = \{x_i^v, b_{i-1}^k \mid i \in \mathbf{N}, v \in I_i, k \in I\}$ and a corresponding relation array $(\beta_{i-1,j}^{k,u})$. Since, for every $k \in I$ the relation array $(\beta_{i-1,j}^{k,u})$ is diagonal, by Theorem 2.6 the indicator $\mathbf{H}(b_0^k)$, $k \in I$, is of ω-type. Moreover, for every $k \in I$ we obtain a submodule $H^k \subset H$ of torsion-free rank 1 defined by $\langle b_{i-1}^k \mid i \in \mathbf{N}\rangle$. The submodules H^k have the torsion submodule $\mathbf{t}H^k = \oplus_{l \in \mathbf{N}} R x_{j(k,l)}^{u(k,l)}$, $k \in I$, while the intersection $\cap_{k \in I} \mathbf{t}H^k$ is zero, as $j : I \times \mathbf{N} \to J$ is injective.

Now assume that G is strictly reduced. For every $k \in I$, let $j_k(n)$ be the Ulm-Kaplansky exponent of $\mathbf{H}(a_0^k)$, and define $N^k := \{j_k(n) \mid n \in \mathbf{N}_0\}$, i.e., N^k is the image of $\mathbf{N}_0$ under j_k, $k \in I$. Then by Proposition 3.5, we infer that

$$T \oplus \bigoplus_{j \in \mathbf{N}} \bigoplus_{k \in J_j} R t_j^k(\alpha) = T \oplus \bigoplus_{k \in I} \bigoplus_{j \in N^k} R t_j^k(\alpha) \cong \mathbf{t}G$$

for some direct sum of cyclic torsion modules T. Now for fixed $k \in I$, let H^k be a module in the class $\mathcal{H}$ of torsion-free rank 1 with a basic generating system

$$C = \{t_j^k(\alpha), b_{i-1}^k \mid i \in \mathbf{N}, j \in N^k\} = \{t_{jk(n)}^k(\alpha), b_{i-1}^k \mid i \in \mathbf{N}, n \in \mathbf{N}_0\}$$

and a corresponding relation array $(\gamma_{i-1,j}^k)$ defined by

$$\gamma_{i_k(n),j_k(n)}^k = 1 \quad \text{and} \quad \gamma_{i,j}^k = 0 \quad \text{otherwise,}$$

where $i_k(n)$ is the height difference of $\mathbf{H}(a_0^k)$, $k \in I$. Then, for every $k \in I$ the relation array $(\gamma_{i-1,j}^k)$ is strictly concave relative to the indicator $\mathbf{H}(a_0^k)$ and by Lemma 2.2 we have

$$\mathbf{H}(b_0^k) \cong \mathbf{H}(a_0^k), \quad k \in I. \tag{3.11}$$

In both cases we obtain a module H given by $H = T = \oplus \bigoplus_{k \in I} H^k$ that is a direct sum of modules of torsion-free rank 1 with a decomposition basis $\{b_0^k \mid k \in I\}$ such that $\mathbf{t}H \cong \mathbf{t}G$. Therefore, we have $f_\sigma(G) = f_\sigma(H)$ for every ordinal σ and $f_\infty(G) = f_\infty(H)$. Since the sets $\{a_0^k \mid k \in I\}$ and $\{b_0^k \mid k \in I\}$ are decomposition bases of the modules G and H, respectively, we have $g(\overline{\mathbf{E}}, G) = g(\overline{\mathbf{E}}, H)$ by (3.10) if G has a torsion zeroth Ulm factor and by (3.11) if G is strictly reduced. Thus, the modules G and H satisfy all the hypotheses of [**8**, Theorem 5.3] and hence are isomorphic. Therefore, G is the direct sum of modules of torsion-free rank 1. □

Now we show that there are Warfield modules in the class $\mathcal{H}$ that are not simply presented. We construct a module of torsion-free rank 2 which contains torsion-free elements whose indicators are of finite and ω-type. This points out the necessity of the hypothesis on the indicators in Theorem 3.7. This example complements the one in [**8**] of a Warfield module which is not simply presented whose torsion-free rank is 1.

Example 3.8. Let G be a module in the class $\mathcal{H}$ with a basic generating system $B = \{x_{2i}, a_{i-1}^k \mid i \in \mathbf{N}, k = 1, 2\}$ and a corresponding restricted relation array $(\alpha_{i-1,j}^k)$ defined by $\alpha_{i,2i}^1 = 1$ and $\alpha_{i,j}^1 = 0$ if $j \neq 2i$, $\alpha_{2i,2i}^2 = 1$ and $\alpha_{i,j}^2 = 0$ otherwise. Hence we have the relations

$$\begin{aligned} pa_{i+1}^1 &= a_i^1 + x_{2i}, \quad i \in \mathbf{N}, & pa_1^1 &= a_0^1, \\ pa_{2i+1}^2 &= a_{2i}^2 + x_{2i}, \quad i \in \mathbf{N}, & pa_{i+1}^2 &= a_i^2, \quad \text{otherwise.} \end{aligned}$$

First we show that G is Warfield by finding a decomposition basis of G. Consider the relation arrays $(\alpha_{i-1,j}^1)$ and $(\alpha_{i-1,j}^2)$. Since the relation array $(\alpha_{i-1,j}^2)$ is diagonal, and by Theorem 2.6, we infer that $\mathbf{H}(a_0^2)$ is of ω-type. Hence there exists a torsion-free element $b \in Ra_0^2 \subset G$ such that $h^*(b) \geq \omega$. As the relation array $(\alpha_{i-1,j}^1)$ is strictly concave, Lemma 2.2 implies that $\mathbf{H}(a_0^1)$ is of finite-type.

Hence we have $h^*(p^n a_0^1) < \omega \leq h^*(b)$ for all $n \in \mathbf{N}_0$. Thus we obtain $h^*(ra_0^1 + sb) = \min\{h^*(ra_0^1), h^*(sb)\}$, $r, s \in R$, i.e., $\{a_0^1, b\}$ is a decomposition basis of G. By [**7**, Theorem 12] the module G is Warfield.

We now show that G is not simply presented. Since $\mathbf{t}G = \oplus_{i\in\mathbf{N}} Rx_{2i}$, the defining relations corresponding to B imply that G is generated by $\{a_{i-1}^k \mid i \in \mathbf{N}, k = 1, 2\}$. If G was simply presented then, by [**8**, Lemma 2.2], it would be a direct sum of modules of at most torsion-free rank 1. This implies that there is a 2×2-matrix $D = (d_{kh})$ with entries in R and two natural numbers $i(1), i(2)$ such that the sets $\{d_{k1}a_{i-1}^1 + d_{k2}a_{i-1}^2 \mid i \geq i(k)\}$, $k = 1, 2$, generate submodules $G^k \subset G$, $k = 1, 2$, where $G^k \neq 0$ for $k = 1, 2$, and $G^1 \cap G^2 = 0$. The $G^k \neq 0$, $k = 1, 2$, implies that D has no zero rows, and $G^1 \cap G^2 = 0$ implies that D has no zero columns. However, we show the nonexistence of such a matrix D thus contradicting our assumption that G is simply presented.

Let $D = (d_{kh})$ be a 2×2-matrix with entries in R such that D has no zero rows or zero columns. For $i \geq 2$ and $k \in \{1, 2\}$, we obtain

$$(3.12)\quad d_{k1}(a_i^1 - a_{i-1}^1) + d_{k2}(a_i^2 - a_{i-1}^2) = \begin{cases} d_{k1}x_{2i-2} & i \in 2\mathbf{N} \\ d_{k1}x_{2i-2} + d_{k2}x_{i-1} & i \notin 2\mathbf{N}. \end{cases}$$

Since D has no zero rows or zero columns, we must distinguish two cases, when $d_{k1} \neq 0$, $k = 1, 2$, or $d_{k1} = 0$ for exactly one $k \in \{1, 2\}$. Let $i(k) \in \mathbf{N}$, $k = 1, 2$. First assume that $d_{k1} \neq 0$ for $k = 1, 2$. Then $h_R^*(d_{k1}) < \infty$, $k = 1, 2$. Choose an even natural number n such that $n > \max\{h_R^*(d_{k1}) + 1, i(k) \mid k = 1, 2\}$. Then $h_R^*(d_{11}, d_{21}) < 2n - 2$ and therefore $d_{11}d_{21}x_{2n-2} \neq 0$. Furthermore, since $n \in 2\mathbf{N}$, and by (3.12) we obtain

$$d_{k1}(a_n^1 - a_{n-1}^1) + d_{k2}(a_n^2 - a_{n-1}^2) = d_{k1}x_{2n-2}, \quad k = 1, 2.$$

Hence we have $0 \neq d_{11}d_{21}x_{2n-2} \in \cap_{k=1,2}\langle d_{k1}a_{i-1}^1 + d_{k2}a_{i-1}^2 \mid i \geq i(k)\rangle$ and G is not a direct sum of modules of torsion-free rank 1.

Now assume that $d_{11} \neq 0$ and $d_{21} = 0$. Since D has no zero rows or zero columns, we infer that $d_{22} \neq 0$. Thus we have $h_R^*(d_{kk}) < \infty$ for $k = 1, 2$. Choose an even natural number n such that $n > \max\{h_R^*(d_{kk}) + 1, i(k) \mid k = 1, 2\}$. Then $h_R^*(d_{11}d_{22}) < 2n - 2$, and therefore $d_{11}d_{22}x_{2n-2} \neq 0$. Furthermore, since $n \in 2\mathbf{N}$, and by (3.12), we obtain

$$d_{11}(a_n^1 - a_{n-1}^1) + d_{12}(a_n^2 - a_{n-1}^2) \;=\; d_{11}x_{2n-2}$$

and

$$d_{21}(a_{2n-1}^1 - a_{2n-2}^1) + d_{22}(a_{2n-1}^2 - a_{2n-2}^2) \underset{d_{21}=0}{=} d_{22}x_{2n-2}.$$

Altogether we have $0 \neq d_{11}d_{22}x_{2n-2} \in \cap_{k=1,2}\langle d_{k_1}a_{i-1}^1 + d_{k2}a_{i-1}^2 \mid i \geq i(k)\rangle$, and again G is not a direct sum of modules of torsion-free rank 1.

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