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PERTURBATION OF FRAMES FOR A SUBSPACE OF A HILBERT SPACE

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ABSTRACT. A frame sequence $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ allows every element in the closed linear span, $[f_i]$, to be written as an infinite linear combination of the frame elements f_i. Thus a frame sequence can be considered to be some kind of "generalized basis." Using an extension of a classical condition, we prove that a perturbation $\{g_i\}_{i=1}^{\infty}$ of a frame sequence $\{f_i\}_{i=1}^{\infty}$ is again a frame sequence whenever the gap from $[g_i]$ to $[f_i]$ is small enough. In the special case of a Riesz sequence $\{f_i\}_{i=1}^{\infty}$ the gap condition may be omitted.

1. Introduction. A frame sequence $\{f_i\}_{i=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ has the property that every element in $[f_i] := \overline{\text{span}}\,\{f_i\}_{i=1}^{\infty}$ has a representation as an infinite linear combination of the frame elements f_i. In contrast with the situation for a basis, the corresponding coefficients are not necessarily unique, which makes frame sequences a very useful tool when more freedom is required. A frame sequence is thus a very natural generalization of the concept of a Riesz sequence (i.e., a sequence that is a Riesz basis for its closed linear span).

Our goal is to prove some perturbation results for frame sequences. To motivate the following, remember that if $\{f_i\}_{i=1}^{\infty}$ is a Riesz sequence, then $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is a Riesz sequence whenever

$$(1)\qquad \left\| \sum_{i=1}^{\infty} c_i(f_i - g_i) \right\| \leq \mu \left(\sum_{i=1}^{\infty} |c_i|^2 \right)^{1/2}, \quad \forall \{c_i\}_{i=1}^{\infty} \in l^2(N),$$

for a sufficiently small constant μ. We prove the same conclusion holds under a weaker condition than (1).

The direct analogue of this last Riesz sequence result for a frame sequence $\{f_i\}_{i=1}^{\infty}$ does not hold unless $[f_i]$ is the whole Hilbert space. This leads us to consider the notion of the gap from one subspace of

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a Hilbert space to another. The gap from a subspace $\mathcal{K}$ to a subspace $\mathcal{L}$ is the supremum over a family of sine values, $\sin(\theta_x)$, where each θ_x is the angle between a given vector x in the unit sphere of $\mathcal{K}$ and the orthogonal projection of x onto $\mathcal{L}$. We prove an analogue for frame sequences of the above-described result for Riesz sequences when the gap from $[g_i]$ to $[f_i]$ is small enough.

Frame perturbation ideas have a long history. Implicitly, the original frame paper of Duffin and Schaeffer [**8**] is concerned with a special class of frame perturbations of the usual trigonometric orthonormal basis of $L^2[-\gamma, \gamma]$. Frame perturbations were first explicitly introduced by Chris Heil in his Ph.D. thesis [**11**]. More recent references are [**3**], [**4**], [**6**], [**9**]. The more technical question of perturbation of a frame sequence was first treated in Lewis' M.S. Thesis [**15**] and in [**5**]. Our main result extends [**15**] along the lines of [**3**].

2. Preliminaries. First we collect some definitions and basic facts that we will need later. In all that follows, $\mathcal{H}$ denotes a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ linear in the first entry, over the scalar field (which is either the real or complex numbers). Throughout, I and J will denote countable index sets, and $\mathbf{N}$ will be the set of all positive integers. Given a sequence $\{f_i\}_{i \in I} \subseteq \mathcal{H}$, we let $[f_i]$ denote the closed linear span of the elements $\{f_i\}_{i \in I}$. The orthogonal complement of a subspace $\mathcal{K}$ in $\mathcal{H}$ and the orthogonal projection onto $\mathcal{K}$ will be denoted by $\mathcal{K}^{\perp}$ and $P_{\mathcal{K}}$, respectively. We will also denote by $l^2(I)$ the usual Hilbert space of all absolutely square-summable, scalar-valued sequences with domain I. As usual and where appropriate, I will also denote the identity operator on $\mathcal{H}$.

A family of elements $\{f_i\}_{i \in I} \subseteq \mathcal{H}$ is called a *frame* if $A, B > 0$ exists such that

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad \forall\, f \in \mathcal{H}. \tag{2}$$

If at least the upper condition is satisfied, $\{f_i\}_{i=1}^{\infty}$ is a *Bessel sequence*. In that case we can define a bounded linear operator

$$T : l^2(I) \longrightarrow \mathcal{H}, \quad T\{c_i\}_{i \in I} = \sum_{i \in I} c_i f_i,$$

and $\|T\| \le \sqrt{B}$. In case $\{f_i\}_{i=1}^{\infty}$ is a frame, the *frame operator* $S := TT^*$ is invertible. This leads to a representation of any $f \in \mathcal{H}$ as an infinite linear combination of the frame elements, the so-called *frame decomposition*:

$$f = SS^{-1}f = \sum_{i \in I} \langle f, S^{-1} f_i \rangle f_i, \quad \forall\, f \in \mathcal{H}.$$

The numbers A, B in (2) are called *frame bounds*. It is well known that $\{S^{-1}f_i\}_{i \in I}$ is also a frame, with bounds B^{-1}, A^{-1}. For more general information about frames, we refer to [**7**], [**12**].

In particular, a frame $\{f_i\}_{i \in I}$ is a total set in $\mathcal{H}$, i.e., $[f_i] = \mathcal{H}$. If a family $\{f_i\}_{i \in I}$ is not total in $\mathcal{H}$ it might still happen that $\{f_i\}_{i \in I}$ is a frame for $[f_i]$, in which case we say that $\{f_i\}_{i \in I}$ is a *frame sequence*. In this case the frame operator S is invertible as an operator from $[f_i]$ onto $[f_i]$.

A *Riesz basis* is a special case of a frame. Recall that $\{f_i\}_{i \in I}$ is a Riesz basis for $\mathcal{H}$ if $[f_i] = \mathcal{H}$ and $A, B > 0$ exists for which

$$A \sum_{i \in I} |c_i|^2 \le \left\| \sum_{i \in I} c_i f_i \right\|^2 \le B \sum_{i \in I} |c_i|^2, \quad \forall\, \{c_i\}_{i \in I} \in l^2(I).$$

The numbers A, B above are actually frame bounds. We say that $\{f_i\}_{i \in I}$ is a *Riesz sequence* if $\{f_i\}_{i \in I}$ is a Riesz basis for $[f_i]$.

Our goal is to prove some perturbation results for frame sequences. The perturbation condition we consider here has several classical predecessors, of which we only mention a few. Paley and Wiener [**17**] proved that if $\{f_i\}_{i \in I}$ is an orthonormal basis for $L^2[a, b]$ and $\lambda \in [0,\ )$ exists such that

$$\left\| \sum_{i \in F} c_i (f_i - g_i) \right\| \le \lambda \left\| \sum_{i \in F} c_i f_i \right\|$$

for all finite scalar sequences $\{c_i\}_{i \in F}$, $F \subseteq I$, then $\{g_i\}_{i \in I}$ is a Riesz basis for $L^2[a, b]$. Later, Boas [**2**] proved that the same holds if $\{f_i\}_{i \in I}$ is a Riesz basis and $L^2[a, b]$ is replaced by any Hilbert space. Pollard [**16**] showed that if $\{f_i\}_{i \in I}$ is total and $\lambda_1, \lambda_2 \in [0, (1/\sqrt{2}))$ exist such that

$$\left\| \sum_{i \in F} c_i (f_i - g_i) \right\| \le \lambda_1 \left\| \sum_{i \in F} c_i f_i \right\| + \lambda_2 \left\| \sum_{i \in F} c_i g_i \right\|$$

for all finite scalar sequences $\{c_i\}_{i\in F}$, $F \subseteq I$, then $\{g_i\}_{i\in I}$ is total. Hilding [**13**] relaxed the condition to $\lambda_1, \lambda_2 \in [0,1)$. Both conditions can be viewed as "boundedness conditions" on the operator $\{c_i\}_{i\in I} \to \sum_{i\in I} c_i(f_i - g_i)$. A different condition concerning this operator can be found in the book of Kato [**14**]: the operator $\{c_i\}_{i\in I} \to \sum_{i\in I} c_i(f_i - g_i)$ is said to be *bounded with respect to the operator* $\{c_i\}_{i\in I} \to \sum_{i\in I} c_i f_i$ if $\lambda, \mu \geq 0$ exists for which

$$\Big\| \sum_{i\in F} c_i(f_i - g_i) \Big\| \leq \lambda \Big\| \sum_{i\in F} c_i f_i \Big\| + \mu \bigg(\sum_{i\in F} |c_i|^2 \bigg)^{1/2}$$

for all finite scalar sequences $\{c_i\}_{i\in F}$, $F \subseteq I$.

Conditions of this type are also important in modern analysis. Based on the consequences of the validity of the above condition, Favier and Zalik [**9**] were able to show that if

$$\{f_i\}_{i\in I} := \left\{ \frac{1}{2^{n/2}}\, \psi(2^n x - bm) \right\}_{m,n\in Z}$$

is a frame for $L^2(R)$ (for a certain $b > 0$ and a function ψ satisfying some mild conditions), then

$$\{g_i\}_{i\in I} := \left\{ \frac{1}{2^{n/2}}\, \psi(2^n x - \tilde{b}m) \right\}_{m,n\in Z}$$

is also a frame for $\tilde{b}$ close to b. This is indeed a surprising result: for $b \neq \tilde{b}$, the two functions $x \to 1/2^{n/2}\psi(2^n x - bm)$ and $x \to 1/2^{n/2}\psi(2^n x - \tilde{b}m)$ are moving far apart from each other for large values of m, so $\{f_i\}_{i\in I}$, $\{g_i\}_{i\in I}$ are not close to each other in the traditional sense. Frames of this special type are called *wavelet frames*.

The condition we consider in the sequel incorporates all the cases discussed above. We assume that $\{f_i\}_{i\in I}$ is a frame sequence in a Hilbert space $\mathcal{H}$, and that constants $\lambda_1, \lambda_2, \mu$ exist such that

$$\Big\| \sum_{i\in F} c_i(f_i - g_i) \Big\| \leq \lambda_1 \Big\| \sum_{i\in F} c_i f_i \Big\| + \lambda_2 \Big\| \sum_{i\in F} c_i g_i \Big\| + \mu \bigg(\sum_{i\in F} |c_i|^2 \bigg)^{1/2}$$

for all finite scalar sequences $\{c_i\}_{i\in F}$, $F \subseteq I$.

Clearly we still have to restrict the values of the parameters in order to get conclusions about $\{g_i\}_{i\in I}$. The complication which appears compared to the situation where a frame $\{f_i\}_{i\in I}$ for $\mathcal{H}$ is perturbed (cf. [**3**], [**4**]), is that a perturbation $\{g_i\}_{i\in I}$ might be outside of $[f_i]$.

The main tool in our generalization is the *gap* from a subspace $\mathcal{K}$ to a subspace $\mathcal{L}$, a notion that can be found in [**14**].

Definition 1. Let $\mathcal{K}$ and $\mathcal{L}$ be subspaces of $\mathcal{H}$. When $\mathcal{K} \neq \{0\}$, the gap from $\mathcal{K}$ to $\mathcal{L}$ is given by

$$\begin{aligned}\delta(\mathcal{K},\mathcal{L}) &:= \sup_{x\in\mathcal{K},\|x\|=1} \inf_{y\in\mathcal{L}} \|x-y\| \\ &= \sup_{x\in\mathcal{K},\|x\|=1} \min_{y\in\mathcal{L}} \|x-y\|.\end{aligned}$$

Also, when $\mathcal{K} = \{0\}$, we define $\delta(\mathcal{K},\mathcal{L}) = 0$.

Note that the gap from a subspace $\mathcal{K}$ to a subspace $\mathcal{L}$ is the supremum over a family of sine values, $\sin(\theta_x)$, where each θ_x is the angle between a given vector x in the unit sphere of $\mathcal{K}$ and the orthogonal projection of x onto $\mathcal{L}$.

We can calculate $\delta(\mathcal{K},\mathcal{L})$ using the orthogonal projections $P_{\mathcal{L}^\perp}$ and $P_{\mathcal{K}}$.

Lemma 2.1. $\delta(\mathcal{K},\mathcal{L}) = \|P_{\mathcal{K}}P_{\mathcal{L}^\perp}\|$, *for all subspaces* $\mathcal{K}$ *and* $\mathcal{L}$ *of* $\mathcal{H}$.

Proof. We may assume that $\mathcal{K} \neq \{0\}$. By definition,

$$\begin{aligned}\delta(\mathcal{K},\mathcal{L}) &= \sup_{\substack{x\in\mathcal{K}\\ \|x\|=1}} \|x - P_{\mathcal{L}}x\| = \sup_{\substack{x\in\mathcal{K}\\ \|x\|=1}} \|P_{\mathcal{L}^\perp}x\| \\ &= \sup_{\substack{x\in\mathcal{H}\\ \|x\|=1}} \|P_{\mathcal{L}^\perp}P_{\mathcal{K}}x\| \\ &= \|P_{\mathcal{L}^\perp}P_{\mathcal{K}}\| = \|P_{\mathcal{K}}P_{\mathcal{L}^\perp}\|. \qquad \square\end{aligned}$$

Generally, $\delta(\mathcal{K},\mathcal{L}) \neq \delta(\mathcal{L},\mathcal{K})$. For example, whenever $\mathcal{K}$ and $\mathcal{L}$ are subspaces of $\mathcal{H}$ with $\mathcal{K}$ properly contained in $\mathcal{L}$, we have that $\delta(\mathcal{K},\mathcal{L}) = 0$ and $\delta(\mathcal{L},\mathcal{K}) = 1$.

A crucial point in the proof of our main theorem in the next section is to show that a certain operator on $\mathcal{H}$ is invertible. We will need the following lemma.

Lemma 2.2. *Suppose that U is a bounded operator on $\mathcal{H}$ and that there exist λ_1 and $\lambda_2 \in [0,1)$ such that*

$$\|x - Ux\| \leq \lambda_1 \|x\| + \lambda_2 \|Ux\|, \quad \forall\, x \in \mathcal{H}.$$

Then U is onto and invertible.

A proof can be found in [**3**]. That paper also contains an example showing that the hypothesis of Lemma 2.2 is weaker than the Neumann condition $\|I - U\| < 1$.

3. Perturbation results. The next theorem is our main result.

Theorem 3.1. *Let $\{f_i\}_{i\in I}$ be a frame sequence with bounds A, B and let $\{g_i\}_{i\in I} \subseteq \mathcal{H}$. Let $\mathcal{K} := [g_i]$, $\mathcal{L} := [f_i]$ and assume that there exist constants $\lambda_2 \in [0,1)$ and λ_1, $\mu \geq 0$ such that*

$$\lambda_1 + \frac{\mu}{\sqrt{A}} < \sqrt{1 - \delta(\mathcal{K}, \mathcal{L})^2}$$

and

$$\Big\|\sum_{i\in F} c_i(f_i - g_i)\Big\| \leq \lambda_1 \Big\|\sum_{i\in F} c_i f_i\Big\| + \lambda_2 \Big\|\sum_{i\in F} c_i g_i\Big\| + \mu\Big(\sum_{i\in F} |c_i|^2\Big)^{1/2}, \tag{3}$$

for all finite scalar sequences $\{c_i\}_{i\in F}$, $F \subseteq I$. Then $\{g_i\}_{i\in I}$ is a frame sequence with bounds

$$A\bigg(1 - \frac{\lambda_1 + \lambda_2 + \mu/\sqrt{A}}{1 + \lambda_2}\bigg)^2 \quad \text{and} \quad B\bigg(1 + \frac{\lambda_1 + \lambda_2 + \mu/\sqrt{B}}{1 - \lambda_2}\bigg)^2. \tag{4}$$

(5)
Moreover, $[g_i]$ is isomorphic to $[f_i]$ and $[g_i]^\perp$ is isomorphic to $[f_i]^\perp$.

Proof. Standard arguments (cf. [**4**], [**3**]) give the existence (and value) of the upper frame bound for $\{g_i\}_{i\in I}$ and that (3) actually holds for all sequences $\{c_i\}_{i\in I} \in l^2(I)$. Let S be the frame operator for $\{f_i\}_{i\in I}$. Since $\{g_i\}_{i\in I}$ is a Bessel sequence, we can define a bounded operator

$$U : \mathcal{H} \longrightarrow \mathcal{H}, \quad Uh = \sum_{i\in I} \langle h, S^{-1} f_i\rangle g_i + P_{\mathcal{K}^\perp} P_{\mathcal{L}^\perp} h.$$

We want to show that U is invertible. Fix $h \in \mathcal{H}$ and write $h = h_1 + h_2$, where $h_1 \in \mathcal{L} = [f_i]$, $h_2 \in [f_i]^\perp$. Then, using the frame representations of h_1 and h_2 we obtain that

$$\begin{aligned}
\|h - Uh\| &\leq \|h_1 - Uh_1\| + \|h_2 - Uh_2\| \\
&= \left\| \sum_{i\in I} \langle h_1, S^{-1} f_i\rangle (f_i - g_i)\right\| + \|(P_{\mathcal{L}^\perp} - P_{\mathcal{K}^\perp} P_{\mathcal{L}^\perp}) h_2\| \\
&\leq \lambda_1 \left\| \sum_{i\in I} \langle h_1, S^{-1} f_i\rangle f_i \right\| + \lambda_2 \left\| \sum_{i\in I} \langle h_1, S^{-1} f_i\rangle g_i \right\| \\
&\quad + \mu \left(\sum_{i\in I} |\langle h_1, S^{-1} f_i\rangle|^2 \right)^{1/2} + \|(I - P_{\mathcal{K}^\perp}) P_{\mathcal{L}^\perp}\| \cdot \|h_2\| \\
&\leq \lambda_1 \|h_1\| + \lambda_2 \|Uh_1\| + \frac{\mu}{\sqrt{A}} \|h_1\| + \delta(\mathcal{K}, \mathcal{L}) \|h_2\| \\
&= \left(\lambda_1 + \frac{\mu}{\sqrt{A}}\right) \|h_1\| + \delta(\mathcal{K}, \mathcal{L}) \|h_2\| + \lambda_2 \|Uh_1\| \\
&\leq \sqrt{\left(\lambda_1 + \frac{\mu}{\sqrt{A}}\right)^2 + \delta(\mathcal{K}, \mathcal{L})^2} \cdot \sqrt{\|h_1\|^2 + \|h_2\|^2} + \lambda_2 \|Uh\| \\
&= \sqrt{\left(\lambda_1 + \frac{\mu}{\sqrt{A}}\right)^2 + \delta(\mathcal{K}, \mathcal{L})^2} \cdot \|h\| + \lambda_2 \|Uh\|.
\end{aligned}$$

By Lemma 2.2 we conclude that U is onto and invertible. Thus, U maps $[f_i]$ onto $[g_i]$ and $[f_i]^\perp$ onto $[g_i]^\perp$ which establishes (5). Now let

$h \in \mathcal{K} = [g_i]$. Since $U^{-1}h \in [f_i]$, we have

$$\begin{aligned}\|h\|^4 &= |\langle UU^{-1}h, h\rangle|^2 \\ &= \Big|\sum_{i\in I}\langle U^{-1}h, S^{-1}f_i\rangle\langle g_i, h\rangle\Big|^2 \\ &\le \sum_{i\in I}|\langle U^{-1}h, S^{-1}f_i\rangle|^2 \cdot \sum_{i\in I}|\langle g_i, h\rangle|^2 \\ &\le \frac{1}{A}\cdot\|U^{-1}h\|^2\cdot\sum_{i\in I}|\langle g_i, h\rangle|^2.\end{aligned}$$

In order to estimate $\|U^{-1}h\|$, we observe that our estimate for $\|h-Uh\|$ above shows that

$$\|k - Uk\| \le \Big(\lambda_1 + \frac{\mu}{\sqrt{A}}\Big)\|k\| + \lambda_2\|Uk\|, \quad \forall\, k \in [f_i].$$

Therefore,

$$\begin{aligned}\|Uk\| &\ge \|k\| - \|Uk - k\| \\ &\ge \Big(1 - \Big(\lambda_1 + \frac{\mu}{\sqrt{A}}\Big)\Big)\|k\| - \lambda_2\|Uk\|, \quad \forall\, k \in [f_i],\end{aligned}$$

and so

$$\|Uk\| \ge \frac{1-(\lambda_1 + \mu/\sqrt{A})}{1+\lambda_2}\|k\|, \quad \forall\, k \in [f_i].$$

Thus, for $h \in [g_i]$, with $k := U^{-1}h$ we have

$$\|U^{-1}h\| \le \frac{1+\lambda_2}{1-(\lambda_1+\mu/\sqrt{A})}\|h\|$$

and therefore by the above calculation

$$\|h\|^4 \le \frac{1}{A}\Big(\frac{1+\lambda_2}{1-(\lambda_1+\mu/\sqrt{A})}\Big)^2\|h\|^2\cdot\sum_{i\in I}|\langle g_i, h\rangle|^2.$$

Hence,

$$\begin{aligned}\sum_{i\in I}|\langle g_i, h\rangle|^2 &\ge A\Big(\frac{1-(\lambda_1+\mu/\sqrt{A})}{1+\lambda_2}\Big)^2\|h\|^2 \\ &= A\Big(1 - \frac{\lambda_1+\lambda_2+\mu/\sqrt{A}}{1+\lambda_2}\Big)^2\cdot\|h\|^2, \quad \forall\, h \in [g_i]. \qquad \square\end{aligned}$$

Observe that $\sqrt{1-\delta(\mathcal{K},\mathcal{L})^2}$ is the infimum over the family of cosine values, $\cos(\theta_x)$, where each θ_x is the angle between a given vector x in the unit sphere of $\mathcal{K}$ and the orthogonal projection of x onto $\mathcal{L}$. Also, as pointed out to us by an anonymous referee, one can show by an argument similar to the proof of Lemma 2.1 that

$$\sqrt{1-\delta(\mathcal{K},\mathcal{L})^2} = \inf\{\|P_{\mathcal{L}}x\| \mid x \in \mathcal{K}, \|x\| \le 1\}.$$

Thus, $\delta(\mathcal{K},\mathcal{L}) < 1$ if and only if $P_{\mathcal{L}}$ is an isomorphism of $\mathcal{K}$ onto $\mathcal{L}$.

If $\{f_i\}_{i\in I}$ is a frame for $\mathcal{H}$, we have $\mathcal{L} := [f_i] = \mathcal{H}$ and therefore $\delta(\mathcal{K},\mathcal{L}) = \|P_{\mathcal{K}}P_{\mathcal{L}^\perp}\| = 0$. Moreover, it follows from (5) in Theorem 3.1 that $\mathcal{K} := [g_i] = \mathcal{H}$ and so $\{g_i\}_{i\in I}$ is a frame for $\mathcal{H}$. Consequently, Theorem 3.1 is an extension of the main result of [**3**].

In the special case where $\{f_i\}_{i\in I}$ is a Riesz sequence the gap $\delta(\mathcal{K},\mathcal{L})$ does not need to be included in the hypotheses.

Theorem 3.2. *Let $\{f_i\}_{i\in I}$ be a Riesz sequence with bounds A, B, and let $\{g_i\}_{i\in I} \subseteq \mathcal{H}$. Assume that there exist constants $\lambda_2 \in [0,1[$ and $\lambda_1, \mu \ge 0$ such that $\lambda_1 + \mu/\sqrt{A} < 1$ and*

$$\Big\|\sum_{i\in F} c_i(f_i-g_i)\Big\| \le \lambda_1\Big\|\sum_{i\in F} c_i f_i\Big\| + \lambda_2\Big\|\sum_{i\in F} c_i g_i\Big\| + \mu\Big(\sum_{i\in F}|c_i|^2\Big)^{1/2},$$

for all finite scalar sequences $\{c_i\}_{i\in F}$, $F \subseteq I$. Then $\{g_i\}_{i\in I}$ is a Riesz sequence with the same bounds as in (4) *of Theorem* 3.1.

Proof. The upper frame condition is equivalent to the upper Riesz basis condition, so this part is proved by the same argument as in the proof of Theorem 3.1. For the proof of the lower Riesz sequence condition, let $\{c_i\}_{i\in I} \in l^2(I)$. Then

$$\begin{aligned}\Big\|\sum_{i\in I} c_i g_i\Big\| &\ge \Big\|\sum_{i\in I} c_i f_i\Big\| - \Big\|\sum_{i\in I} c_i(f_i-g_i)\Big\| \\ &\ge (1-\lambda_1)\Big\|\sum_{i\in I} c_i f_i\Big\| - \lambda_2\Big\|\sum_{i\in I} c_i g_i\Big\| - \mu\Big(\sum_{i\in I}|c_i|^2\Big)^{1/2}. \\ &\ge (1-\lambda_1)\sqrt{A}\Big(\sum_{i\in I}|c_i|^2\Big)^{1/2} - \lambda_2\Big\|\sum_{i\in I} c_i g_i\Big\| - \mu\Big(\sum_{i\in I}|c_i|^2\Big)^{1/2}.\end{aligned}$$

So

$$\left\| \sum_{i\in I} c_i g_i \right\| \geq \frac{(1-\lambda_1)\sqrt{A}-\mu}{1+\lambda_2} \left(\sum_{i=1}^{\infty} |c_i|^2 \right)^{1/2}.$$

By assumption $((1-\lambda_1)\sqrt{A}-\mu)(1+\lambda_2)^{-1}$ is positive, from which the result follows. $\square$

However, in the case that $\{f_i\}_{i\in I}$ is a frame sequence, the condition in Theorem 3.2 is not enough to guarantee stability.

Example. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. Fix $\mu \in (0, \sqrt{2})$ and a sequence $\{\alpha_i\}_{i=1}^{\infty} \subseteq \mathbf{C}\backslash\{0\}$, converging to zero and with $\max_i |\alpha_i| \leq \mu$. Let

$$\{f_i\}_{i=1}^{\infty} = \{e_1, e_1, e_3, e_3, \dots, e_{2k-1}, e_{2k-1}, \dots\},$$

and

$$\{g_i\}_{i=1}^{\infty} = \{e_1, e_1+\alpha_1 e_2, e_3, e_3+\alpha_2 e_4, \dots, e_{2k-1}, e_{2k-1}+\alpha_k e_{2k}, \dots\}.$$

Then

$$\left\| \sum_{i=1}^{\infty} c_i (f_i - g_i) \right\| \leq \mu \left(\sum_{i\in I} |c_i|^2 \right)^{1/2}, \quad \forall \{c_i\}_{i\in I} \in l^2(N).$$

Moreover, $\{f_i\}_{i=1}^{\infty}$ is a frame sequence with bounds $A = B = 2$. The condition in Theorem 3.2 is satisfied, but $\{g_i\}_{i=1}^{\infty}$ is not a frame sequence. The example corresponds to the situation $\delta(\mathcal{K}, \mathcal{L}) = 1$ that cannot be handled by Theorem 3.1.

Our final theorem describes a way in which the gap $\delta(\mathcal{K}, \mathcal{L})$ in Theorem 3.1 can be calculated from $\{g_i\}_{i\in I}$ and $\{f_i\}_{i\in I}$. Henceforth we will assume that the index set $I = \mathbf{N}$ or $\{1, \dots, m\}$ for some $m \in \mathbf{N}$. Let us define the function $\nu : \mathcal{H} \to \mathcal{H}$ by

$$\nu(0) := 0 \quad \text{and} \quad \nu(x) := \frac{x}{\|x\|} \quad \text{if} \quad x \neq 0.$$

The next lemma is a simple variation on the Gram-Schmidt orthogonalization process. A proof is therefore omitted.

Lemma 3.3. *Let* $\{h_i\}_{i\in I}$ *be a sequence in* $\mathcal{H}$. *Then the sequence* $\{\tilde{h}_i\}_{i\in I}$ *defined by*

$$\tilde{h}_i := \nu(h_i - Q_{i-1}h_i),$$

where

$$Q_{i-1} = Q_{i-1}(\{h_i\}_{i\in I}) := \sum_{j=1}^{i-1} \langle \cdot, \tilde{h}_j\rangle \tilde{h}_j$$

and $Q_0 := 0$, *is a normalized tight frame for* $\mathcal{M} := [h_i]$ *whose nonzero elements form an orthonormal basis for* $\mathcal{M}$.

Theorem 3.4. *Suppose that* $\{f_i\}_{i\in I}$ *and* $\{g_i\}_{i\in I}$ *are sequences in* $\mathcal{H}$. *Let* $\mathcal{K} := [g_i]$, $\mathcal{L} := [f_i]$, *and for each* $i \in I$ *define*

$$h_i := P_{\mathcal{L}}\tilde{g}_i = \sum_{j\in I} \langle \tilde{g}_i, \tilde{f}_j\rangle \tilde{f}_j.$$

Then

$$\delta(\mathcal{K}, \mathcal{L}) = \sup_{x\in\mathcal{H}, \|x\|=1} \left(\sum_{i\in I} |\langle x, \tilde{g}_i - h_i\rangle|^2 \right)^{1/2}.$$

Proof. We know that $\delta(\mathcal{K}, \mathcal{L}) = \|P_{\mathcal{K}} P_{\mathcal{L}^\perp}\|$. Now fix $x \in \mathcal{H}$. By Lemma 3.3,

$$\begin{aligned} \|P_{\mathcal{K}} P_{\mathcal{L}^\perp} x\|^2 &= \left\| \sum_{i\in I} \langle P_{\mathcal{L}^\perp} x, \tilde{g}_i\rangle \tilde{g}_i \right\|^2 \\ &= \sum_{i\in I} |\langle x, P_{\mathcal{L}^\perp} \tilde{g}_i\rangle|^2 \\ &= \sum_{i\in I} |\langle x, \tilde{g}_i - h_i\rangle|^2. \qquad \square \end{aligned}$$

Frames for Banach spaces were introduced by Gröchenig [**10**]. Extensions of Theorem 3.1 are considered by the authors in a paper in preparation.

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REFERENCES

1. C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. **17** (1958), 151–164.

2. R.P. Boas, Jr., *Expansions of analytic functions*, Trans. Amer. Math. Soc. **48** (1940), 467–487.

3. P.G. Casazza and O. Christensen, *Perturbation of operators and applications to frame theory*, J. Fourier Anal. Appl. **3** (1997), 543–557.

4. O. Christensen, *A Paley-Wiener theorem for frames*, Proc. Amer. Math. Soc. **123** (1995), 2199–2202.

5. ———, *Operators with closed range, pseudo-inverses and perturbation of frames for a subspace*, Canad. Bull. Math. **42** (1999), 37–45.

6. O. Christensen and C. Heil, *Perturbations of Banach frames and atomic decompositions*, Math. Nachr. **185** (1997), 33–47.

7. I. Daubechies, *Ten lectures on wavelets*, CBMS-NSF Regional Conf. Ser. in Appl. Math., SIAM, 1992.

8. R.J. Duffin and A.C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. **72** (1952), 341–366.

9. S.J. Favier and R.A. Zalik, *On the stability of frames and Riesz bases*, Appl. Comp. Harmonic Anal. **2** (1995), 160–173.

10. K. Gröchenig, *Describing functions*: *Atomic decompositions versus frames*, Monatsh. Math. **112** (1991), 1–41.

11. C. Heil, *Wiener amalgam space in generalized harmonic analysis and wavelet theory*, Ph.D. Thesis, University of Maryland, College Park, 1990.

12. C. Heil and D. Walnut, *Continuous and discrete wavelet transforms*, SIAM Review **31** (1989), 628–666.

13. S. Hilding, *Note on completeness theorems of Paley-Wiener type*, Ann. of Math. **49** (1948), 953–955.

14. T. Kato, *Perturbation theory for linear operators*, Springer, New York, 1976.

15. C.P. Lewis, *Frames and frame perturbations in Hilbert spaces*, M.S. Thesis, University of Pittsburgh, 1996.

16. H. Pollard, *Completeness theorems of Paley-Wiener type*, Ann. of Math. **45** (1944), 738–739.

17. R.E.A.C. Paley and N. Wiener, *Fourier transforms in the complex domain*, Amer. Math. Soc. Colloq. Publ. **19** (1934).

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