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UPPER AND LOWER SOLUTIONS FOR A HOMOGENEOUS DIRICHLET PROBLEM WITH NONLINEAR DIFFUSION AND THE PRINCIPLE OF LINEARIZED STABILITY

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ABSTRACT. We consider a class of quasilinear elliptic equations on a bounded domain subject to homogeneous Dirichlet boundary data. We establish a means of constructing upper and lower solutions in a neighborhood of a given solution to the quasilinear boundary value problem, leading to a principle of linearized stability–instability for the solution viewed as an equilibrium to the corresponding parabolic problem.

1. Introduction. The purpose of this note is to demonstrate a means of constructing upper and lower solutions in a neighborhood of a given solution to a homogeneous Dirichlet problem with nonlinear diffusion of the form

$$\nabla \cdot (d(x,u)\nabla u) + f(x,u) = 0 \quad \text{in } \Omega \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a smooth bounded domain in $\mathbf{R}^n$. The construction requires only minimal assumptions on the coefficients d and f, namely, that both are sufficiently smooth and that d is positive. One then formulates the linearization of (1.1) about u and considers the principal eigenvalue σ and a positive principal eigenfunction ϕ of the linearization. If

$$\sigma > 0, \quad z = u + \varepsilon\phi - \varepsilon^2 \frac{d_u(x,u)}{2d(x,u)}\phi^2$$

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gives a lower solution to (1.1)–(1.2) for small positive ε while

$$w = u - \varepsilon\phi - \frac{\varepsilon^2 d_u(x,u)}{2d(x,u)}\phi^2$$

gives an upper solution for small positive ε. (If $\sigma < 0$, the roles of z and w are reversed.) If $d_u(x,u) \equiv 0$, (1.1) reduces to a semi-linear problem and z and w reduce to $u + \varepsilon\phi$ and $u - \varepsilon\phi$, respectively. It is well known that $u \pm \varepsilon\phi$ may be employed as lower and upper (or upper and lower) solutions for (1.1)–(1.2) in the semilinear context. Moreover, the simpler constructions remain valid in the quasilinear setting ($d_u(x,u) \not\equiv 0$) so long as the Dirichlet boundary condition (1.2) is replaced with a Neumann boundary condition, for example. However, there is a fundamental obstacle to using $u \pm \varepsilon\phi$ in the context of (1.1)–(1.2). Namely, the eigenfunction ϕ vanishes on $\partial\Omega$ while $|\nabla\phi|^2$ does not. As a consequence, the lowest order, in ε, term in $\nabla \cdot (d(x,z)\nabla z) + f(x,z)$, namely $\varepsilon\sigma\phi$, does not overpower the higher order, in ε, term $\varepsilon^2 d_u(x,u)|\nabla\phi|^2$ on all of Ω. The more complicated construction enables one to circumvent this obstacle, as we demonstrate in the next section.

The problem of constructing lower or upper solutions for (1.1)–(1.2) arises in a number of contexts. In our case, we encountered a need for lower solutions above a positive equilibrium solution to

$$\begin{aligned} &\frac{\partial u}{\partial t} = \nabla \cdot (d(x,u)\nabla u) + \lambda(m(x) - u)u \quad \text{in } \Omega \times (0,\infty) \\ &u = 0 \quad \text{on } \partial\Omega \times (0,\infty) \end{aligned} \tag{1.3}$$

while studying persistence phenomena in ecological models that incorporate aggregation into the dispersal mechanism of the population ([**6**]). See [**7**] also for background. It seemed to us the construction of upper and lower solutions to (1.1)–(1.2) would be of interest to the mathematical community but of lesser interest per se to the mathematical biology community. Consequently, we have presented the construction in some generality in this note. We employ the results of this note in our analysis of (1.3) in [**6**].

The remainder of this article is as follows. As noted previously, we demonstrate that z and w as defined above are lower and upper (or upper and lower) solutions for (1.1)–(1.2) in Section 2. We then observe in Section 3 the principles of linearized stability–instability for the corresponding parabolic problems which arise naturally as consequences of the construction. These versions of the principle of linearized stability for quasilinear parabolic problems are less general than those obtained in [**8**] or [**10**], for example. However, arising as they do from the method of upper and lower solutions, the versions in Section 3 provide more quantitative information regarding the basin of attraction for the given solution to (1.1)–(1.2) than would be feasible in a more general context.

2. Upper and lower solutions.

We may now establish our main result.

Theorem 2.1. *Suppose that* $d, f \in C^3(\overline{\Omega} \times \mathbf{R})$ *and that* $d(x,s) > d_1 > 0$ *for all* $x \in \overline{\Omega}$ *and* $s \in \mathbf{R}$, *where* Ω *is a smooth bounded domain in* $\mathbf{R}^n$. *Suppose that*

$$\begin{aligned} \nabla \cdot (d(x,u)\nabla u) + f(x,u) &= 0 \quad \textit{in } \Omega \\ u &= 0 \quad \textit{on } \partial\Omega \end{aligned} \tag{2.1}$$

and that σ *is the principal eigenvalue of the linearization of* (2.1) *about* u, *i.e.,* ϕ *exists so that*

$$\begin{aligned} \nabla \cdot [d(x,u)\nabla\phi + d_u(x,u)\phi\nabla u] + f_u(x,u)\phi &= \sigma\phi \quad \text{in } \Omega \\ \phi &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.2}$$

with $\phi > 0$ *in* Ω.

(i) *If* $\sigma > 0$, *then* $z = u + \varepsilon\phi - \varepsilon^2(d_u(x,u)/(2d(x,u)))\phi^2$ *is a lower solution to* (2.1) *for all sufficiently small* $\varepsilon > 0$ *and* $w = u - \varepsilon\phi - \varepsilon^2(d_u(x,u)/(2d(x,u)))\phi^2$ *is an upper solution to* (2.1) *for all sufficiently small* $\varepsilon > 0$.

(ii) *If* $\sigma < 0$ *and* z *and* w *are given in* (i), z *is an upper solution to* (2.1) *and* w *is a lower solution to* (2.1) *for all sufficiently small* $\varepsilon > 0$.

Proof. We shall demonstrate that z is a lower solution to (2.1) when $\sigma > 0$. The remaining three arguments proceed analogously and are left to the reader. Expanding d and f via Taylor's Theorem, we have that

$$\begin{aligned}
&\nabla \cdot d(x,z)\nabla z + f(x,z)\\
&\quad = \nabla \cdot \bigg[\bigg(d(x,u) + d_u(x,u)\bigg[\varepsilon\phi - \varepsilon^2 \frac{d_u(x,u)}{2d(x,u)}\phi^2\bigg]\\
&\qquad\qquad + d^*(x,z)\varepsilon^2\phi^2\bigg) \cdot \nabla\bigg(u + \varepsilon\phi - \varepsilon^2 \frac{d_u(x,u)}{2d(x,u)}\phi^2\bigg)\bigg]\\
&\qquad + f(x,u) + f_u(x,u)\bigg(\varepsilon\phi - \varepsilon^2 \frac{d_u(x,u)}{2d(x,u)}\phi^2\bigg) + q^*(x,z)\varepsilon^2\phi^2,
\end{aligned}$$

where $d^*(x,z)$ and $q^*(x,z)$ are smooth and bounded.

Rearranging terms, we have that

(2.3)

$$\begin{aligned}
&\nabla \cdot d(x,z)\nabla z + f(x,z)\\
&\quad = \nabla \cdot (d(x,u)\nabla u) + f(x,u)\\
&\qquad + \varepsilon[\nabla \cdot (d(x,u)\nabla\phi + d_u(x,u)\phi\nabla u) + f_u(x,u)\phi]\\
&\qquad + \varepsilon^2\bigg[\nabla \cdot \bigg(d_u(x,u)\phi\nabla\phi - d(x,u)\nabla\bigg(\frac{d_u(x,u)}{2d(x,u)}\phi^2\bigg)\bigg)\bigg]\\
&\qquad + \varepsilon^2\bigg[\nabla \cdot \bigg(\phi^2 d^*(x,z)\nabla u - \frac{d_u^2(x,u)}{2d(x,u)}\phi^2\nabla u\bigg) + q^*(x,z)\phi^2\\
&\qquad\qquad - f_u(x,u)\frac{d_u(x,u)}{2d(x,u)}\phi^2\bigg]\\
&\qquad + \varepsilon^3\bigg[\nabla \cdot \bigg(\frac{-d_u^2(x,u)\phi^2}{2d(x,u)} + d^*(x,z)\phi^2\bigg)\nabla\bigg(\phi - \frac{\varepsilon d_u(x,u)}{2d(x,u)}\phi^2\bigg)\\
&\qquad\qquad - \nabla \cdot d_u(x,u)\phi\nabla\bigg(\frac{d_u(x,u)}{2d(x,u)}\phi^2\bigg)\bigg].
\end{aligned}$$

The first term on the righthand side of (2.3) vanishes since u satisfies (2.1). By (2.2) the second term equals $\varepsilon\sigma\phi$. Since $\sigma > 0$ and the remaining three terms are higher in order in ε, z will be a lower solution provided the remaining terms on the righthand side of (2.3) vanish on $\partial\Omega$. It is evident that such is the case for the fourth and fifth terms since for $\vec{F}(x)$ a smooth vector field on $\overline{\Omega}$, $\nabla \cdot (\phi^2\vec{F}) = 2\phi\nabla\phi \cdot \vec{F} + \phi^2\nabla \cdot \vec{F}$, which vanishes on $\partial\Omega$ by virtue of the fact that ϕ does. Hence, to establish that z is a lower solution to (2.1), we need only show that

$$\left[\nabla \cdot \left(d_u(x,u)\phi\nabla\phi - d(x,u)\nabla\left(\frac{d_u(x,u)}{2d(x,u)}\phi^2\right)\right)\right]$$

vanishes on $\partial\Omega$. But now

(2.4)

$$\begin{aligned}
&\left[\nabla \cdot \left(d_u(x,u)\phi\nabla\phi - d(x,u)\nabla\left(\frac{d_u(x,u)}{2d(x,u)}\phi^2\right)\right)\right] \\
&= \left[\nabla \cdot \left(d_u(x,u)\phi\nabla\phi - d(x,u)\phi^2\nabla\left(\frac{d_u(x,u)}{2d(x,u)}\right) - \frac{d(x,u)d_u(x,u)}{2d(x,u)}\nabla\phi^2\right)\right] \\
&= \left[\nabla \cdot \left(d_u(x,u)\phi\nabla\phi - \phi^2 d(x,u)\nabla\left(\frac{d_u(x,u)}{2d(x,u)}\right) - d_u(x,u)\phi\nabla\phi\right)\right] \\
&= -\nabla \cdot \left(\phi^2 d(x,u)\nabla\left(\frac{d_u(x,u)}{2d(x,u)}\right)\right).
\end{aligned}$$

Consequently,

$$\left[\nabla \cdot \left(d_u(x,u)\phi\nabla\phi - d(x,u)\nabla\left(\frac{d_u(x,u)}{2d(x,u)}\phi^2\right)\right)\right]$$

vanishes on $\partial\Omega$, and z is a lower solution to (2.1) as claimed.

Remark. Theorem 2.1 remains valid when auxiliary parameters are included, e.g., when we consider $\lambda f(x,u)$ in place of $f(x,u)$. Moreover, the smoothness assumptions in the hypotheses are intended to make the proof of the construction as simple as possible. We do not claim that they are sharp.

3. Linearized stability and instability. The results of the preceding section provide a basis for a linearized stability analysis of the parabolic quasilinear problem corresponding to (2.1), i.e.,

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \nabla \cdot (d(x,u)\nabla u) + f(x,u) \quad \text{in } \Omega \times (0,\infty) \\
&u = 0 \quad \text{on } \partial\Omega \times (0,\infty).
\end{aligned} \tag{3.1}$$

Suppose that u_0 is an equilibrium solution to (3.1) and that there is a neighborhood U of u_0 in an appropriate Banach space ($C_0^1(\overline{\Omega})$ e.g.) so that any solution to (3.1) with initial data in the neighborhood is defined for all $t > 0$. Such an assumption is met, for instance, if u_0 is positive in Ω and $f(x,u) = u\tilde{f}(x,u)$ with $(\partial\tilde{f}/\partial u)(x,u) \leq 0$ for $x \in \overline{\Omega}$ and $u \geq 0$ and $\tilde{f}(x,u) < 0$ for $x \in \overline{\Omega}$ and $u \geq k > 0$. (In this case, solutions are also asymptotically bounded. See [**7**], for example.)

Theorem 12 of [**11**, Chapter 3] guarantees that if u_1 and u_2 are solutions to (3.1) with $u_1(x,0)$, $u_2(x,0)$ in U such that $u_1(x,0) \overset{<}{\not\equiv} u_2(x,0)$ on Ω, then $u_1(x,t) < u_2(x,t)$ for all $t > 0$. As in [**5**], if u is a solution to (3.1) with $u(x,0)$ in U a lower solution to (2.1), $u(x,t)$ is monotonically increasing in t for all $x \in \Omega$; likewise, if $u(x,0)$ in U is an upper solution to (2.1), $u(x,t)$ is monotonically decreasing in t for all $x \in \Omega$. If in either case u is asymptotically bounded, then it converges pointwise on $\overline{\Omega}$ as $t \to \infty$. The regularity results of [**1**]–[**3**] guarantee that the pointwise limit is an equilibrium to (3.1), i.e., a solution to (2.1), and moreover that the convergence of $u(x,t)$ to the equilibrium as $t \to \infty$ may be taken in the $C^1(\overline{\Omega})$ topology.

Suppose now that ϕ, z and w are as in the statement of Theorem 2.1, corresponding to $u = u_0$. Since $\nabla\phi \cdot \eta < 0$ on $\partial\Omega$, where η is a unit outer normal to $\partial\Omega$, $z - u$ and $u - w$ are both positive in Ω with $\Delta(z-u)\cdot\eta$ and $\Delta(u-w)\cdot\eta$ negative on $\partial\Omega$ for $\varepsilon > 0$ sufficiently small. (In other words, $z-u$ and $u-w$, lie in $[C_0^1(\overline{\Omega})^+]^0$, the interior of the positive cone $C_0^1(\overline{\Omega})^+$ of $C_0^1(\overline{\Omega})$. See [**4**], for example.) Combining this observation with the information of the preceding paragraph, one readily obtains the following stability–instability principle for positive solutions to (3.1).

Theorem 3.1. *Suppose u_0 is a positive equilibrium solution to* (3.1), *and assume U is a neighborhood of u_0 in $C_0^1(\overline{\Omega})$ so that solutions to* (3.1) *with initial data in U exist for all $t > 0$. Let ϕ and σ be as in* (2.2) *with $u = u_0$.*

(i) *Suppose $\sigma < 0$. Then u_0 is locally asymptotically stable viewed as a solution to* (3.1).

(ii) *Suppose $\sigma > 0$. Let ρ be a solution to* (3.1) *with $\rho(x,0) \in U$ and $\rho(x,0) - u \in (C_0^1(\overline{\Omega}^+)^0$.*

Then for all $t > 0$, $\rho(x,t) > z(x,t)$ *in* Ω, *where* $z(x,t)$ *is a solution to* (3.1) *having* $z(x,0) = z$ *for some* $\varepsilon > 0$ *and sufficiently small. Consequently,* $\rho(x,t)$ *is bounded from below away from* u_0. *In particular, either* $\lim_{t\to\infty} \rho(x,t) = +\infty$ *for some* $x \in \Omega$ *or* $\liminf_{t\to\infty} \rho(x,t) \geq u^*(x)$, *where* u^* *is an equilibrium solution to* (3.1) *with* $u^*(x) > u_0(x)$ *on* Ω. *Similarly, if* ρ *is a solution to* (3.1) *with* $\rho(x,0) \in U$ *and* $u - \rho(x,0) \in (C_0^1(\overline{\Omega})^+)^0$, $\rho(x,t)$ *is bounded from above away from* u_0, *with either* $\lim_{t\to\infty} \rho(x,t) = -\infty$ *for some* $x \in \Omega$ *or* $\limsup_{t\to\infty} \rho(x,t) \leq u_*(x)$, *where* u_* *is an equilibrium solution to* (3.1) *with* $u_*(x) < u_0(x)$ *on* Ω.

Remark. The results of this section can be viewed as an extension of the principle of linearized stability for the semi-linear case (i.e., $(\partial/\partial u)d(x,u) \equiv 0$) as established in [**9**].

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