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THE EQUIVARIANT DIRAC CYCLIC COCYCLE

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ABSTRACT. In this paper we compute the equivariant Chern character associated with the Dirac operator using the cyclic cocycle formula developed by Connes and Moscovici, when a countable discrete group acts properly on a smooth compact spin Riemannian manifold of even dimension. Canonical order calculus which is due to Simon is used to simplify the computations. Finally observing that this equivariant Dirac cyclic cocycle is a well-defined element of the delocalized cohomology, we pair it with an equivariant K-theory idempotent.

0. Introduction. The Chern character theory of K-cycles over an algebra $\mathcal{A}$, developed as an analogue of the classical index theory associated to the elliptic differential operators on smooth compact manifolds, plays a key role in noncommutative geometry.

Connes and Moscovici in their recent paper [**8**] introduced a universal local index formula, based on (generalized) Wodziciki residues, for the Chern character in the cyclic cohomology for finitely summable Fredholm modules. This formula has numerous implications in many areas, such as quantization and index theorem on infinite dimensional manifolds. When using a cyclic cocycle formula, for example the formula in [**14**] by Jaffe, Lesniewski and Osterwalder, one of the complexities we face is the commutation of the operator $[D, f]$ with the heat operators e^{-sD^2}, whereas Connes and Moscovici's formula overcomes this difficulty.

The main result of this paper is the computation of the equivariant Chern character for the entire cyclic cohomology associated with the Dirac operator and the smooth crossed product algebra $C^\infty(M, G)$, where M is a smooth compact spin Riemannian manifold endowed with a G-invariant Riemannian metric, and G is a countable discrete group acting properly on M.

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Over the past few years many proofs have appeared concerning the computation of the equivariant index of the Dirac operator (cf. [**4**], **15**]). In [**15**], Lafferty, Yu and Zhang present a very simple and direct geometric proof for such a computation. They develop a Clifford asymptotics for the heat kernel and then use the fact that the geodesic moving frame is related by an infinitesimal holonomy to the natural trivialization of the normal bundle. Their method is not directly compatible with our computation of the equivariant Dirac cyclic cocycle. We face two problems: the first problem arises from our need to consider the action of the group G on M with each $g \in G$ acting as an isometry on M, whereas they had only one isometry. Secondly, we are computing a cocycle which involves composition of operators $[D, f]$'s and a heat operator e^{-tD^2}, whereas in their case they only deal with a single heat operator composed with an isometry.

To overcome these difficulties, canonical order calculus that was developed by Simon [**10**] and the asymptotic expansion of the heat kernel together with the cyclic cocycle formula of Connes and Moscovici, are used to compute the equivariant Chern character associated with the Dirac operator. The process is as follows:

First, we show that the equivariant cyclic cocycle takes the form of a single isometry composed with $[D, f]$'s and a heat operator e^{-tD^2}. Next, we localize the computation of the cocycle; thus, we only need to consider a small tubular neighborhood of the fixed submanifold. Restricting the computation to this tubular neighborhood and using the relation between normal and orthogonal coordinates and frame as in [**15**], we compute the equivariant Dirac cocycle. In every step of our simplification process, some extra terms appear. Canonical order calculus is used to show that these extra terms have trivial contribution in the computation of the cocycle. The part which has a nontrivial contribution is expressed in terms of topological data related to the manifold and the group action. Finally, we prove that this equivariant Dirac cocycle is a well defined element of the delocalized cohomology that was developed by Baum and Connes [**2**] and compute its pairing with an equivariant K-theory idempotent.

Although we compute the equivariant cyclic cocycle associated with the Dirac operator, our method can be carried out for the computation of the equivariant cocycle associated with the twisted Dirac operator and for any geometric type operators, for example De Rham or Signature operators.

The paper is organized as follows: Section 1 starts with preliminary and some background material, a brief review of spectral triple and the main result in Connes and Moscovici's paper [**8**]; in Section 2 we briefly mention the canonical order calculus, and in Section 3 we mention the normal and orthogonal coordinates and frames and their relation as in [**15**]. Section 4 deals with approximating the heat kernel for D^2 with the heat kernel for the harmonic oscillator, using Duhamel's expansion together with canonical order calculus; most of the computation of the cyclic cocycle is carried out in Section 5; then in Section 6 we reach the main results in this paper which is divided into two parts. Part 1 deals with the computation of the equivariant Dirac cyclic cocycle and in part 2 we review briefly the delocalized cohomology as in [**2**], then compute explicitly the index pairing of this equivariant Dirac cocycle with an idempotent in the equivariant K-theory.

1. Preliminary material. For the convenience of the reader and in the process of making this paper self-contained, we briefly mention the spectral triple and the cyclic cocycle attached to it. For more details one can consult [**8**].

The spectral triple is a triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is an involutive algebra represented in the Hilbert space $\mathcal{H}$, and D is a self-adjoint operator in $\mathcal{H}$ with compact resolvent, which almost commutes with any $a \in \mathcal{A}$, with $[D, a]$ being bounded for any $a \in \mathcal{A}$. The notion of dimension for spectral triples, provided by the degree of summability $D^{-1} \in \mathcal{L}^{(p,\infty)}$ gives only an upper bound on dimension and cannot detect the individual dimensions of the various pieces of a space, which is a union of pieces of different dimensions $(\mathcal{A}_k, \mathcal{H}_k, D_k)$, $k = 1, \dots, N$,

$$\mathcal{A} = \oplus \mathcal{A}_k, \qquad \mathcal{H} = \oplus \mathcal{H}_k, \qquad D = \oplus D_k.$$

Consequently, Connes and Moscovici present a new notion of dimension called the dimension spectrum Sd, where $Sd \subset \mathbf{C}$.

Definition 1.1. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has discrete dimension spectrum Sd if

1. $Sd \subset \mathbf{C}$.

2. The function
$$\zeta_b(z) = \mathrm{Trace}\,(b|D|^{-z})$$
extends holomorphically to $\mathbf{C}-Sd$, where b lies in the algebra generated by $\delta^n(a)$, $a \in \mathcal{A}$; here δ denotes the derivation $\delta(P) = [|D|, P]$ and we assume that $\mathcal{A} \subset \cap_{n>0} \mathrm{Dom}\,\delta^n$.

The operator $b\,|D|^{-z}$ is then of trace class for $\mathrm{Re}\,z > p$. On the technical side they assume that the analytic continuation of ζ_b is such that $\Gamma(z)\zeta_b(z)$ is of rapid decay on the vertical lines $z = s + ir$, for any s with Re $s > 0$.

The dimension spectrum Sd is simple, when the singularities of the function $\zeta_b(z)$ at $z \in Sd$ are at most simple poles.

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with discrete dimension spectrum and $D^{-1} \in \mathcal{L}^{(p,\infty)}$; then with the following notation

$$T_0(a) = [D, a], \qquad T(a) = [D^2, a], \qquad a^{(k)} = T^k(a),$$
$$\forall\, a \quad \text{operators in} \quad \mathcal{H}$$

The cyclic cocycle attached to the spectral triple is given in the following theorem, [**8**, p. 230].

Theorem 1.2. *The following formula defines an even cocycle in the (b, B) bicomplex of $\mathcal{A}$, which is cohomologous to the cyclic cohomology Chern character* $\mathrm{ch}_*(\mathcal{A}, \mathcal{H}, D)$:

$$(1) \quad \varphi_p(a_0, \dots, a_p) = \sum_{k,q} \frac{(-1)^{|k|}}{k_1! \cdots k_p!} \alpha_k \sigma_q(m) \tau_q\big(\gamma a_0 (T_0(a_1))^{(k_1)} \cdots (T_0(a_p))^{(k_p)} |D|^{-(2|k|+p)}\big)$$

for $p \neq 0$ even, while
$$\varphi_0(a_0) = \tau_{-1}(\gamma a_0).$$
with $m = |k| + (p/2)$, $|k| = k_1 + \cdots + k_p$, $\alpha_k^{-1} = (k_1 + 1) \cdots (k_1 + \cdots + k_p + p)$, and $\prod_{l=0}^{m-1}(z + (2l+1)/2) = \sum z^j \sigma_{m-j}(m)$. The $\tau_q(P)$ denotes the residue at $z = 0$ of $z^q \mathrm{Trace}\,(P|D|^{-2z})$.

The above theorem is the main result in Connes and Moscovici's paper. Before using it, we will elaborate on our data.

M is a smooth compact spin Riemannian manifold of dimension $2n$, with a fixed spin $(2n)$ structure. Let $SO(M)$ be the principal $SO(2n)$ bundle over M, with ρ a double covering map $\rho : \mathrm{spin}\,(2n) \to SO(2n)$. Since M is a spin manifold, the coordinate transformations $g_{ij}(x) \in SO(2n)$, can be lifted in a smooth way to $\tilde{g}_{ij}(x) \in \mathrm{spin}\,(2n)$, such that the compatibility condition $\tilde{g}_{ij}(x)\,\tilde{g}_{jk}(x) = \tilde{g}_{ik}(x)$ is satisfied.

Let G be a countable discrete group acting in a smooth proper way on M. And M is endowed with a G-invariant Riemannian metric [**2**]. We also assume the action of G on M preserves the spin structure of M, i.e, the action of G lifts to an action on the principal spin $(2n)$ bundle $\mathrm{Spin}\,(M)$, for $g \in G$ then $g^* : \mathrm{Spin}\,(M) \to \mathrm{Spin}\,(M)$ is induced by the map $g : M \to M$. D is the Dirac operator on the Z_2 graded Hilbert space $\mathcal{H} = L^2(\mathcal{E}) = L^2(\mathcal{E}^+) \oplus L^2(\mathcal{E}^-)$, where $\mathcal{E}^\pm$ is the Z_2 graded spinor bundle over M, i.e, $\mathcal{E}^\pm = \mathrm{Spin}\,(M) \times_{\mathrm{spin}\,(2n)} \mathcal{S}^\pm$, with $\mathcal{S}^\pm$ being the Z_2 graded spinor space.

Since the action of G preserves the spin structure, it acts on the spinor bundle $\mathcal{E}$ and commutes with the Dirac operator D.

Definition 1.3. The smooth crossed product algebra $C^\infty(M;G)$ is defined by:

1. As a complex vector space, $C^\infty(M;G)$ is identified with $C^\infty(M \times G)$. Thus, any $f \in C^\infty(M;G)$ can be written uniquely as a finite formal sum $f = \sum_{g\in G} f_g\,[g]$, where $f_g \in C^\infty(M)$ and $g \in G$.

2. Addition and multiplication are given by

$$f_1 + f_2 = \sum_{g\in G} (f_g^1 + f_g^2)\,[g]$$

where $f_1 = \sum_g f_g^1\,[g]$ and $f_2 = \sum_g f_g^2\,[g]$, and the multiplication is given by

$$(f_g\, g)(f_h\, h) = f_g(g.f_h)\,[gh]$$

where $g, h \in G$ and $g.f_h$ denotes the action of g on the smooth function f_h which is given by $(g.f_h)(x) = f_h(g^{-1}x)$.

3. The differential is defined by $df = \sum_g df_g\,[g]$, where df_g is the differential of the smooth function f_g on M.

To simplify the notation, from now on we will denote $[g]$ by g, thus $f \in C^{\infty}(M;G)$ will be denoted by $f = \sum_{g\in G} f_g\, g$.

Let $\mathcal{A}$ be the above algebra, under the norm $\|f\| = \|f\|_{\infty} + \|[D,f]\|_{\infty}$, where $\|.\|_{\infty}$ is the operator norm on $\mathcal{H}$, and there is a representation of $\mathcal{A}$ in $\mathcal{H}$. Then $(\mathcal{A},\mathcal{H},D)$ is a spectral triple, the dimension spectrum is simple and contained in the set

$$\{r \in Z \text{ such that } r \leq 2n\}.$$

Since we have simple poles, thus we only need to consider τ_0 when computing the cocycle.

Remarks 1.4. Although we don't assume the Dirac operator D to be invertible, in order to use formula (1) we replace D by an invertible operator $D_\alpha = D\widehat{\otimes}I + \alpha I\widehat{\otimes}F_C$, for $0 \neq \alpha \in \mathbf{R}$, and $F_C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (cf. [**9**]). Then $(\mathcal{A}, \mathcal{H}\widehat{\otimes}H_C, D_\alpha)$ is a spectral triple with simple spectrum dimension, here H_C is the Z_2 graded Hilbert space $H_C^{\pm} = \mathbf{C}$. Thus, the local formula for the cocycle takes the form

$$(2) \quad \sum_k \frac{(-1)^{|k|}}{k_1!\cdots k_{2r}!}\alpha_k \sigma_0(|k|+r)\tau_0\big(\gamma \tilde{f}_0 (T_0(\tilde{f}_1))^{(k_1)} \cdots (T_0(\tilde{f}_{2r}))^{(k_{2r})}|D_\alpha|^{-(2|k|+2r)}\big)$$

where the f_i's are in $\mathcal{A}$ and $\tilde{f} = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}$.

Using the Mellin transform we relate $|D_\alpha|^{-s}$ to the heat operator $e^{-tD_\alpha{}^2}$ by

$$(3) \quad |D_\alpha|^{-s} = ({D_\alpha}^2)^{-s/2} = \frac{1}{\Gamma(s/2)}\int_0^{\infty} t^{s/2-1} e^{-t(D^2\widehat{\otimes}I)} e^{-t(\alpha^2\widehat{\otimes}I)}dt.$$

For fixed $\xi \in M$ and in terms of normal coordinates at ξ, the heat operator e^{-tD^2} has an asymptotic expansion

$$(4) \qquad e^{-tD^2}(\xi,x) = \frac{e^{-\rho^2(\xi,x)/4t}}{(4\pi t)^n}\sum_{i=0}^{N} t^i U_i(\xi,x) + O(t^{\epsilon}).$$

Where $N > n$, $\rho(\xi, x)$ is the Riemannian distance between ξ and x, and the $U_i(\xi, x) : \mathcal{E}_\xi \longrightarrow \mathcal{E}_x$ is a linear transformation satisfying some properties.

Given an initial value $U_0(\xi, \xi) = id$, there exists a unique formal solution to the heat equation $(\partial/\partial t) + D^2$, with D^2 in its local expression. Therefore, applying the operators $(\partial/\partial t)$ and D^2 to the asymptotic expansion in (4) and then equating the coefficients of powers of t, we conclude that the number of Clifford variables in each $U_j(\xi, x)$ and $U_j(x, y)$ (for x, y in a small neighborhood of ξ) is *at most* $2j$ (cf. [**17**]).

Proposition 1.5.

$$\tau_0\big(\gamma \tilde{f}_0 (T_0(\tilde{f}_1))^{(k_1)} \cdots (T_0(\tilde{f}_{2r}))^{(k_{2r})} |D_\alpha|^{-(2|k|+2r)}\big) = 0.$$

for $|k| = k_1 + \cdots + k_{2r} \neq 0$.

Proof. It is enough to prove this for $k_1 \neq 0$ and $k_j = 0$ for all $j > 1$. Using the definition of τ_0, we want to show

$$\lim_{z \to 0} z \operatorname{Tr}\left(\gamma \tilde{f}_0 (T_0(\tilde{f}_1))^{(k_1)} [D_\alpha, \tilde{f}_2] \cdots [D_\alpha, \tilde{f}_{2r}] |D_\alpha|^{-(2k_1+2r+2z)}\right) = 0.$$

Let Tr_s denote the supertrace, i.e., $\operatorname{Tr}_s(A) = \operatorname{Tr}(\gamma A)$. Thus applying the Mellin transform we get:

$$\begin{aligned} (5)\quad & \operatorname{Tr}_s\big(\tilde{f}_0 (T_0(\tilde{f}_1))^{(k_1)} [D_\alpha, \tilde{f}_2] \cdots [D_\alpha, \tilde{f}_{2r}] |D_\alpha|^{-(2k_1+2r+2z)}\big) \\ &= \frac{1}{\Gamma(r+k_1+z)} \int_0^\infty t^{r+k_1+z-1} \operatorname{Tr}_s\big(\tilde{f}_0 (T_0(\tilde{f}_1))^{(k_1)} \\ &\qquad \cdots [D_\alpha, \tilde{f}_{2r}] e^{-tD_\alpha{}^2}\big)\, dt \end{aligned}$$

Split the above integral into $\int_0^1 + \int_1^\infty$. The second integral is an analytic function of z; hence, it will vanish when computing the residue at $z = 0$.

In the first integral replace $e^{-tD_\alpha^2}$ by its asymptotic expansion (4). To have a nonzero supertrace we need at least $2n$ Clifford variables, each U_j has at most $2j$ Clifford variables and each $[D_\alpha, \tilde{f}_i]$ and $(T_0(\tilde{f}_1))^{(k_1)}$

has at most one. Therefore we need $2r+2j \geq 2n$, i.e., $r+j-n = \beta_j \geq 0$. Consequently,

$$\begin{aligned} &\left|\mathrm{Tr}_s\big(\tilde{f}_0(T_0(\tilde{f}_1))^{(k_1)}[D_\alpha, \tilde{f}_2]\cdots[D_\alpha, \tilde{f}_{2r}]|D_\alpha|^{-(2k_1+2r+2z)}\big)\right| \\ &\qquad \leq \sum_{j=n-r}^{N} \left| \int_0^1 \frac{t^{k_1+z+\beta_j-1}}{\Gamma(r+k_1+z)}\, A\, dt \right| \\ &\qquad = \sum_{j=n-r}^{N} \left| \frac{1}{\Gamma(r+k_1+z)} \left(\frac{A}{k_1+z+\beta_j} \right) \right| \end{aligned} \tag{6}$$

where $A = \int_M \mathrm{tr}_s(\tilde{f}_0(T_0(\tilde{f}_1))^{(k_1)} \cdots [D_\alpha, \tilde{f}_{2r}]U_j)(\xi,\xi)d\xi$ (here $d\xi$ is the volume form on M and tr_s is the pointwise supertrace). Because $k_1 + \beta_j > 0$, formula (6) is an analytic function of z, which remains bounded as z tends to zero. Hence the residue at $z = 0$ vanishes. □

By Proposition 1.5, the cocycle formula (1) reduces to

$$\varphi_{2r}(f_0, \dots, f_{2r}) = \sigma_0(r)\tau_0\big(\gamma\tilde{f}_0[D_\alpha, \tilde{f}_1]\cdots[D_\alpha, \tilde{f}_{2r}]|D_\alpha|^{-2r}\big) \tag{7}$$

where τ_0 is the residue at $z = 0$, i.e.,

$$\begin{aligned} &\tau_0\big(\gamma\tilde{f}_0\cdots[D_\alpha, \tilde{f}_{2r}]|D_\alpha|^{-2r}\big) \\ &\qquad = \lim_{z\to 0} z\, \mathrm{Tr}_s\big(\tilde{f}_0[D_\alpha, \tilde{f}_1]\cdots[D_\alpha, \tilde{f}_{2r}]|D_\alpha|^{-(2r+2z)}\big) \end{aligned}$$

which will be denoted by

$$\mathrm{Re}\, s_{z=0}\, \mathrm{Tr}_s\big(\tilde{f}_0[D_\alpha, \tilde{f}_1]\cdots[D_\alpha, \tilde{f}_{2r}]|D_\alpha|^{-(2r+2z)}\big).$$

By the Mellin transform we get:

$$\begin{aligned} &\mathrm{Tr}_s\big(\tilde{f}_0[D_\alpha, \tilde{f}_1]\cdots[D_\alpha, \tilde{f}_{2r}]|D_\alpha|^{-(2r+2z)}\big) \\ &\qquad = \frac{1}{\Gamma(r+z)} \int_0^\infty t^{r+z-1}\, \mathrm{Tr}_s\big(\tilde{f}_0[D_\alpha, \tilde{f}_1]\cdots[D_\alpha, \tilde{f}_{2r}]e^{-tD_\alpha{}^2}\big)\, dt. \end{aligned} \tag{8}$$

Split the above integral into $\int_0^1 + \int_1^\infty$. As before the second integral vanishes and the first integral in (8) becomes:

$$\frac{1}{\Gamma(r+z)} \int_0^1 t^{r+z-1}\, \mathrm{Tr}_s \left[\begin{pmatrix} f_0[D,f_1]\cdots[D,f_{2r}] & 0 \\ 0 & 0 \end{pmatrix} e^{-t(D^2 \widehat{\otimes} I)} e^{-t(\alpha^2 \widehat{\otimes} I)} \right] dt \tag{9}$$

$$+ \frac{1}{\Gamma(r+z)} \int_0^1 t^{r+z-1}\, \mathrm{Tr}_s\big(E(\alpha)\, e^{-t(D^2 \widehat{\otimes} I)}\, e^{-t(\alpha^2 \widehat{\otimes} I)}\big)\, dt \tag{10}$$

where $E(\alpha)$ is a sum of 2×2 matrices with the parameter α showing up in every nonzero entry. Since D_α was defined for any nonzero $\alpha \in \mathbf{R}$, the above formula holds for any such α. Taking the limit as α tends to zero, formula (10) vanishes, whereas formula (9) is identified with

$$\frac{1}{\Gamma(r+z)} \int_0^1 t^{r+z-1}\, \mathrm{Tr}_s\big(f_0[D,f_1]\cdots[D,f_{2r}]\, e^{-tD^2}\big)\, dt.$$

Our goal in this paper is to find an explicit expression of

$$t^r\, \mathrm{Tr}_s\big(f_0[D,f_1]\cdots[D,f_{2r}]\, e^{-tD^2}\big) \tag{11}$$

in terms of topological data related to the manifold and the group action.

Express each $f_i \in \mathcal{A}$ in formula (11) as $f_i = \sum_{g_i \in G} f_{g_i}\, g_i$, where $f_{g_i} \in C^\infty(M)$, then from the properties of the crossed product algebra $[D, f\, g] = g\,[D,(g.f)]$ (here $g.f$ denotes the action of g on f) we get the following theorem.

Theorem 1.6.

$$\begin{aligned} &\sum_{g_0,\cdots,g_{2r}} t^r\, \mathrm{Tr}_s\big(f_{g_0} g_0 [D, f_{g_1} g_1]\cdots[D, f_{g_{2r}} g_{2r}]\, e^{-tD^2}\big) \\ &= \sum_{g_0,\cdots,g_{2r}} t^r\, \mathrm{Tr}_s\big(g_0\cdots g_{2r}\, ((g_0\cdots g_{2r}).f_{g_0})[D,((g_1\cdots g_{2k}).f_{g_1})] \\ &\qquad \cdots [D,(g_{2r}.f_{g_{2r}})]\, e^{-tD^2}\big) \end{aligned} \tag{12}$$

where the f_{g_i}'s are smooth functions on M, g_i's in G and

$$(g_i g_{i+1} \cdots g_{2k}).f_{g_i}(x) = f_{g_i}((g_i \cdots g_{2k})^{-1}x).$$

We simplify the notation by letting

$$\check{f}_j = (g_j g_{j+1} \cdots g_{2k}).f_{g_j} \quad \text{for } j = 0, \dots, 2k. \tag{13}$$

Then each $\check{f}_j$ is a smooth function on M, and $[D, \check{f}_i] = c(d\check{f}_i)$ is Clifford multiplication by $d\check{f}_i$. Let $g = g_0 g_1 \cdots g_{2r} \in G$. Consequently, with this new notation formula (12) takes the form

$$\sum_{g\in G} \sum_{g=g_0\cdots g_{2r}} t^r \operatorname{Tr}_s\big(g\, \check{f}_0 c(d\check{f}_1) \cdots c(d\check{f}_{2r}) e^{-tD^2}\big) = \sum_{g\in G} \sum_{g=g_0\cdots g_{2r}} \int_M K_t^r(g,x)\, d\operatorname{vol}(x)$$

where $K_t^r(g,x)$ is the kernel

(14)

$$K_t^r(g,x) = t^r \operatorname{tr}_s\big(g^*(x)\, \check{f}_0(gx) c(d\check{f}_1)(gx) \cdots c(d\check{f}_{2r})(gx) e^{-tD^2}(gx,x)\big).$$

Here $g^*(x) : \mathcal{E}_{gx} \to \mathcal{E}_x$ is a smooth linear transformation.

The localization process. The group G acts in a proper smooth way on M, which is endowed with a G-invariant Riemannian metric. Thus, each $g \in G$ is an isometry of M. Its fixed point set $F_g = \{x \in M \mid gx = x\}$ is a submanifold of M which may consist of several connected components of different dimensions, i.e., $F_g = \cup_{i=1}^p (F_g)_i$, where each $(F_g)_i$ is a connected component of F_g, which are totally geodesic closed submanifolds of M of even dimension (cf. [**1**]). Without loss of generality, we assume that F_g is a connected closed submanifold of dimension $2m$, and for simplicity we assume F_g is oriented. Let TF_g and $\nu(F_g)$ be the tangent and normal bundles along F_g. Using the Riemannian metric on M, we have the orthogonal decomposition

$$TM|_{F_g} \cong TF_g \oplus \nu(F_g). \tag{15}$$

The next proposition shows that we only need to consider a small tubular neighborhood $\tilde{U}$ of the fixed submanifold F_g in M.

Proposition 1.7.

$$\lim_{z\to 0} \frac{z}{\Gamma(z+r)} \int_0^1 t^{z-1} \left(\int_{M-\tilde{U}} K_t^r(g,x)\, d\,\mathrm{vol}\,(x) \right) dt = 0.$$

Proof. Using the asymptotic expansion of the heat kernel, we get

$$\begin{aligned} &\int_{M-\tilde{U}} K_t^r(g,x)\, d\,\mathrm{vol}\,(x) \\ &\quad = \int_{M-\tilde{U}} \sum_{j=0}^{N} t^{r+j} \frac{e^{-\rho^2(gx,x)/4t}}{(4\pi t)^n} \operatorname{tr}_s\big(g^*(x)\check{f}_0(gx) \\ &\qquad\qquad \cdots c(d\check{f}_{2r})(gx)U_j(gx,x)\big) d\,\mathrm{vol}\,(x). \end{aligned}$$

On $M-\tilde{U}$, $\rho^2(gx,x)\geq\varepsilon$ for some $\varepsilon>0$, and as t approaches zero, $(1/(4\pi t)^n)e^{-\varepsilon/4t}$ remains bounded by some constant C independent of t. Consequently,

$$\begin{aligned} &\Bigg| \sum_{j=0}^{N} \int_0^1 \frac{t^{r+z+j-1}}{\Gamma(r+z)} \int_{M-\tilde{U}} \frac{e^{-\rho^2(gx,x)/4t}}{(4\pi t)^n} \\ &\qquad \cdot \operatorname{tr}_s\big(g^*(x)\cdots c(d\check{f}_{2r})(gx)U_j(gx,x)\big) d\,\mathrm{vol}\,(x)\, dt \Bigg| \\ &\qquad \leq \sum_{j=0}^{N} \left| \int_0^1 C \frac{t^{r+z+j-1}}{\Gamma(r+z)} (A_j)\, dt \right| \\ &\qquad = \sum_{j=0}^{N} \left| \frac{C}{\Gamma(r+z)} \left(\frac{A_j}{r+z+j} \right) \right| \end{aligned}$$

where A_j denotes $\int_{M-\tilde{U}} \operatorname{tr}_s(g^*(x)\check{f}_0(gx)\cdots c(d\check{f}_{2r})(gx)U_j(gx,x))d\mathrm{vol}(x)$. Since $r+j>0$, the above formula is an analytic function of z. Hence the residue at $z=0$ vanishes. □

2. Canonical order calculus. For the purpose of making this paper self-contained we define this calculus and mention some of its properties briefly without any proofs. For details and proofs, consult [**10**, Chapter 12]; see also [**1**].

Definition 2.1. A family of operators $\{A_t\}_{t>0}$ on $L^2(\mathbf{R}^{2n}, dx)$, or on $L^2(N, dx)$ where N is a compact $2n$-dimensional Riemannian manifold, is said to have canonical order $p \in \mathbf{R}$ if and only if:

1. For each $t > 0$, A_t maps $\mathbf{H}_{-\infty}$ to $\mathbf{H}_{\infty}$, here $\mathbf{H}_{-\infty} = \cup_s \mathbf{H}_s$, $\mathbf{H}_{\infty} = \cap_s \mathbf{H}_s$, where $\mathbf{H}_s$ is the s-Sobolev space.

2. For any $q \geq l \in \{0, \pm 1, \pm 2, \dots\}$, there is a constant c so that, for $0 < t < 1$,

$$||A_t\, u||_k \leq c\, t^{-a}\, ||u||_l, \quad \text{with} \quad a = \frac{q-l}{2} - p.$$

Remarks 2.2. 1. If the operator A_t has canonical order r, then it has canonical order p for all $p \leq r$.

2. If $A_t = B_t^1 + \cdots + B_t^r$, where each B_t^i is an operator of canonical order p_i, then A_t has canonical order p, where $p = \min_{1 \leq j \leq r}(p_j)$.

Proposition 2.3. *Let* Δ *be the Laplacian on* $\mathbf{R}^{2n}$ *or on* $\wedge(N)$, *then* (*assuming all the operators make sense*)

1. $e^{-t\Delta}$ *has canonical order zero.*

2. *The operators* $(\partial^p/\partial x_i^p)e^{-t\Delta}$ *and* $e^{-t\Delta}(\partial^p/\partial y_i^p)$ *have canonical order* $-p/2$, *where* p *is some nonnegative integer.*

Proposition 2.4. *For* x, y *in a small neighborhood* W *of the origin in* $\mathbf{R}^{2n}$ *with coordinates* x_i*'s and* y_i*'s, respectively, then*

1. *The operator* $(x_i - y_i)^p\, e^{-t\Delta}$ *has canonical order* $p/2$, *where* p *is some nonnegative integer.*

2. *The operator* $(x_i - y_i)^p\, (\partial^q/\partial x_j^q)\, e^{-s_k\Delta}$ *has canonical order* $p/2 - q/2$.

The following theorem will be used frequently as it deals with composition of operators.

Theorem 2.5. *Let $A_t^0, A_t^1, \cdots, A_t^p$ be operators of canonical orders $m_0, m_1, \ldots, m_p$ respectively. Let*

$$B_t = \int_{\sum_{i=1}^p s_i \le t} A_{s_0}^0 A_{s_1}^1 \cdots A_{s_p}^p ds_1 ds_2 \cdots ds_p$$

Then the above integral is a convergent integral, where $s_0 = t - s_1 \cdots - s_p$ and B_t is an operator of canonical order $= p + \sum_{j=0}^p m_j$.

All our computations will be carried out in a small neighborhood about some point in M, in terms of normal or orthogonal coordinates at that point. This neighborhood can be identified via the exponential map with a small neighborhood of the origin in $\mathbf{R}^{2n}$. Therefore the difference between the heat kernels on these two neighborhoods vanishes as t tends to zero (cf. [**10**, p. 283]).

3. Normal and orthogonal coordinates and frame. Recall that F_g is the fixed submanifold of M of dimension $2m$ and $\nu(\varepsilon)$ is an ε neighborhood of the zero section of the normal bundle $\nu(F_g)$ along F_g, i.e., $\nu(\varepsilon) = \{x \in \nu(F_g) \mid \|x\| < \varepsilon\}$. Let $\pi : \nu(\varepsilon) \to F_g$ be the projection map, denote by $\nu_\xi(\varepsilon)$ the fiber $\pi^{-1}(\xi)$.

3.1 The orthogonal coordinates [**15**]. Let $\xi \in F_g$. Then an oriented orthonormal frame field $E = (E_1, \ldots, E_{2n})$ exists defined in the neighborhood of ξ such that

1. for $\eta \in F_g$, $E_1(\eta), \ldots, E_{2m}(\eta)$ are tangent to F_g at η, and $E_{2m+1}(\eta), \ldots, E_{2n}(\eta)$ are normal to F_g at η.

2. The frame E is *parallel* along geodesics *tangent* to F_g, and along geodesics *normal* to F_g.

3. The map $dg : SO(M)_x \to SO(M)_{gx}$ is expressed as a matrix-valued function $\mathcal{T}$ by

$$dg(E_1(x), \ldots, E_{2n}(x)) = (E_1(gx), \ldots, E_{2n}(gx))\mathcal{T}(x).$$

We will denote this by $dgE(x) = E(gx)\mathcal{T}(x)$ such that, at the fixed point ξ, the matrix takes the form,

$$\mathcal{T}(\xi) = e^{\Psi(\xi)} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & e^{\Theta(\xi)} \end{pmatrix}$$

where $\Theta(\xi) \in \mathrm{so}\,(2n-2m)$, and

$$e^{\Theta(\xi)} = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & \\ -sin\theta_1 & \cos\theta_1 & & & \\ & & \ddots & & \\ & & & \cos\theta_{n-m} & \sin\theta_{n-m} \\ & & & -\sin\theta_{n-m} & \cos\theta_{n-m} \end{pmatrix}$$

where $0 < \theta_i < 2\pi$, for $i = 1, \dots n-m$.

4. The orientation of E is that of M.

There is a neighborhood V of ξ in F_g such that E is defined on $U = \exp(\nu|_V \cap \nu(\varepsilon))$ for sufficiently small ε. Let $B_0(\varepsilon)$ be a ball of radius ε in $\mathbf{R}^{2n-2m}$; then we have homeomorphism $\phi : V \times B_0(\varepsilon) \to U$ given by

$$\phi(\eta, c) = \exp_\eta\left(\sum_{\alpha=1}^{2n-2m} c_\alpha(x) E_{2m+\alpha}(\eta)\right) = x. \tag{16}$$

This map gives the orthogonal coordinate c_α's of x in the normal direction at any fixed point η in F_g with respect to the frame E.

Lemma 3.1. *Under the above notation and homeomorphism, the action of g is constant along fibers of $\nu(F_g)$, i.e., $\mathcal{T}(x) = \mathcal{T}(\eta)$, where $x = (\eta, c)$, and the isometry g assumes the form $g(\eta, c) = (\eta, c\, e^{-\Theta(\eta)}) = (\eta, \bar{c})$.*

Let $E^{gx} = (E_1^{gx}, \dots, E_{2n}^{gx})$ be an oriented normal frame field defined over U by requiring that $E_i^{gx}(gx) = E_i(gx)$ for all i, and that E^{gx} be parallel along geodesics through gx. Therefore at any point on U the two frames differ by some rotation, i.e., there is a map $\Phi : U \to \mathrm{so}\,(2n)$ such that

$$E^{gx}(x) = E(x)e^{\Phi(x)}. \tag{17}$$

The normal frame E^{gx} will provide us with normal coordinates y_i's of $z \in U$ at gx by

$$\exp_{gx}\left(\sum_{i=1}^{2n} y_i E_i^{gx}(gx)\right) = z.$$

The relation between normal and orthogonal coordinates and frames plays one of the key roles in the computation of the cocycle.

Lemma 3.2. *For $x = (\xi, c)$ then,*

$$\begin{aligned} \frac{\partial}{\partial y_i} &= E_i^{gx} + 0(|y|) && \text{for } i = 1, \dots, 2n, \\ \frac{\partial}{\partial c_\alpha} &= E_{2m+\alpha} + 0(|c|) && \text{for } \alpha = 1, \dots, 2n-2m \\ y_i &= 0(|c|) && \text{for } i = 1, \dots, 2m, \end{aligned}$$

and

$$y_{2m+\alpha} = c_\alpha - \bar{c}_\alpha + 0(|c|) \quad \text{for } \alpha = 1, \dots, 2n-2m.$$

For the proof consult [**15**] and [**1**]. □

Remarks 3.3. For any $\xi \in F_g$, then $g^*(\xi) \in \mathrm{End}\,(\mathcal{E}_\xi)$. Because $g^*(\xi)$ commutes with any section $s \in \Gamma(F_g, TM|F_g)$, this implies that $g^*(\xi) \in \mathrm{Cl}\,(\nu_\xi(F_g))$; hence, it has at most $2n - 2m = \dim(\nu_\xi(F_g))$ Clifford variables [**3**, Chapter 6]. Similarly, for any $x \in U$ with $x = (\xi, c)$ then the number of Clifford variables in $g^*(x)$ is at most $2n - 2m$.

Consider an operator $k_t = g\,Pe^{-tD^2}$, where g is an isometry and P is some bounded operator which contributes to the canonical order. The kernel of k_t has the form $k_t(g,x) = g^*(x)\,P(gx)\,e^{-tD^2}(gx,x)$. The next proposition is one of the key results which we will refer to frequently, and in those situations the operator P will be given explicitly.

Proposition 3.4. *Let $k_t(g,x)$ be the kernel of the operator k_t as above. If the canonical order of terms with at least $2n$ Clifford variables* *in the asymptotic expansion is greater than* $m = (\dim F_g)/2$*, then*

$$\mathrm{Res}_{z=0} \int_0^1 t^{z-1} \left(\int_M \mathrm{tr}_s(k_t(g,x))\, d\,\mathrm{vol}\,(x) \right) dt = 0.$$

Proof. From the localization process it is enough to consider a small neighborhood $\tilde{U}$ of the fixed submanifold F_g. Thus

$$\begin{aligned} &\int_M \mathrm{tr}_s(k_t(g,x))\, d\,\mathrm{vol}\,(x) \\ &\qquad \sim \int_{\tilde{U}} \mathrm{tr}_s(k_t(g,x))\, d\,\mathrm{vol}\,(x) + 0(t^\beta) \\ &\qquad \sim \int_{F_g} \left(\int_{\nu_\xi(\varepsilon)} \mathrm{tr}_s(k_t(g, \exp_\xi c)\, \psi(c))\, dc \right) d\xi + 0(t^\beta) \end{aligned}$$

with $x = (\xi, c)$, i.e., $\exp_\xi c = x$, $d\xi$ is the volume form on F_g, $dc_1 \cdots dc_{2n-2m}$ is the Lebesgue measure on the fiber $\nu_\xi(\varepsilon)$ and ψ is a smooth function. The integral

$$I_t(g,\xi) = \int_{\nu_\xi(\varepsilon)} k_t(g, \exp_\xi c)\, \psi(c)\, dc$$

has an asymptotic expansion of the form [**3**, Chapter 6]

$$I_t(g,\xi) \sim (4\pi t)^{-m} \sum_{j=0}^{\tilde{N}} t^{\,j+q} \Upsilon_j(\xi) + o(t^\varepsilon)$$

where $\varepsilon > 0$, $\tilde{N} > m$ and $\Upsilon_j(\xi) \in \mathrm{End}\,(\mathcal{E}_\xi)$ has at most $2j + (2n - 2m)$ Clifford variables. The operator P contributes t^q power of t, for some q. To have a nonzero supertrace we need at least $2n$ Clifford variables, i.e., $2j + (2n - 2m) \geq 2n$, from the hypothesis we know that such a term has canonical order greater than m and this implies that $j + q - m > m$. Hence

$$\begin{aligned} \int_{F_g} \mathrm{tr}_s(I_t(g,\xi))\, d\xi &= \sum_{j=m}^{\tilde{N}} \int_{F_g} (4\pi t)^{-m}\, t^{\,j+q} \mathrm{tr}_s(\Upsilon_j(\xi))\, d\xi \\ &= \sum_{j=m}^{\tilde{N}} t^{\,j+q-m} A_j \end{aligned}$$

where $A_j = \int_{F_g} (4\pi)^{-m} \operatorname{tr}_s(\Upsilon_j(\xi))\, d\xi$. Consequently,

$$\begin{aligned}\left| \int_0^1 t^{z-1} \left(\int_M \operatorname{tr}_s(k_t(g,x))\, d\operatorname{vol}(x) \right) dt \right| \\ \leq \sum_{j=m}^{N} \left| \int_0^1 t^{z+j+q-m-1} A_j\, dt \right| \\ = \sum_{j=m}^{N} \left| \frac{A_j}{z+j+q-m} \right|\end{aligned}$$

This is an analytic function, which remains bounded as z tends to zero. Therefore the residue at $z = 0$ vanishes. □

4. Approximation of the heat kernel e^{-tD^2}. In this section we approximate the heat kernel for D^2 by the heat kernel for the harmonic oscillator. From now on $x = (\xi, c)$ is fixed, the local expression of D^2 in terms of the normal coordinates y_i's at gx is given by (for details see the Appendix):

$$D^2 = -\Delta + a + b + B$$

where

$$\begin{aligned}\Delta &= \sum_i \frac{\partial^2}{\partial y_i^2} \\ a &= \frac{1}{4} \sum_{i,j,\alpha,\beta} \tilde{R}_{ij\alpha\beta}(gx)\, y_i \frac{\partial}{\partial y_j}\, e_\alpha e_\beta \\ b &= \frac{1}{64} \sum_{i,j,k,\alpha,\beta,\gamma,\eta} y_i y_j\, \tilde{R}_{ik\alpha\beta}(gx)\, \tilde{R}_{kj\gamma\eta}(gx)\, e_\alpha e_\beta e_\gamma e_\eta\end{aligned}$$

where $\tilde{R}_{ij\alpha\beta}(gx) = -(R(E_i^{gx}, E_j^{gx})E_\alpha^{gx},\ E_\beta^{gx})(gx)$, with R being the curvature operator on M, and the term B being of the form $\phi_1 + \phi_2$, where

$$\phi_1 = \sum_{i_l,q,p,\alpha,\beta} k_p y_{i_1} \cdots y_{i_j} \frac{\partial}{\partial y_q}\, e_\alpha e_\beta,$$

and

$$\phi_2 = \sum_{i_l,q,p} k_p\, y_{i_1} \cdots y_{i_j} \frac{\partial}{\partial y_q}, \quad \text{for } j \geq 2$$

with k_l's some constants. Before using Duhamel's expansion to approximate locally the heat kernel we replace the y_i's that appear in the local expression of D^2 by bounded smooth functions $h_i(x)$ with bounded derivatives, so that $h_i(x) = y_i$ in a neighborhood of ξ. We will continue using the notation y_i to denote $h_i(x)$. Thus, by Duhamel's expansion we get

$$\begin{aligned} t^r \mathrm{tr}_s\big(g^*(x) \cdots c(d\check{f}_{2r})(gx)\, e^{-tD^2}(gx,x)\big) \\ - t^r\, \mathrm{tr}_s\big(g^*(x) \cdots c(d\check{f}_{2r})(gx)\, e^{-t\Delta}(gx,x)\big) \\ = t^r\, \mathrm{tr}_s\Bigg(g^*(x)\check{f}_0(gx) \cdots c(d\check{f}_{2r})(gx) \\ \cdot \int_0^t \big(e^{-(t-s_1)\Delta}(-a-b+B)\, e^{-s_1\Delta} ds_1\big)(gx,x)\Bigg) \\ + t^r\, \mathrm{tr}_s\Bigg(g^*(x)\check{f}_0(gx) \cdots c(d\check{f}_{2r})(gx) \\ \cdot \int_0^t \int_0^{s_1} \big(e^{-(t-s_1-s_2)\Delta}(-a-b+B)\, e^{-s_1\Delta} \\ (-a-b+B)\, e^{-s_2\Delta} ds_1 ds_2\big)(gx,x)\Bigg) + \cdots \end{aligned} \tag{18}$$

Our first step is to show that if any term in the above expansion contains only the operator (a), then the whole term will vanish.

Lemma 4.1. *The commutator* $[e^{-t\Delta}, a] = 0$, *where the operator* (a) *is as above.*

Proof. This follows from the fact that the $\tilde{R}_{ij\alpha\beta}$'s are skew-symmetric with respect to i and j variables. □

Remarks 4.2. 1. The operator (a) has canonical order zero, of course by the canonical order of the operator (a) we mean of the operator $a\, e^{-s_i\Delta}$ and has at most 2 Clifford variables.

2. The operator (b) has canonical order 1 and has at most 4 Clifford variables.

3. ϕ_1 and ϕ_2 both have canonical order greater or equal to $1/2$ and ϕ_1 has 2 Clifford variables, whereas ϕ_2 has none.

4. Each $c(df_i)$ has one Clifford variable and $g^*(x)$ has at most $2n-2m$ Clifford variables by Remark 3.3.

Concerning the expansion (18), we only need to consider those terms with greater than or equal to $2n$ Clifford variables (to have a nonzero supertrace) and if the canonical order of these terms is greater than m, then by Proposition 3.4 they vanish.

Suppose the operator (a) appears q times and (b) appears s times, ϕ_1 appears p times whereas ϕ_2 appears l times in expansion (18), such that they contribute to a total of $2n$ Clifford variables; hence, by Remark 4.2 we get the equation

$$2n - 2m + 2r + 2q + 4s + 2p \leq 2n. \tag{19}$$

By Theorem 2.5 and Remark 4.2, the canonical order of that term is

$$r + 2s + 3/2\,p + l/2 + q > m \qquad \text{by (19)}.$$

Theorem 4.3.

$$\operatorname{Res}_{z=0} \int_0^1 t^{z-1} \left(\int_M \operatorname{tr}_s(H_t^r(g,x))\, d\operatorname{vol}(x) \right) dt = 0$$

where

$$H_t^r(g,x) = \big(t^r\, g^*(x)\check{f}_0 \cdots c(d\check{f}_{2r})\, e^{-tD^2}(gx,x) \\ - t^r\, g^*(x)\check{f}_0 \cdots c(d\check{f}_{2r})\, e^{-t(\Delta-b)}(gx,x)\big).$$

Proof. Expand both operators e^{-tD^2} and $e^{-t(-\Delta+b)}$ as perturbations of $e^{-t\Delta}$, using Duhamel's expansion. All the terms that contain the operator (b) will cancel out, so what remains are terms containing operator (a) and the ϕ_i's $\in B$. From the above computations, we showed that for terms having at least $2n$ Clifford variables their canonical order is greater than m. Consequently their residue at $z=0$ vanishes by Proposition 3.4. $\square$

5. Computation of the equivariant cocycle. Recall the two frames E and E^{gx} were related by an infinitesimal holonomy, i.e., $E^{gx}(x) = E(x)e^{\Phi(x)}$. The next lemma will give an explicit description of $\Phi(x) = (\Phi_{ij}(x)) \in$ so $(2n)$. Also, under the identification of $\mathcal{E}_x$ with $\mathcal{E}_{gx}$ via parallel translation along the unique geodesic joining gx to x we have $\bar{g^*}(x) \in \mathrm{End}\,(\mathcal{E}_x) \sim \mathrm{End}\,(\mathcal{S})$.

Lemma 5.1. *Let $x = (\xi, c)$, then*

$$\Phi_{ij}(x) = -\frac{1}{2}\sum_{\alpha,\beta=1}^{2n-2m} \bar{c_\alpha} c_\beta R_{\alpha\beta ij}(\xi) + 0(|c|^2)$$

and

$$\bar{g^*}(x) = \exp\left(\frac{1}{2}\sum_{1\le i,j\le 2n} \Phi_{ij}(x) e_i e_j\right) \cdot \exp\left(-\frac{1}{2}\sum_{\alpha=1}^{n-m} \theta_\alpha e_{2m+2\alpha-1} e_{2m+2\alpha}\right). \tag{20}$$

Here $R_{\alpha\beta ij}(\xi) = -(R(E_{2m+\alpha}, E_{2m+\beta})E_i, E_j)(\xi)$, where R is the curvature operator on M.

For the proof consult [**15**] and [**1**].

Proposition 5.2. *With $x = (\xi, c)$, then*

$$\bar{g}^*(x) = \bar{g}_1^*(x) + \bar{g}_2^*(x)$$

where

$$\bar{g}_1^*(x) = \left(\prod_{\alpha}^{n-m} -\sin\frac{\theta_\alpha}{2}\right) \cdot \exp\left(-\frac{1}{4}\sum_{\alpha,\beta=1}^{2n-2m} \bar{c_\alpha} c_\beta \sum_{i,j=1}^{2m} R_{\alpha\beta ij}(\xi) e_i e_j\right) e_{2m+1}\cdots e_{2n}$$

and $\bar{g}_2^*(x) = H_1 + H_2$ *where* H_1 *is a sum of terms of the form* $f(\theta_\alpha, \theta_\beta)\, c_i c_j e^{2n-2m-2p+2}$ *with* $1 \le p \le n-m$*, and* H_2 *is a sum of terms of the form* $c_i c_j c_\alpha c_\beta e^{2n-2m+2}$*; here* e^j *means* j *Clifford variables* $e_{i_1} \cdots e_{i_j}$ *either from normal or tangent direction and* $f(\theta_\alpha, \theta_\beta)$ *is a product of* $\sin(\theta_\alpha)$ *and* $\cos(\theta_\beta)$*.*

Proof. This follows from Lemma 5.1 and the fact that

$$\exp\left(-\frac{1}{2}\sum_{\alpha=1}^{n-m}\theta_\alpha e_{2m+2\alpha-1}e_{2m+2\alpha}\right) = \left(\prod_{\alpha=1}^{n-m} -\sin\frac{\theta_\alpha}{2}\right) e_{2m+1}e_{2m+2}\dots e_{2n} + \mathcal{N}$$

The term $\mathcal{N}$ is a sum of terms of the form $\sin(\theta_\alpha/2)\cos(\theta_\beta/2)e^{2n-2m-2p}$ with $1 \le p \le n-m$. □

The heat kernel $e^{-t(-\Delta+b)}$ has an asymptotic expansion

$$e^{-t(-\Delta+b)}(gx,x) = \frac{e^{-\rho^2(gx,x)/4t}}{(4\pi t)^n}\sum_{i=0}^{N} t^i\, U_i(gx,x) + O(t^\varepsilon) \tag{21}$$

where the $U_i(gx,x) = \mathcal{E}_x \to \mathcal{E}_{gx}$ are linear maps. Under the identification of $\mathcal{E}_x$ with $\mathcal{E}_{gx}$ via parallel translation along the unique geodesic joining gx to x we can consider $U_i \in \mathrm{End}\,(\mathcal{E}_x) \sim \mathrm{End}\,(\mathcal{S})$. We let K_t denote the kernel

$$K_t(x) = \frac{e^{-\rho^2(gx,x)/4t}}{(4\pi t)^n}\sum_{j=0}^{N} t^j\, U_j(x) + 0(t^\alpha). \tag{22}$$

With respect to the normal coordinates y_i's of x at gx and frame E^{gx}, we write the operator $-\Delta + b$ as

$$-\Delta + b = -\sum_{i=1}^{2n}\frac{\partial^2}{\partial y_i^2} + \frac{1}{16}\tilde{A}_{ij}^2\, y_i\, y_j.$$

Here

$$\tilde{A}_{ij} = -1/2 \sum_{\alpha,\beta=1}^{2n} \tilde{R}_{ij\alpha\beta}(gx)\, e_\alpha e_\beta, \quad \text{and} \quad \tilde{A}^2_{ij} = \sum_{k=1}^{2n} \tilde{A}_{ik}\, \tilde{A}_{kj}.$$

Also, let

$$\tilde{A}^k(x) = \sum_{i,j} y_i\, y_j\, \tilde{A}^k_{ij}. \tag{23}$$

Lemma 5.3 [**15**]. *Let* $A_{ij} = -1/2 \sum_{\alpha,\beta=1}^{2n} R_{ij\alpha\beta}(\xi) e_\alpha e_\beta$, *for* $1 \leq i, j \leq 2n$, *where* $R_{ijkl}(\xi) = -(R(E_i, E_j)E_k, E_l)(\xi)$. *Then*

$$A = (A_{ij}) = \begin{pmatrix} A^\top & 0 \\ 0 & A^\perp \end{pmatrix}$$

where

$$(A_{ij})^\top = -1/2 \sum_{\alpha,\beta=1}^{2m} R_{ij\alpha\beta}(\xi)\, e_\alpha e_\beta \quad i, j = 1, \ldots, 2m$$
$$(A_{ij})^\perp = -1/2 \sum_{\alpha,\beta=1}^{2m} R_{ij\alpha\beta}(\xi)\, e_\alpha e_\beta \quad i, j = 2m+1, \ldots, 2n.$$

Lemma 5.4. *With* $\tilde{A}$ *and* A *as above, then*

$$\operatorname{tr} \tilde{A}^{2k} = \operatorname{tr} (A^\top)^{2k} + \operatorname{tr} (A^\perp)^{2k} + S_1^k + S_2^k$$

and

$$\tilde{A}^{2k}(x) = (A^\perp)^{2k}(c - \bar{c}) + T_1^k + T_2^k$$

where S_1^k *contains at most* $4k$ *Clifford variables with at least one from normal direction, whereas* S_2^k *has* $4k$ *Clifford variables and two* c_i*'s,* T_1^k *contains* $4k$ *Clifford variables with at least one from normal direction and two* c_i*'s, and* T_2^k *has at most* $4k$ *Clifford variables and four* c_j*'s.*

Proof. The proof is straightforward. It follows from Lemma 3.2 and the fact

$$\tilde{R}_{i\,j\,\alpha\,\beta}(gx) = R_{i\,j\,\alpha\,\beta}(gx) = R_{ij\,\alpha\,\beta}(\xi) + o(|c|). \qquad \square$$

The next lemma is also a consequence of Lemma 3.2.

Lemma 5.5. 1. $c(d\check{f}_\alpha)(gx) = c(d\check{f}_\alpha)(\xi) + 0(|c|)\sum_j e_j$.
2. $c(d\check{f}_\alpha)(\xi) = c(d\check{f}_\alpha)^\top(\xi) + c(d\check{f}_\alpha)^\perp(\xi)$, *where*

$$c(d\check{f}_\alpha)^\top = \sum_{j=1}^{2m} e_j E_j(\check{f}_\alpha) \quad \textit{and} \quad c(d\check{f}_\alpha)^\perp = \sum_{j=2m+1}^{2n} e_j E_j(\check{f}_\alpha).$$

Lemma 5.6 [**18**]. *There exists a polynomial* $F(z_1, z_2, \dots, w_1, w_2, \dots)$ *such that*

$$K_t(x) = \frac{e^{-\rho^2(gx,x)/4t}}{(4\pi t)^n} \cdot \sum_{j=0}^{N} t^j\, F\big((\operatorname{tr}\tilde{A}^2, \dots, \operatorname{tr}\tilde{A}^{2s_j}), (\tilde{A}^2(x), \dots \tilde{A}^{2l_j}(x))\big) + 0(t^\alpha);$$

moreover, the polynomial F *is determined by*

$$F\Bigg(-2\sum_{i=1}^{n} z_i^2, ..., (-1)^s 2\sum_{i=1}^{2n} z_i^{2s}, ...; (-1)\sum_{k=1}^{n}(z_{2k-1}^2 + z_{2k}^2)x_k^2, ..., (-1)^s \cdot \sum_{k=1}^{n}(z_{2k-1}^2 + z_{2k}^2)x_k^{2s}, ...,\Bigg)$$
$$= e^{\|z\|^2/4t}\prod_{k=1}^{n}\frac{ix_k t/2}{\sinh(ix_k t/2)}\exp\left(\frac{-ix_k t}{2}\,\frac{(z_{2k-1}^2 + z_{2k}^2)}{4t}\,\coth\left(\frac{ix_k t}{2}\right)\right).$$

From Lemmas 5.3, 5.4, and 5.6 (using the definition of the polynomial F as in [**18**]) one can show that, in terms of the orthogonal coordinates at ξ, we have

(24)

$$\sum_{j=0}^{N} t^j\, F\big((\operatorname{tr}\tilde{A}^2,\dots,\operatorname{tr}\tilde{A}^{2s_j}),(\tilde{A}^2(x),\dots\tilde{A}^{2l_j}(x)\big)$$
$$=\sum_{k=0}^{N} t^k F\big((\operatorname{tr}A^2,\dots,\operatorname{tr}A^{2s_k}),((A^{\perp})^2(c-\bar{c}),\dots(A^{\perp})^{2l_k}(c-\bar{c}))\big)$$
$$+\sum_{p=0}^{N} t^p \hbar_p$$

where $\hbar_p$ is sum of terms of the form $R_{ijkl}(\xi)\cdots R_{hqrs}(\xi)e_{2m+\alpha}e_{i_1}\cdots e_{i_{2p-1}}$, i.e., terms with at most $2p$ Clifford variables with at least one from the normal direction. $\hbar_p$ also contains sums of terms of the form $c_{j_1}\cdots c_{j_\alpha}R_{ijkl}(\xi)\cdots R_{hqrs}(\xi)e_{i_1}\cdots e_{i_{2p}}$ with $\alpha\geq 2$, i.e., terms with at least two c_i's and at most $2p$ Clifford variables.

Theorem 5.7.

$$\operatorname{Res}_{z=0}\int_0^1 \frac{t^{z+r-1}}{\Gamma(z+r)}\int_{\tilde{U}}\operatorname{tr}_s\big(g^*(x)\check{f}_0(gx)\cdots c(d\check{f}_{2r})(gx)\,e^{-t(-\Delta+b)}(gx,x)\big)d\operatorname{vol}(x)\,dt$$
$$=\operatorname{Res}_{z=0}\frac{1}{\Gamma(z+r)}\int_0^1 t^{z+r-1}\Bigg[\int_{F_g}\int_{\nu_\xi(\varepsilon)}\operatorname{tr}_s\Bigg(\bar{g}_1^*\check{f}_0(\xi)c(d\check{f}_1)^{\top}(\xi)\cdots c(d\check{f}_{2r})^{\top}(\xi)\,\frac{e^{\frac{-\|\bar{c}-c\|^2}{4t}}}{(4\pi t)^n}\sum_{k=0}^{N}t^kF\big((\operatorname{tr}A^2,\dots,\operatorname{tr}A^{2s_k}),$$
$$((A^{\perp})^2(c-\bar{c}),\dots,(A^{\perp})^{2l_k}(c-\bar{c}))\big)\Bigg)\,dc\,d\xi\Bigg]\,dt.$$

Proof. The proof of this theorem is in several steps and it relies heavily on the use of Proposition 3.4. Under the identification of $\mathcal{E}_x$ with $\mathcal{E}_{gx}$ via parallel translation along the unique geodesic joining gx to x, we replace $e^{-t(-\Delta+b)}$ by the asymptotic expansion (22) and also replace g^* by $\bar{g}^*(x) = \bar{g}_1^*(x) + \bar{g}_2^*(x)$ as in Proposition 5.2. Our first claim is that the $\bar{g}_2^*$ part can be ignored in the computation of the cyclic cocycle. By Taylor expansion, we express $\rho^2(gx, x)$, the square of the Riemannian distance between gx and x, in terms of orthogonal coordinates at ξ as in [**11**]

$$\rho^2(gx, x) = \|c - \bar{c}\|^2 + T(c) \tag{25}$$

where $T(c)$ is sum of terms of the form $R_{ijkl}(\xi)c_{i_1}c_{i_2}\cdots c_{i_\alpha}(\bar{c}_p - c_p)(\bar{c}_q - c_q)$ for $\alpha \geq 2$. Thus we have

$$\int_{\tilde{U}} t^r \mathrm{tr}_s\big(\bar{g}_2^*(x)\check{f}_0(gx)\cdots c(d\check{f}_{2r})(gx)K_t(x)\big)\, d\,\mathrm{vol}\,(x)$$

$$\sim \sum_{j=0}^{N} \int_{F_g} \left(\int_{\nu_\xi(\varepsilon)} t^{r+j}\, \mathrm{tr}_s\bigg(\bar{g}_2^*(x) \cdots c(d\check{f}_{2r})\, \frac{e^{-\|\mathbf{c}-\bar{\mathbf{c}}\|^2/4t}}{(4\pi t)^n}\, U_j(c,\bar{c})\psi(c)\bigg)\, dc\right) d\xi$$

$$+ \sum_j \int_{F_g} \left[\int_{\nu_\xi(\varepsilon)} t^{r+j}\, \mathrm{tr}_s\bigg(\bar{g}_2^*(x)\cdots c(d\check{f}_{2r})\, \frac{e^{-\|\mathbf{c}-\bar{\mathbf{c}}\|^2/4t}}{(4\pi t)^n}\, (T(c) + T^2(c) + \cdots)U_j\psi(c)\bigg)\, dc\right] d\xi$$

where the smooth function ψ has an expansion in $\nu_\xi(\varepsilon)$ [**11**],

$$\psi(c) = 1 - \frac{1}{6}\sum_{k,h,i} R_{kihi}(\xi)c_k c_h - \frac{1}{2}\sum_{k,h,\alpha} R_{k\alpha h\alpha}(\xi)c_k c_h + 0(c^3). \tag{26}$$

And the result follows from Proposition 3.4, because a term with $2n$ Clifford variables has canonical order greater than m.

Next, write each $c(d\check{f}_i)(gx)$ as in part(1) of Lemma 5.5; thus, we get

$$\check{f}_0(gx)\cdots c(d\check{f}_{2r})(gx) = \check{f}_0(\xi)\cdots c(d\check{f}_{2r})(\xi) + P$$

where P has at least two c_i's and at most $2r$ Clifford variables. Similar steps as above can be used to show that P has a trivial contribution in the computation. Then express each $c(d\check{f}_i)$ as a sum of tangent and normal direction as in part (2) of Lemma 5.5. Then from Proposition 5.2 observe that $\bar{g}_1^*$ contains all the normal direction Clifford variables. Thus if we consider a term with $2n$ Clifford variable then the presence of $c(d\check{f}_i)^\perp(\xi)$ will drop the number of Clifford variables and the supertrace will be zero. Finally by Proposition 3.4, $\hbar_p$ part can be ignored in the computation of the cyclic cocycle. □

6. The main results.

1) Evaluation of the Dirac cyclic cocycle. Let $\omega = (\omega_1, \dots, \omega_{2n})$ be a dual frame to the orthogonal frame $E = (E_1, \dots, E_{2n})$. And let

$$\Omega = (\Omega_{ij}) = \begin{pmatrix} \Omega^\top & 0 \\ 0 & \Omega^\perp \end{pmatrix}$$

where

$$(\Omega^\top)_{ij} = -\frac{1}{2}\sum_{\alpha,\beta=1}^{2m} R_{ij\alpha\beta}(\xi)\omega_\alpha \wedge \omega_\beta \qquad 1 \le i,j \le 2m$$

$$(\Omega^\perp)_{ij} = -\frac{1}{2}\sum_{\alpha,\beta=1}^{2m} R_{ij\alpha\beta}(\xi)\omega_\alpha \wedge \omega_\beta \quad 2m{+}1 \le i,j \le 2n.$$

As a complex vector space the Clifford algebra $\mathrm{Cl}\,(\mathbf{R}^{2n}) \otimes \mathbf{C}$ is isomorphic to the complexified exterior algebra $\wedge^*(R^{2n})$. Thus we can replace $A^\top$ and $A^\perp$ by $\Omega^\top$ and $\Omega^\perp$, and also $\mathrm{tr}\,A^{2k}$, $(A^\perp)^{2k}(c-\bar{c})$ by $\mathrm{tr}\,\Omega^{2k}$, $(\Omega^\perp)^{2k}(c-\bar{c})$ in Theorem 5.7.

The matrix Ω is a skew-symmetric matrix, thus it is similar to a block diagonal matrix. Without loss of generality we can assume that Ω is of the form

$$\Omega = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & \ddots & & \\ & & & & x_n \\ & & & -x_n & 0 \end{pmatrix}$$

with $x_i = u_i$ for $i = 1, \dots, m$ and $x_i = v_i$ for $i = 1, \dots, n-m$, where u_i's and v_j's are indeterminants.

Hence we have

$$-\frac{1}{4}\sum_{\alpha,\beta=1}^{2n-2m}\bar{c_\alpha}c_\beta\sum_{i,j=1}^{2m}R_{\alpha\beta ij}(\xi)\omega_i\wedge\omega_j = -\frac{1}{4}\sum_{\alpha=1}^{n-m}\sin\theta_\alpha.v_\alpha(c_{2\alpha-1}^2+c_{2\alpha}^2) \tag{27}$$

$$\operatorname{Tr}\Omega^{2k} = 2(-1)^k\left(\sum_{i=1}^{m}u_i^{2k}+\sum_{j=1}^{n-m}v_j^{2k}\right) \tag{28}$$

and

$$(\Omega^\perp)^{2k}(c-\bar{c}) = (-1)^k\sum_{\alpha=1}^{n-m}4\sin^2\,\frac{\theta_\alpha}{2}\,v_\alpha^{2k}\,(c_{2\alpha-1}^2+c_{2\alpha}^2). \tag{29}$$

Finally, let

$$(d\check{f}_\alpha)^\top = \sum_{i=1}^{2m}\omega_i\,E_i(\check{f}_\alpha)\quad\text{where}\quad c(d\check{f}_\alpha)^\top = \sum_{i=1}^{2m}e_i\,E_i(\check{f}_\alpha).$$

Let

(30)

$$\begin{aligned} &L_{2r}(g)(\xi)\\ &=\int_{\nu_\xi(\varepsilon)}t^r\operatorname{tr}_s\Big(\bar{g}_1^*(\exp_\xi c)\,\check{f}_0(\xi)c(d\check{f}_1)^\top(\xi)\cdots c(d\check{f}_{2r})^\top(\xi)\,\frac{e^{(-\|\bar{c}-c\|^2/4t)}}{(4\pi t)^n}\\ &\quad\cdot\sum_{k=0}^{N}t^kF\big((\operatorname{tr}\Omega^2,\dots,\operatorname{tr}\Omega^{2s_k});(\Omega^\perp)^2(c-\bar{c}),\dots(\Omega^\perp)^{2l_k}(c-\bar{c})\big)\Big)\,dc.\end{aligned}$$

Computing the supertrace in Theorem 5.7 is equivalent to computing the form of top order $2m$ on F_g, if we multiply by $(2/i)^n$, (this constant shows up from $\operatorname{Tr}_s(e_1e_2\cdots e_{2n}) = (2/i)^n$). Consequently, using the above observations together with Lemma 5.6 and (30) we get

(31)

$$
\begin{aligned}
L_{2r}(g)(\xi) = & \int_{\nu_\xi(\varepsilon)} \left(\frac{2}{i}\right)^n \frac{t^r}{(4\pi t)^n} \check{f}_0(\xi)(d\check{f}_1)^\top(\xi) \wedge \cdots \wedge (d\check{f}_{2r})^\top(\xi) \\
& \cdot \left(\prod_{\alpha}^{n-m} -\sin\frac{\theta_\alpha}{2}\right)\left(\prod_{\alpha=1}^{m} \frac{iu_\alpha t/2}{\sinh(iu_\alpha t/2)}\right)\left(\prod_{\beta=1}^{n-m} \frac{iv_\beta t/2}{\sinh(iv_\beta t/2)}\right) \\
& \cdot \exp\left(-\frac{1}{4}\sum_{\alpha=1}^{n-m} \sin\theta_\alpha . v_\alpha (c_{2\alpha-1}^2 + c_{2\alpha}^2)\right) \\
& \cdot \exp\left(\sum_{\beta=1}^{n-m} \frac{-iv_\beta t}{2} 4\sin^2\frac{\theta_\beta}{2} \frac{(c_{2\beta-1}^2 + c_{2\beta}^2)}{4t} \coth\left(\frac{iv_\beta t}{2}\right)\right) dc
\end{aligned}
$$

After some simplifications (for details consult [**15**] and [**1**]) formula (31) becomes

$$
\begin{aligned}
L_{2r}(g)(\xi) = & \frac{(-\pi)^{n-m}}{(2\pi i)^n} \frac{1}{t^{m-k}} \check{f}_0(\xi)(d\check{f}_1)^\top(\xi) \wedge \cdots \wedge (d\check{f}_{2k})^\top(\xi) \\
& \cdot \left(\prod_{\alpha=1}^{m} \frac{iu_\alpha t/2}{\sinh(iu_\alpha t/2)}\right) \left[\prod_{\beta=1}^{n-m} \sin\left(\frac{v_\beta t}{2} + \frac{\theta_\beta}{2}\right)\right]^{-1}.
\end{aligned}
$$

Computing the residue at $z = 0$, the only nontrivial term is the one which contains a $2m - 2r$ form. Therefore

$$
\begin{aligned}
& \mathrm{Res}_{z=0} \frac{1}{\Gamma(z+r)} \int_0^1 t^{z-1} \left(\int_{F_g} L_{2r}(g)(\xi)\, d\xi\right) dt \\
& \quad = \frac{1}{\Gamma(r)} \int_{F_g} \frac{(-\pi)^{n-m}}{(2\pi i)^n} \check{f}_0(\xi)(d\check{f}_1)^\top(\xi) \wedge \cdots \wedge (d\check{f}_{2r})^\top(\xi)\, d\xi \\
& \qquad \cdot \left[\left(\prod_{\alpha=1}^{m} \frac{iu_\alpha/2}{\sinh(iu_\alpha/2)}\right)\left(\prod_{\beta=1}^{n-m} \sin\left(\frac{v_\beta}{2} + \frac{\theta_\beta}{2}\right)\right)^{-1}\right](\widehat{m-r}).
\end{aligned}
$$

Where the notation $\big[\big(\prod_{\alpha=1}^m ((iu_\alpha/2)/\sinh(iu_\alpha/2))\big)\big(\prod_{\beta=1}^{n-m} \sin((v_\beta/2) +(\theta_\beta/2))\big)^{-1}\big](\widehat{m-r})$ means the $(m-r)$th term of the Taylor series expansion of $((iu_\alpha t/2)/\sinh(iu_\alpha t/2))$ and $\sin((v_\beta t/2) + (\theta_\beta/2))$ with respect to t at $t = 0$. Thus we have proved the main theorem in this article.

Theorem 6.1. *Let M be a smooth compact oriented Riemannian spin manifold, with fixed spin structure, of dimension $2n$. Let G be a countable discrete group acting in a proper smooth way on M, where M is endowed with a G-invariant Riemannian metric. Assume the action is a spin action on M. Let $\mathcal{A}$ be the $C^\infty(M, G)$ algebra under a certain norm and D the Dirac operator. Then the $2r$th component of the equivariant Chern character of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ as an entire cyclic cocycle is given by*

$$\begin{aligned}
\varphi_{2r}\bigg(\sum_{g_0} f_{g_0} g_0, \dots, \sum_{g_{2r}} f_{g_{2r}} g_{2r}\bigg) &= \bigg(Ch(D, \mathcal{H}), \bigg(\sum_{g_0} f_{g_0} g_0, \dots, \sum_{g_{2r}} f_{g_{2r}} g_{2r}\bigg)\bigg) \\
&= \sum_{g \in G} \sum_{g = g_0 \cdots g_{2r}} \bigg\{ \sum_j \int_{(F_g)_j} \big[L_{2r}(g)\big]_j \bigg\}
\end{aligned}$$

where

$$\begin{aligned}
\big[L_{2r}(g)\big]_j = \frac{(-\pi)^{n-m_j}}{(2\pi i)^n} \frac{1}{\Gamma(r)} &((g_0 \cdots g_{2r}).f_{g_0}) \, (d(g_1 \cdots g_{2r}).f_{g_1})^\top \\
&\wedge \cdots \wedge \, (d(g_{2r}).f_{g_{2r}})^\top \\
&\cdot \, \hat{A}(T(F_g)_j) \left[Pf\big(\sin(i\Omega/4\pi + \Theta(g)/2)(\nu(F_g)_j)\big)\right]^{-1}
\end{aligned}$$

The $(F_g)_j$'s are the connected components of the fixed submanifold F_g in M with $\dim (F_g)_j = m_j$. $T(F_g)_j$ is the tangent bundle, whereas $(\nu(F_g)_j)$ is the normal bundle of the fixed submanifold $(F_g)_j$. The $\hat{A}$-genus restricted to the tangent bundle is

$$\hat{A}\big(T(F_g)_j\big) = \bigg(\prod_{\alpha=1}^{m} \frac{iu_\alpha/2}{\sinh(iu_\alpha/2)} \bigg)$$

and the Pfaffian restricted to the normal bundle is

$$\left[Pf\big(\sin(i\Omega/4\pi + \Theta(g)/2)(\nu(F_g)_j)\big)\right]^{-1} = \bigg(\prod_{\beta=1}^{n-m} \sin\bigg(\frac{v_\beta}{2} + \frac{\theta_\beta}{2}\bigg)\bigg)^{-1}.$$

2) The index pairing. The cyclic cocycle that we computed in Theorem 6.1 is an element in the entire cyclic cohomology. Before pairing it with a K-theory element, we would like to view it as an element in the delocalized cohomology that was developed by Baum and Connes.

Let us briefly recall the construction of the delocalized cohomology, for more details one can consult [**2**].

Let G be a countable discrete group acting in a proper smooth way on M, where M is endowed with a G-invariant Riemannian metric. Define $\widehat{M} \subset G \times M$ by:

$$\widehat{M} = \{(g, x) \in G \times M | gx = x\}$$

then G acts on $\widehat{M}$ by $h(g, x) = (hgh^{-1}, hx)$ where $(g, x) \in \widehat{M}, h \in G$.

The quotient space under the group action is denoted by $\widehat{M}/G$. The action of G on $\widehat{M}$ is proper, so $\widehat{M}/G$ is an orbifold. One can also view the space $\widehat{M}$ as the disjoint union of F_g's, i.e.,

$$\widehat{M} = \sqcup_{g \in G} F_g$$

where $F_g = \{x \in M | gx = x\}$, the fixed submanifold of M by g.

Let $L = \{g_1, g_2, \dots\}$ be elements of finite order in G such that any element of finite order in G is conjugate to one and only one of the g_i. Then $\widehat{M}/G = \sqcup_{g \in L} F_g/Z(g)$, where $Z(g) = \{h \in G | hgh^{-1} = g\}$ is the centralizer of g in G.

We denote by $\Omega^*(\widehat{M})^G$ the space of all G-invariant differential forms on $\widehat{M}$, and by $H^*(\widehat{M} : C)^G$ the de Rham cohomology of G-invariant forms on $\widehat{M}$. The delocalized cohomology is the cohomology of the quotient space $\widehat{M}/G$, and we have

$$H_c^*(\widehat{M}/G; C) = H^*(\widehat{M}; C)^G = \oplus_{g \in L} H^*(F_g/Z(g); C).$$

For g and h conjugates in G. Then $H^*(F_g/Z(g); C) \simeq H^*(F_h/Z(h); C)$. Let Λ_g denote the cocycle as in Theorem 6.1

$$\begin{aligned} \Lambda_g &= \hat{A}(T(F_g)) \left[\mathrm{Pf}\left(\sin(i\Omega/4\pi + \Theta(g)/2)(\nu(F_g))\right)\right]^{-1} \\ &= \left(\prod_{j=1}^{m} \frac{iu_j/2}{\sinh(iu_j/2)}\right) \left(\prod_{l=1}^{n-m} \sin\left(\frac{v_l}{2} + \frac{\theta_l}{2}\right)\right)^{-1} \end{aligned}$$

(for simplicity we assume F_g has only one connected component). The claim is that Λ_g is a well-defined element of the delocalized cohomology $H^*(F_g/Z(g); C)$. To prove the claim:

Note that the expression of $\hat{A}$ and the Pfaffian involves the u_i's and v_j's, which are the Chern roots in the normal and tangent direction to F_g defined in terms of the connection. The Levi-Civita connection ∇ on TM preserves the decomposition in (15). Therefore it induces connections ∇_0 and ∇_1 on TF_g and $\nu(F_g)$, respectively, which preserves the metrics on these bundles; hence,

$$\nabla = \nabla_0 \oplus \nabla_1.$$

Let $\alpha \in Z(g)$, then α induces a diffeomorphism

$$\alpha : F_g \to F_g, \quad \text{by sending} \quad x \text{ to } \alpha\, x.$$

Therefore, $(\alpha^{-1})^* u_i = u_i$ and $(\alpha^{-1})^* v_j = v_j$ for all i and j. The linear map $dg : \nu(F_g)_x \to \nu(F_g)_x$ gives rise to an orthogonal decomposition $\nu(F_g)_x = \oplus_{i=1}^{l} N_x(\theta_i) \oplus N_x(\pi)$ which is constant on each connected component of F_g [**16**]. Hence it follows easily that $\alpha^* \Lambda_g = \Lambda_g$, and Λ_g is a well-defined element in $H^*(F_g/Z(g))$.

Let A be an algebra over $\mathbf{C}$ with a unit and $\mathcal{E}$ is a finitely generated A-module. Let e be the idempotent corresponding to the finitely generated A-module $\mathcal{E}$; then the Chern character map takes the form (cf. [**5**] and [**13**])

$$\mathrm{Ch}\,(e) = \sum_{k=0}^{\infty} \frac{1}{k!} \,\mathrm{Tr}\left[\left(\frac{i}{2\pi}\, e\, de\, de \right)^k \right].$$

Using the fact that $e(de)e = 0$ and $e(de)^2 = (de)^2 e$, the above formula becomes

$$\mathrm{Ch}\,(e) = \sum_{k=0}^{\infty} \frac{1}{k!} \,\mathrm{Tr}\left[\left(\frac{i}{2\pi}\, e\, de\, de \cdots de \right) \right]^k. \tag{32}$$

As in [**2**], the equivariant K-theory is defined by

$$K_G^0(M) = K_0(C_0(M) \times G)$$

The C^* algebra $C_0(M) \times G$ contains the smooth crossed product algebra $C^\infty(M,G)$ as a dense subalgebra. In fact it was shown in [**2**] that $C^\infty(M,G)$ is closed under holomorphic functional calculus and that the inclusion $C^\infty(M,G) \subset C_0(M) \times G$ induces an isomorphism $K_*(C^\infty(M,G)) \sim K_*(C_0(M) \times G)$.

Let e be an idempotent in the matrix algebra $M_s(C^\infty(M,G))$; then $e = (e_{ij})$ where each $e_{ij} \in C^\infty(M,G)$. Therefore it has an expression of the form $e_{ij} = \sum_{g_l \in G} f_l^{ij} g_l^{ij}$ with $f_l^{ij} \in C^\infty(M)$. The Chern character map $Ch_G : K_0(C^\infty(M,G)) \to H^{ev}(\widehat{M}/G)$ is given by:

$$\begin{aligned} Ch_G(e) &= \sum_{r=0}^{\infty} \frac{1}{r!} \operatorname{Tr}\left[\left(\frac{i}{2\pi}\, e\, de\, de\right)^r\right] \\ &= \sum_{r=0}^{\infty} \sum_{i_0, i_1, \cdots, i_{2k}} \frac{1}{r!}\left(\frac{i}{2\pi}\right)^r e_{i_0 i_1}\, de_{i_1 i_2} \cdots de_{i_{2r} i_0} \\ &= \sum_{r=0}^{\infty} \sum_{i_0, i_1, \cdots, i_{2r}} \sum_{g_0^{i_0 i_1}, \dots, g_{2r}^{i_{2r} i_0}} \frac{1}{r!}\left(\frac{i}{2\pi}\right)^r \\ &\quad \cdot\ (f_0^{i_0 i_1} g_0^{i_0 i_1})(df_1^{i_1 i_2} g_1^{i_1 i_2}) \cdots (df_{2r}^{i_{2r} i_0} g_{2r}^{i_{2r} i_0}). \end{aligned}$$

Then using the fact that

$$\begin{aligned} &f_0^{i_0 i_1} g_0^{i_0 i_1} \cdots df_{2r}^{i_{2r} i_0} g_{2r}^{i_{2r} i_0} \\ &\quad = ((g_0^{i_0 i_1} \cdots g_{2r}^{i_{2r} i_0}).f_0^{i_0 i_1})\, d((g_1^{i_1 i_2} \cdots g_{2r}^{i_{2r} i_0}).f_1^{i_1 i_2}) \cdots d(g_{2r}^{i_{2r} i_0}.f_{2r}^{i_{2r} i_0}) \end{aligned}$$

which follows from the properties of the smooth crossed product algebra, here $g.f$ means the action of g on the function f. Thus $Ch_G(e)$ becomes:

$$\begin{aligned} Ch_G(e) = &\sum_{r=0}^{\infty} \sum_{i_0, \cdots, i_{2r}} \sum_{g^I \in G} \sum_{g^I = g_0^{i_0 i_1} \cdots g_{2r}^{i_{2r} i_0}} \\ &\quad \cdot \frac{1}{r!}\left(\frac{i}{2\pi}\right)^r ((g_0^{i_0 i_1} \cdots g_{2r}^{i_{2r} i_0}).f_0^{i_0 i_1}) \cdots d(g_{2r}^{i_{2r} i_0}.f_{2r}^{i_{2r} i_0}). \end{aligned}$$

where $g^I = g_0^{i_0 i_1} \cdots g_{2r}^{i_{2r} i_0}$ and $I = \{i_0, i_1, \dots, i_{2r}\}$. Before we compute the pairing, we extend the map φ_{2r} into the matrix-valued map $\tilde{\varphi}_{2r}$, taking values in the $s \times s$ matrix algebra $M_s(\mathcal{A})$ as follows

$$\tilde{\varphi}_{2r}(\tilde{f^0}, \tilde{f^1}, \cdots, \tilde{f^{2r}}) = \sum_{i_0, i_1, \dots, i_{2r}} \varphi_{2r}((f^0)^{i_0 i_1}, (f^1)^{i_1 i_2}, \dots, (f^{2r})^{i_{2r} i_0}),$$

where $\tilde{f^p} = ((f^p)^{ij})$, with $(f^p)^{ij} \in \mathcal{A}$.

Theorem 6.2. *With all the assumptions as in Theorem* 6.1*, let* $e \in K_0(C^\infty(M;G))$ *be an idempotent,* $e = e^2 \in M_s(C^\infty(M;G))$*. Then the* $2r$*th component of the pairing is given by*:

$$\left\langle Ch(D, \mathcal{H}), [e] \right\rangle_{2r} = \sum_j \sum_{i_0, \dots, i_{2r}} \sum_{g^I \in G} \sum_{g^I = g_0^{i_0 i_1} \dots g_{2r}^{i_{2r} i_0}} \frac{(-\pi)^{n-m_j}}{(2\pi i)^n} \cdot C_r \left\langle \Lambda^j_{g^I} \wedge (Ch^r_G(e))^\top|_{(F_{g^I})_j}, [(F_{g^I})_j] \right\rangle$$

where C_r *is some numerical constant and* $(F_{g^I})_j$*'s are the connected components of the fixed submanifold* F_{g^I} *in* M *with* $\dim((F_{g^I})_j) = 2m_j$*,*

$$\Lambda^j_{g^I} = \hat{A}(T(F_{g^I})_j) \big[Pf\big(\sin(i\Omega/4\pi + \Theta(g^I)/2)(\nu(F_{g^I})_j) \big) \big]^{-1}$$

and

$$\begin{aligned} &(Ch^r_G(e))^\top|_{(F_{g^I})_j} \\ &\quad = \frac{1}{r!} \left(\frac{i}{2\pi} \right)^r ((g_0^{i_0 i_1} \cdots g_{2r}^{i_{2r} i_0}).f_0^{i_0 i_1})(d(g_1^{i_1 i_2} \cdots g_{2r}^{i_{2r} i_0}).f_1^{i_1 i_2})^\top \\ &\qquad \cdots (d\, g_{2r}^{i_{2r} i_0}.f_{2r}^{i_{2r} i_0})^\top|_{(F_{g^I})_j}. \end{aligned}$$

Appendix

Local expression of D^2. With respect to the orthogonal frame E^{gx}, the Dirac operator D is given by $D = \sum_{i=1}^{2n} e_i \nabla^{\mathcal{E}}_{E_i^{gx}}$, where $\nabla^{\mathcal{E}}_{E_i^{gx}}$ is the spinor connection on the spin bundle $\mathcal{E}$ and the e_i's are Clifford variables. Thus

$$\begin{aligned} D^2 &= \left(\sum_{i=1}^{2n} e_i \nabla^{\mathcal{E}}_{E_i^{gx}} \right) \left(\sum_{j=1}^{2n} e_j \nabla^{\mathcal{E}}_{E_j^{gx}} \right) = \sum_{i,j} e_i e_j\, \nabla^{\mathcal{E}}_{E_i^{gx}}\, \nabla^{\mathcal{E}}_{E_j^{gx}} \\ &= -\sum_i (\nabla^{\mathcal{E}}_{E_i^{gx}})^2 + \sum_{i<j} (\nabla^{\mathcal{E}}_{E_i^{gx}}\, \nabla^{\mathcal{E}}_{E_j^{gx}} - \nabla^{\mathcal{E}}_{E_j^{gx}}\, \nabla^{\mathcal{E}}_{E_i^{gx}}) \end{aligned}$$

after some simplification we get (cf. [**17**])

$$D^2 = -\sum_i (\nabla^{\mathcal{E}}_{E_i^{gx}})^2 + \mathcal{K}/4$$

where $\mathcal{K}$ is the scalar curvature. Next we express the Laplacian $(\nabla^{\mathcal{E}}_{E_i^{gx}})^2$ in terms of normal coordinates y_i's at gx and frame E^{gx}. But first observe that

$$\nabla^{\mathcal{E}}_{\frac{\partial}{\partial y_i}} = \frac{\partial}{\partial y_i} + \frac{1}{4}\sum_{j,k} \Gamma^k_{i\,j}\, e_j e_k, \quad \text{and} \quad \frac{\partial}{\partial y_i} = E_i^{gx} + 0|y|.$$

Therefore

$$\begin{aligned} \nabla^{\mathcal{E}}_{E_i^{gx}} &= \nabla^{\mathcal{E}}_{(\partial/\partial y_i)} + 0|y| \\ &= \frac{\partial}{\partial y_i} - \frac{1}{8}\sum_{jkl} \tilde{R}_{ilkj}(gx)\, y_l\, e_j e_k + h_1 + h_2 \end{aligned}$$

where h_1 is the sum of terms of the form $k_1\, y_{i_1}\cdots y_{i_j} e_\alpha e_\beta$, and h_2 is the sum of terms of the form $k_2\, y_{i_1}\cdots y_{i_j}$, for $j \geq 2$ where k_i's are some constants. Thus, after simplification we get

$$\begin{aligned} D^2 = &-\sum_i \frac{\partial^2}{\partial y_i^2} + \frac{1}{4}\sum_{ij\alpha\beta} \tilde{R}_{ij\alpha\beta}(gx)\, y_i\, \frac{\partial}{\partial y_j}\, e_\alpha e_\beta \\ &+ \frac{1}{64}\sum_{ijk\,\alpha\beta\,\gamma\eta} y_i y_j\, \tilde{R}_{ik\alpha\beta}(gx)\, \tilde{R}_{kj\gamma\eta}(gx)\, e_\alpha e_\beta e_\gamma e_\eta + B. \end{aligned}$$

And the term B is of the form $\phi_1 + \phi_2$, where

$$\phi_1 = \sum_{i_l,k,p,\alpha,\beta} k_p y_{i_1}\cdots y_{i_j}\, \frac{\partial}{\partial y_k}\, e_\alpha e_\beta,$$

and

$$\phi_2 = \sum_{i_l,k,p} k_p\, y_{i_1}\cdots y_{i_j}\, \frac{\partial}{\partial y_k}, \quad \text{for } j \geq 2$$

where k_l's are some constants.

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