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EXACT LOCATION OF α-BLOCH SPACES IN L_a^p AND H^p OF A COMPLEX UNIT BALL

WEISHENG YANG AND CAIHENG OUYANG

ABSTRACT. In this paper we prove that, on the unit ball of $\mathbf{C}^n$, (i) for $f \in H(B)$ and $0 < \alpha < \infty$, $f \in \mathcal{B}^\alpha \Leftrightarrow \sup_{z\in B} |\mathcal{R}f(z)|(1-|z|^2)^\alpha < \infty$; as a corollary, $\mathcal{B}^\alpha = A(B) \cap \mathrm{Lip}\,(1-\alpha)$ for $0 < \alpha < 1$. (ii) $B^{\alpha(<1+(1/p))} \subset L_a^p \subset \mathcal{B}^{1+((n+1)/p)}$, $\mathcal{B}^{\alpha(<1)} \subset H^p \subset \mathcal{B}^{1+(n/p)}$ for $n > 1$ and $0 < p < \infty$, where L_a^p, H^p denote the Bergman spaces and Hardy spaces, respectively. And $\mathcal{B}^1 \subset \cap_{0<p<\infty} L_a^p \subset \mathcal{B}^{\alpha(>1)}$, $\mathcal{B}^{\alpha(<1)} \subset \cap_{0<p<\infty} H^p \subset \mathcal{B}^{\alpha(>1)}$. Further, it is proved with constructive methods that all of the above containments are strict and best possible.

1. Introduction. Let $H(B)$ denote the class of all holomorphic functions in the unit ball B of $\mathbf{C}^n$. We say that $f \in \mathcal{B}^\alpha$, α-Bloch, if

$$\|f\|_{\mathcal{B}^\alpha(B)} = \sup_{z\in B} |\nabla f(z)|(1-|z|^2)^\alpha < \infty, \quad 0 < \alpha < \infty.$$

It is clear that $\mathcal{B}^\alpha$ is a normed linear space, modulo constant functions, and $\mathcal{B}^{\alpha_1} \subset \mathcal{B}^{\alpha_2}$ for $\alpha_1 < \alpha_2$. When $n = 1$, replace them by $H(D)$ and $\mathcal{B}^\alpha(D)$, where D denotes the unit disk of complex plane.

Hardy and Littlewood proved that [**3**], [**2**]: $\mathcal{B}^\alpha(D) = \mathrm{Lip}\,(1-\alpha)$. We know that $\mathrm{Lip}\,\beta$ can be used to describe the dual space of Hardy space $H^p(D)$ for $0 < p < 1$ [**2**]. So $\mathcal{B}^\alpha$ are important in the theory of Hardy spaces. In [**15**] we gave some invariant gradient characterizations and Bergman-Carleson measure characterization of $\mathcal{B}^\alpha$ on the unit ball.

For $\mathcal{B}^1 = \mathrm{Bloch}\,(B)$, Timoney showed that $H_p \not\subset \mathrm{Bloch}\,(B)$ for any $p \in (0,\infty)$, but he did not know whether there were Bloch functions which were not in H^p or not, see Example 3.7(3) of [**12**]. Later on, in [**10**], Ryll and Wojtaszczyk pointed out that $\mathrm{Bloch}\,(B) \not\subset H^p$;

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therefore, there is no containment between H^p and Bloch. Naturally we want to know the relationships between α-Bloch and some classes of holomorphic functions, such as the exact location of α-Bloch spaces in L^p_a and H^p.

In this paper we will prove that (i) $f \in \mathcal{B}^\alpha \Leftrightarrow \sup_{z\in B} |\mathcal{R}f(z)|(1-|z|^2)^\alpha < \infty$. $\mathcal{B}^\alpha = A(B) \cap \mathrm{Lip}\,(1-\alpha)$ for $0<\alpha<1$. (ii) $\mathcal{B}^{\alpha(<1+(1/p))} \subset L^p_a \subset \mathcal{B}^{1+((n+1)/p)}$, $\mathcal{B}^{\alpha(<1)} \subset H^p \subset \mathcal{B}^{1+(n/p)}$ for $n>1$ and $0<p<\infty$. Further, $\mathcal{B}^1 \subset \cap_{0<p<\infty} L^p_a \subset \mathcal{B}^{\alpha(>1)}$, $\mathcal{B}^{\alpha(<1)} \subset \cap_{0<p<\infty} H^p \subset \mathcal{B}^{\alpha(>1)}$. All of the above containments are strict and best possible. For the inclusion chain $\mathcal{B}^{\alpha(<1+(1/p))} \subset L^p_a \subset \mathcal{B}^{1+((n+1)/p)}$, the strictness at the left side and the possibility at the right side show that, for each p, at least one $f(z)$ exists, $f \in L^p_a$, whose growth rate of gradient, or radial derivative, will be larger than, or equal to, $(1-|z|^2)^{-(1+(1/p))}$, and go so far as to $(1-|z|^2)^{-(1+((n+1)/p))}$. There is a similar conclusion for H^p in the other inclusion chain. Especially in the proof of the strictness and best possibility in (ii), we will use constructive methods.

2. Radial growth of α-Bloch functions. For $y \in S$, the unit sphere in $\mathbf{C^n}$, $\langle z,y\rangle = 0$, let $T_y f(z) = \sum_{j=1}^n y_j(\partial f/\partial z_j)(z)$ denote the complex tangential derivative of f in z and $\mathcal{R}f(z) = \sum_{j=1}^n z_j(\partial f/\partial z_j)(z)$ the radial derivative of f.

Lemma 1. *Suppose that $f \in H(B)$, $z \in B$, $y \in S$, $\langle z,y\rangle = 0$, $\gamma \geq 0$.*
(a) *If $|f(z)| \leq (1-|z|^2)^{-\gamma}$, then*

$$|T_y f(z)| \leq C(1-|z|^2)^{-\gamma-(1/2)}.$$

(b) *If $|T_y f(z)| \leq (1-|z|^2)^{-\gamma}$, then*

$$|\mathcal{R}f(z)| \leq C(1-|z|^2)^{-\gamma-(1/2)}.$$

(c) *If $|f(z)| \leq (1-|z|^2)^{-\gamma}$, then*

$$|\mathcal{R}f(z)| \leq C(1-|z|^2)^{-\gamma-1}.$$

Proof. (a) and (b) are Lemma 1 and Lemma 2 of [**17**], respectively. In fact, the method to prove (a) is similar to 6.4.6 of [**9**] and the idea to prove (b) is due to Lemma 4.8 of [**12**].

Combining (a) with (b), we can get (c).

Lemma 2. *Suppose that $f \in H(B)$, $y \in S$, $\langle z, y\rangle = 0$, $\gamma \geq 0$. If f satisfies*

$$|(T_y\mathcal{R})f(z)| \leq (1-|z|^2)^{-\gamma-(1/2)} \tag{1}$$

when $1/2 < |z| < 1$, then

$$|T_y f(z)|(1-|z|^2)^{\gamma} < C,$$

where C is a positive constant depending only on f.

Proof. When $\xi, y \in S$ and $\langle \xi, y\rangle = 0$ by Lemma 6.4.5 of [**9**], we have

$$\begin{aligned} r(D_j f)(r\xi) &= \int_0^r (D_j\mathcal{R}f)(t\xi)\,dt,\\ rT_y f(r\xi) &= r\sum_{j=1}^n (D_j f)(r\xi) y_j\\ &= \int_0^r \sum_{j=1}^n (D_j\mathcal{R}f)(t\xi) y_j\,dt\\ &= \int_0^r (T_y\mathcal{R}f)(t\xi)\,dt. \end{aligned}$$

Let $z = r\xi$, then by (1), when $1/2 < |z| < 1$, we have

$$\begin{aligned} |T_y f(z)| &\leq \frac{1}{|z|}\int_0^{|z|} \left|(T_y\mathcal{R}f)\left(t\frac{z}{|z|}\right)\right| dt\\ &= \frac{1}{|z|}\left(\int_{0\leq t\leq 1/2} + \int_{1/2<t\leq |z|}\right)\left|(T_y\mathcal{R}f)\left(t\frac{z}{|z|}\right)\right| dt\\ &\leq 2\int_{0\leq t\leq 1/2}\left|(\nabla\mathcal{R}f)\left(t\frac{z}{|z|}\right)\right| dt + 2\int_{1/2}^{|z|}(1-t^2)^{-\gamma-(1/2)}\,dt\\ &\leq C_1 + 2\int_{1/2}^{|z|}(1-t^2)^{-\gamma-(1/2)}\,dt, \end{aligned}$$

since $\mathcal{R}f$ is holomorphic in B. Thus,

$$\begin{aligned}(1-|z|^2)^\gamma |T_y f(z)| &\le 2\int_{1/2}^{|z|}(1-|z|^2)^\gamma(1-t^2)^{-\gamma-1/2}\,dt + C_1(1-|z|^2)^\gamma \\ &\le 2\int_{1/2}^{|z|}(1-t)^{-1/2}\,dt + C_1\left(\frac{3}{4}\right)^\gamma \\ &\le 2\sqrt{2}+C_1 = C,\end{aligned}$$

noticing that $\gamma \ge 0$ implies that $(3/4)^\gamma \le 1$.

In the following, C denotes a positive constant which is not necessarily the same on each appearance.

Proposition 1. *For $f \in H(B)$ and $0<\alpha<\infty$,*

$$f \in \mathcal{B}^\alpha \Longleftrightarrow \sup_{z\in B}|\mathcal{R}f(z)|(1-|z|^2)^\alpha < \infty.$$

Proof. Because $|\mathcal{R}f(z)| \le |\nabla f(z)|$, it is easy to see

$$f \in \mathcal{B}^\alpha \Longrightarrow \sup_{z\in B}|\mathcal{R}f(z)|(1-|z|^2)^\alpha < \infty.$$

On the other hand, suppose $\sup_{z\in B}|\mathcal{R}f(z)|(1-|z|^2)^\alpha < \infty$. When $|z| \le 1/2$, because f is holomorphic in B, it is clear that

$$\sup_{|z|\le 1/2}|\nabla f(z)|(1-|z|^2)^\alpha < \infty. \tag{2}$$

Now, let $1/2 < |z| < 1$. For each fixed z, from the vector space $\{y \in \mathbf{C}^n : \langle z, y\rangle = 0\}$, we can find unit vectors $y_2, \dots, y_n$ so that $z/|z|$, $y_2, \dots, y_n$ form a base of vector space $\mathbf{C}^n$. Of course, $\bar{z}/|z|, \overline{y_2}, \dots, \overline{y_n}$ form another base of $\mathbf{C}^n$. Therefore,

$$\begin{aligned}|\nabla f(z)|^2 &= |\langle \nabla f(z), (\bar{z}/|z|)\rangle|^2 + |\langle \nabla f(z), \overline{y_2}\rangle|^2 + \cdots + |\langle \nabla f(z), \overline{y_n}\rangle|^2 \\ &= \frac{1}{|z|^2}|\mathcal{R}f(z)|^2 + |T_{y_2}f(z)|^2 + \cdots + |T_{y_n}f(z)|^2. \qquad (3)\end{aligned}$$

By the hypothesis $\sup_{z\in B}|\mathcal{R}f(z)|(1-|z|^2)^\alpha<\infty$ and $1/2<|z|<1$, obviously

$$\frac{1}{|z|^2}|\mathcal{R}f(z)|^2\le C(1-|z|^2)^{-2\alpha}. \tag{4}$$

By the hypothesis $\sup_{z\in B}|\mathcal{R}f(z)|(1-|z|^2)^\alpha<\infty$ and Lemma 1(a) for $2\le j\le n$,

$$|T_{y_j}\mathcal{R}f(z)|\le C(1-|z|^2)^{-\alpha-(1/2)}.$$

By Lemma 2,

$$|T_{y_j}f(z)|(1-|z|^2)^\alpha<C. \tag{5}$$

Therefore, by (3), (4) and (5),

$$\sup_{1/2<|z|<1}(1-|z|^2)^{2\alpha}|\nabla f(z)|^2\le C<\infty. \tag{6}$$

By (2) and (6), we know

$$\sup_{z\in B}(1-|z|^2)^\alpha|\nabla f(z)|\le C<\infty.$$

Corollary 1. *$\mathcal{B}^\alpha=A(B)\cap \mathrm{Lip}\,(1-\alpha)$, for $0<\alpha<1$, where $A(B)$ is the ball algebra, see* [**9**].

Proof. If $f\in\mathcal{B}^\alpha$, by Proposition 1,

$$|\mathcal{R}f(z)|\le C(1-|z|^2)^{-\alpha}=C(1-|z|^2)^{(1-\alpha)-1}.$$

By Theorem 6.4.10 of [**9**] and $0<1-\alpha<1$,

$$f\in A(B)\cap \mathrm{Lip}\,(1-\alpha).$$

If $f\in A(B)\cap \mathrm{Lip}\,(1-\alpha)$, then by Theorem 6.4.9 and the Remark of 6.4.9 of [**9**], we can get

$$|\mathcal{R}f(z)|\le C(1-|z|^2)^{(1-\alpha)-1}=C(1-|z|^2)^{-\alpha}.$$

By Proposition 1, $f \in \mathcal{B}^\alpha$.

For $\xi \in S$, $\lambda \in D$, let $f_\xi(\lambda) = f(\xi\lambda)$ denote the slice function of f.

Corollary 2. $f \in \mathcal{B}^\alpha \Leftrightarrow \sup_{\xi \in S} \|f_\xi\|_{B^\alpha(D)} < \infty$.

Proof. If $f \in \mathcal{B}^\alpha$, then $|\mathcal{R}f(z)|(1-|z|^2)^\alpha \le C$ by Proposition 1. Thus, for each $\xi \in S$, $|\mathcal{R}f(\lambda\xi)|(1-|\lambda\xi|^2)^\alpha \le C$ and so $|f'_\xi(\lambda)|(1-|\lambda|^2)^\alpha \le C$. Taking $\sup_{\lambda\in D}$ and $\sup_{\xi\in S}$ in order, we get $\sup_{\xi\in S}\|f_\xi\|_{\mathcal{B}^\alpha(D)} < \infty$.

The converse is a similar process.

3. Power series with Hadamard gaps and α-Bloch, L^p_a. Propositions 2 and 3 will be used in the proof of the Theorem and Corollary 3, and are of independent interest.

It is proved in [**14**] that, if $f(z) = \sum_{k=1}^\infty a_k z^{n_k} \in H(D)$ with $n_{k+1}/n_k \ge q$, $k \ge 1$, $q > 1$, then for $\alpha > 0$,

$$f \in \mathcal{B}^\alpha(D) \Longleftrightarrow \limsup_{k\to\infty} |a_k| n_k^{1-\alpha} < \infty. \tag{7}$$

From [**18**], we know that, if $0 < p < \infty$, $\{n_k\}$ is an increasing sequence of positive integers satisfying $n_{k+1}/n_k \ge q > 1$ for all k, then there is a constant A depending only on p and q such that

$$\begin{aligned} A^{-1}\bigg(\sum_{k=1}^\infty |a_k|^2\bigg)^{1/2} &\le \bigg(\frac{1}{2\pi}\int_0^{2\pi}\bigg|\sum_{k=1}^\infty a_k e^{in_k\theta}\bigg|^p d\theta\bigg)^{1/p} \\ &\le A\bigg(\sum_{k=1}^\infty |a_k|^2\bigg)^{1/2}, \end{aligned} \tag{8}$$

for any number a_k, $k = 1, 2, \dots$.

In [**4**], it is proved that if $\alpha > 0$, $p > 0$, $n \ge 0$, $a_n \ge 0$, $I_n = \{k : 2^n \le k < 2^{n+1}, k \in \mathbf{N}\}$, $t_n = \sum_{k\in I_n} a_k$ and $f(x) = \sum_{n=1}^\infty a_n x^n$. Then there is a constant K depending only on p and α such that

$$\frac{1}{K}\sum_{n=0}^\infty 2^{-n\alpha} t_n^p \le \int_0^1 (1-x)^{\alpha-1} f(x)^p\, dx \le K \sum_{n=0}^\infty 2^{-n\alpha} t_n^p. \tag{9}$$

A holomorphic function $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$ on B, P_{n_k} is a homogeneous polynomial of degree $n_k \in \mathbf{N}$, the set of natural numbers is said to have Hadamard gaps if $n_{k+1}/n_k \geq q > 1$ for all $k = 1, 2, \ldots$.

Based on (7) and Corollary 2, we can give a sufficient condition for a power series in B with Hadamard gaps, to belong to α-Bloch spaces $\mathcal{B}^\alpha(B)$.

Proposition 2. *Let $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$ be a power series on B with Hadamard gaps. Suppose that*

$$\|P_{n_k}\|_\infty = \sup\{|P_{n_k}(\xi)| : \xi \in S\} \leq n_k^{\alpha-1}$$

for all $k \geq 1$. Then $f \in \mathcal{B}^\alpha(B)$, $0 < \alpha < \infty$.

Proof. Considering $\lim_{k\to\infty} \sup A_k = \inf_k \sup_{j\geq k} A_j$ for sequence $\{A_k\}_{k=1}^{\infty}$, the condition of (7) can be written as

$$\inf_k \sup_{j\geq k} |a_j| n_j^{1-\alpha} < \infty$$

for all $k \geq 1$. For each $\xi \in S$, observe that the slice function

$$f_\xi(\lambda) = \sum_{k=1}^{\infty} P_{n_k}(\xi)\lambda^{n_k}, \quad \lambda \in D.$$

If $\|P_{n_k}\|_\infty \leq n_k^{\alpha-1}$ for all $k \geq 1$, then

$$\inf_k \sup_{j\geq k} |P_{n_j}(\xi)| n_j^{1-\alpha} \leq \inf_k \sup_{j\geq k} \|P_{n_j}\|_\infty n_j^{1-\alpha} \leq 1.$$

Therefore, by (7), $\|f_\xi\|_{\mathcal{B}^\alpha(D)} \leq C$; here C is a positive constant depending only on q and α, not on f. Taking $\sup_{\xi\in S}$, we see $\sup_{\xi\in S} \|f_\xi\|_{\mathcal{B}^\alpha(D)} < \infty$, and so $f \in \mathcal{B}^\alpha(B)$ by Corollary 2.

Remark 1. This result generalizes Proposition 4.16 of [**12**].

Next we give a necessary and sufficient condition for a function on B, with Hadamard gaps, to belong to Bergman spaces $L_a^p(B)$.

Proposition 3. *Let* $f(z) = \sum_{k=1}^{\infty} P_{n_k}(z)$ *be a power series on* B *with Hadamard gaps. Then the following are equivalent*:

(i) $f \in L_a^p$, $0 < p < \infty$;

(ii) $\sum_{k=0}^{\infty} 2^{-k} \sum_{n_j \in I_k} \|P_{n_j}\|_p^p < \infty$,

where $I_k = \{n_j : 2^k \le n_j < 2^{k+1}, n_j \in \mathbf{N}\}$, $\|P_{n_j}\|_p^p = \int_S |P_{n_j}(\xi)|^p \, d\sigma(\xi)$.

Proof. By integration in polar coordinates and 1.4.7 Proposition (1) of [**9**],

$$\begin{aligned}\|f\|_{L_a^p}^p &= 2n \int_0^1 r^{2n-1} \, dr \int_S |f(r\xi)|^p \, d\sigma(\xi) \\ &= 2n \int_0^1 r^{2n-1} \, dr \int_S d\sigma(\xi) \int_0^{2\pi} |f(re^{i\theta}\xi)|^p \frac{d\theta}{2\pi} \\ &= 2n \int_S d\sigma(\xi) \int_0^1 r^{2n-1} \, dr \int_0^{2\pi} \left| \sum_{k=1}^{\infty} P_{n_k}(\xi) r^{n_k} e^{in_k\theta} \right|^p \frac{d\theta}{2\pi}.\end{aligned}$$

Applying (8) to the end of the above, we get

$$\|f\|_{L_a^p}^p \le nA^p \int_S d\sigma(\xi) \int_0^1 \left(\sum_{k=1}^{\infty} |P_{n_k}(\xi)|^2 (r^2)^{n_k} \right)^{p/2} dr^2. \tag{10}$$

On the other hand, applying (8) once more and integrating by parts twice, we have

$$\begin{aligned}\|f\|_{L_a^p}^p &\ge nA^{-p} \int_S d\sigma(\xi) \int_0^1 (r^2)^{n-1} \bigg(\sum_{k=1}^{\infty} |P_{n_k}(\xi)|^2 (r^2)^{n_k} \bigg)^{p/2} dr^2 \\ &= A^{-p} \int_S d\sigma(\xi) \int_0^1 \left(\sum_{k=1}^{\infty} |P_{n_k}(\xi)|^2 x^{n_k} \right)^{p/2} dx^n \\ &= A^{-p} \int_S d\sigma(\xi) \bigg[\bigg(\sum_{k=1}^{\infty} |P_{n_k}(\xi)|^2 x^{n_k} \bigg)^{p/2} x^n \bigg|_0^1 \\ &\qquad - \int_0^1 x^n d \bigg(\sum_{k=1}^{\infty} |P_{n_k}(\xi)|^2 x^{n_k} \bigg)^{p/2} \bigg]\end{aligned}$$

$$\geq A^{-p}\int_S d\sigma(\xi)\bigg[\bigg(\sum_{k=1}^{\infty}|P_{n_k}(\xi)|^2x^{n_k}\bigg)^{p/2}x\bigg|_0^1 - \int_0^1 xd\bigg(\sum_{k=1}^{\infty}|P_{n_k}(\xi)|^2x^{n_k}\bigg)^{p/2}\bigg]$$

$$= A^{-p}\int_S d\sigma(\xi)\int_0^1\bigg(\sum_{k=1}^{\infty}|P_{n_k}(\xi)|^2x^{n_k}\bigg)^{p/2}dx. \tag{11}$$

Combining (10) and (11), we get

$$\|f\|_{L_a^p}^p \approx \int_S d\sigma(\xi)\int_0^1\bigg(\sum_{k=1}^{\infty}|P_{n_k}(\xi)|^2x^{n_k}\bigg)^{p/2}dx.$$

Using (9), we have

$$\|f\|_{L_a^p}^p \cong \int_S\bigg(\sum_{k=1}^{\infty}2^{-k}t_k^{p/2}\bigg)d\sigma(\xi),$$

where

$$t_k = \sum_{n_j\in I_k}|P_{n_j}(\xi)|^2.$$

Since $n_{j+1}\geq qn_j\geq q2^k$, so $q^N2^k\leq n_{j+N}<2^{k+1}$. Thus the number N of P_{n_j} when $n_j\in I_k$ is at most $[\log_q 2]+1$ for $k=0,1,2,\dots$. Therefore, by (9) for $p<2$ and (10) for $p\geq 2$ of [**5**],

$$\|f\|_{L_a^p}^p \approx \int_S\bigg(\sum_{k=1}^{\infty}2^{-k}\bigg(\sum_{n_j\in I_k}|P_{n_j}(\xi)|^2\bigg)^{p/2}\bigg)d\sigma(\xi)$$

$$\approx \sum_{k=0}^{\infty}2^{-k}\sum_{n_j\in I_k}\|P_{n_j}\|_p^p.$$

This proves Proposition 3. $\square$

Remark 2. In [**6**], we proved Proposition 3 for $p=2$ by a slightly different method.

4. Strict and best possible inclusions for α-Bloch and L_a^p, H^p.

Theorem. *When $0 < p < \infty$ and $n > 1$, we have*

(a)

$$\mathcal{B}^{\alpha(<1+(1/p))} \subset L_a^p \subset \mathcal{B}^{1+((n+1)/p)};$$
$$\mathcal{B}^{\alpha(<1)} \subset H^p \subset \mathcal{B}^{1+(n/p)}.$$

(b) *For L_a^p, H^p and $\mathcal{B}^\alpha$, all of the inclusion relationships in (a) are strict and best possible, where "best possible" means that, for each p, the indices α of $\mathcal{B}^\alpha$ at the left sides cannot be larger and those at the right sides cannot be smaller.*

Proof. Since

$$f(z) - f(0) = \int_0^1 \nabla f(tz) z \, dt,$$

thus

$$|f(z)|^p \leq C\bigg\{|f(0)|^p + \bigg(\int_0^1 |\nabla f(tz)||z| \, dt\bigg)^p\bigg\}.$$

If $f \in \mathcal{B}^\alpha_{1<\alpha<1+(1/p)}$, then

$$\begin{aligned}\int_B |f(z)|^p \, dv(z) &\leq C|f(0)|^p + C\int_B \bigg(\int_0^1 |\nabla f(tz)||z| \, dt\bigg)^p dv(z)\\ &\leq C|f(0)|^p + C\int_B \bigg(\int_0^1 (1-t^2|z|^2)^{-\alpha}|z| \, dt\bigg)^p dv(z)\\ &\leq C|f(0)|^p + C(\alpha-1)^{-1}\int_B (1-|z|)^{p(1-\alpha)} \, dv(z)\\ &\leq C|f(0)|^p + 2nC(\alpha-1)^{-1}\\ &\quad \cdot \int_0^1 r^{2n-1}(1-r)^{p(1-\alpha)} \, dr < \infty.\end{aligned}$$

Thus, $f \in L_a^p$. This means $\mathcal{B}^\alpha_{1<\alpha<1+(1/p))} \subset L_a^p$. By the monotonicity of α-Bloch, we get

$$\mathcal{B}^\alpha_{0<\alpha<1+(1/p)} \subset L_a^p, \quad 0 < p < \infty.$$

Let $\alpha = 1$; then $\mathcal{B}^\alpha = \mathcal{B}^1$, the usual Bloch space. By this conclusion we see $\mathcal{B}^1 \subset L_a^p$, for $0 < p < \infty$. This is a well-known result.

By Corollary 1, we can easily see that

$$\mathcal{B}^\alpha_{0<\alpha<1} \subset H^p, \quad 0 < p < \infty.$$

Lemma 2 of [**11**] states that, let $f \in H(B)$ and $0 < p < \infty$, $s \geq 0$, $n+s+1 \geq p$. Then, for $z \in B$,

$$|\nabla f(z)|^p \leq K \int_B |f(w)|^p \frac{(1-|w|^2)^s}{|1-\langle z,w\rangle|^{n+s+p+1}}\, dv(w).$$

Using this lemma we have

$$\begin{aligned}(1-|z|^2)^{1+((n+1)/p)}&|\nabla f(z)| \\ &\leq K^{1/p}\bigg(\int_B |f(w)|^p \frac{(1-|z|^2)^{p+n+1}(1-|w|^2)^s}{|1-\langle z,w\rangle|^{n+s+p+1}}\, dv(w)\bigg)^{1/p} \\ &\leq (2^{p+n+1+s}K)^{1/p}\bigg(\int_B |f(w)|^p\, dv(w)\bigg)^{1/p}.\end{aligned}$$

If $f \in L_a^p$, then $(2^{p+n+1+s}K)^{1/p}(\int_B |f(w)|^p\, dv(w))^{1/p} \leq M < \infty$, thus $\sup_{z\in B}(1-|z|^2)^{1+((n+1)/p)}|\nabla f(z)| \leq M < \infty$, $f \in \mathcal{B}^{1+((n+1)/p)}$.

Suppose $f \in H^p$, by Theorem 7.2.5(a) of [**9**],

$$|f(z)| \leq 2^{n/p}\|f\|_p(1-|z|)^{-n/p}.$$

By Lemma 1(c),

$$|\mathcal{R}f(z)| \leq 2^{n/p}C\|f\|_p(1-|z|^2)^{-(n/p)-1}.$$

By Proposition 1,

$$f \in \mathcal{B}^{1+(n/p)}.$$

Therefore, when $0 < p < \infty$ and $n > 1$, $H^p \subset \mathcal{B}^{1+(n/p)}$.

The proof of Theorem (a) is completed.

Next we construct some functions to show that the conclusion (b) is true. Let

$$f_t(z) = (1-z_1)^{-t}, \quad t > 0.$$

(i)

$$\frac{\partial f_t}{\partial z_j} = 0 \quad \text{for } j = 2, \dots, n;$$
$$\frac{\partial f_t}{\partial z_1} = t(1 - z_1)^{-t-1}.$$

Then,

$$(1 - |z|^2)^\alpha |\nabla f_t(z)| = t(1 - |z|^2)^\alpha |1 - z_1|^{-t-1}.$$

Noting

$$|1 - z_1|^{-t-1} \leq (1 - |z_1|)^{-t-1} \leq C(1 - |z|^2)^{-t-1},$$

thus, when $\alpha \geq t + 1$,

$$(1 - |z|^2)^\alpha |\nabla f_t(z)| \leq C(1 - |z|^2)^{\alpha-t-1} < C < \infty.$$

Therefore,

$$f_t \in \mathcal{B}^\alpha \quad \text{for } \alpha \geq t + 1. \tag{12}$$

When $\alpha < t + 1$, put $z = (y, 0, \dots, 0)$, where $0 < y < 1$,

$$(1 - |z|^2)^\alpha |\nabla f_t(z)| = t(1 + y)^\alpha (1 - y)^{\alpha-t-1}.$$

Let $y \to 1$. Then

$$(1 - |z|^2)^\alpha |\nabla f_t(z)| \longrightarrow \infty;$$

therefore,

$$f_t \notin \mathcal{B}^\alpha \quad \text{for } \alpha < t + 1. \tag{13}$$

(ii)

$$\int_S |f_t(r\xi)|^p \, d\sigma(\xi) = \int_S \frac{d\sigma(\xi)}{|1 - r\xi_1|^{tp}}$$
$$= \int_S \frac{d\sigma(\xi)}{|1 - \langle re_1, \xi\rangle|^{tp}},$$

where $e_1 = (1, 0, \dots, 0) \in \mathbf{C}^n$. By Proposition 1.4.10 of [**9**],

$$\int_S \frac{d\sigma(\xi)}{|1 - \langle re_1, \xi\rangle|^{tp}} \le C < \infty, \quad \text{when } t < \frac{n}{p},$$
$$\int_S \frac{d\sigma(\xi)}{|1 - \langle re_1, \xi\rangle|^{tp}} \approx \log \frac{1}{1-r^2} \longrightarrow \infty, \quad \text{when } t = \frac{n}{p},\ r \longrightarrow 1.$$

Thus, for $0 < p < \infty$,

$$f_t \in H^p \quad \text{when } t < \frac{n}{p}; \tag{14}$$

$$f_t \notin H^p \quad \text{when } t = \frac{n}{p}. \tag{15}$$

(iii) Let P be the orthogonal projection of $\mathbf{C}^n$ onto $\mathbf{C}^1 : \xi = (\xi_1, \xi_2, \dots, \xi_n) \to \xi_1$.

$$J = \int_B |f_t(z)|^p \, dv(z) = 2n \int_0^1 r^{2n-1} \, dr \int_S \frac{d\sigma(\xi)}{|1 - rP(\xi)|^{tp}}.$$

Using 1.4.4(1) of [**9**], we get

$$\begin{aligned} \int_S \frac{d\sigma(\xi)}{|1 - rP(\xi)|^{tp}} &= \binom{n-1}{1} \int_D \frac{(1-|w|^2)^{n-2}}{|1 - rw|^{tp}} \, dv_1(w) \\ &= (n-1) \int_D \frac{(1-|w|^2)^{n-2} \, dv_1(w)}{|1 - \langle r, w\rangle|^{2+(n-2)+(tp-n)}}. \end{aligned}$$

By Lemma 4.2.2 of [**16**], when $tp - n < 0$, the integral at the end of the above equation is finite, and so $J \le C < \infty$; when $tp - n = 0$,

$$\int_D \frac{(1-|w|^2)^{n-2}}{|1 - \langle r, w\rangle|^n} \, dv_1(w) \approx \log \frac{1}{1-r^2}.$$

Thus

$$\begin{aligned} J &= 2n \int_0^1 r^{2n-1} \, dr \int_S \frac{d\sigma(\xi)}{|1 - rP(\xi)|^n} \\ &\le C \int_0^1 r^{2n-1} \log \frac{1}{1-r^2} \, dr \\ &\le C \int_0^\infty \tau (1 - e^{-\tau})^{n-1} e^{-\tau} \, d\tau. \end{aligned}$$

The integral at the end of the above expression is a finite linear combination of gamma functions without poles; therefore, we also have $J \le C < \infty$. When $tp - n > 0$,

$$\int_D \frac{(1-|w|^2)^{n-2}\, dv_1(w)}{|1-\langle r, w\rangle|^{2+(n-2)+(tp-n)}} \approx (1-r^2)^{n-tp}.$$

Thus when $tp - n > 0$ and $n - tp > -1$,

$$J \le C \int_0^1 r^{2n-1}(1-r^2)^{n-tp}\, dr \le C < \infty;$$

when $n - tp = -1$,

$$J \approx \int_0^1 r^{2n-1}(1-r^2)^{-1}\, dr = \infty.$$

Therefore

$$f_t \in L_a^p \quad \text{when } t < \frac{n+1}{p}; \tag{16}$$

$$f_t \notin L_a^p \quad \text{when } t = \frac{n+1}{p}. \tag{17}$$

(iv) For arbitrary $\varepsilon > 0$, let $t = (n/p) - (1/2)\varepsilon$. Then $t < n/p$ and $1 + (n/p) - \varepsilon < t + 1$ by (14) and (13), we get

$$f_t \in H^p \quad \text{but } f_t \notin \mathcal{B}^{1+(n/p)-\varepsilon}.$$

That means the inclusion $H^p \subset \mathcal{B}^{1+(n/p)}$ is best possible. At the same time, it also shows that the inclusion $\mathcal{B}^{\alpha(<1)} \subset H^p$ is strict because

$$\mathcal{B}^{\alpha(<1)} \subset \mathcal{B}^{1+(n/p)-\varepsilon} \quad \text{for } \varepsilon \le \frac{n}{p}$$

leads to $f_t \notin \mathcal{B}^{\alpha(<1)}$.

For another arbitrary $\varepsilon > 0$, let $t = ((n+1)/p) - (1/2)\varepsilon$, then $t < (n+1)/p$ and $1 + ((n+1)/p) - \varepsilon < t + 1$ by (16) and (13), it is easy to see that

$$f_t \in L_a^p \quad \text{but } f_t \notin \mathcal{B}^{1+((n+1)/p)-\varepsilon}.$$

Thus the inclusion $L_a^p \subset \mathcal{B}^{1+((n+1)/p)}$ is best possible. At the same time, it also shows that the inclusion $B^{\alpha(<1+(1/p))} \subset L_a^p$ is strict, because

$$\mathcal{B}^{\alpha(<1+(1/p))} \subset \mathcal{B}^{1+((n+1)/p)-\varepsilon} \quad \text{for } \varepsilon \leq \frac{n}{p}$$

leads to $f_t \notin \mathcal{B}^{\alpha(<1+(1/p))}$.

Let $t = n/p$. By (12) and (15), we get

$$f_{n/p} \notin H^p \quad \text{but } f_{n/p} \in \mathcal{B}^{1+(n/p)}.$$

They mean the inclusion $H^p \subset \mathcal{B}^{1+(n/p)}$ is strict.

Let $t = (n+1)/p$. By (12) and (17), we get

$$f_{(n+1)/p} \notin L_a^P \quad \text{but } f_{(n+1)/p} \in \mathcal{B}^{1+((n+1)/p)}.$$

Thus, the inclusion $L_a^p \subset \mathcal{B}^{1+((n+1)/p)}$ is strict.

Finally we prove the inclusions at the left sides of (a) are best possible.

Corollary 1.9 of [**10**] states that $\mathcal{B}^1$ is not contained in H^p. Therefore $\mathcal{B}^{\alpha(<1)} \subset H^p$ is best possible for H^p and $\mathcal{B}^\alpha$. In fact, we will see that some spaces are inserted between H^p and $\mathcal{B}^\alpha_{0<\alpha<1}$ later.

For $0 < p < \infty$, let

$$f_p(z) = \sum_{k=1}^{\infty} P_{n_k}(z) = \sum_{k=1}^{\infty} 2^{k/p} W_{2^k}(z)$$

where $\{W_{2^k}(z)\}$ is a sequence of Ryll-Wojtaszczyk polynomials with Hadamard gaps in Theorem 1.2 of [**10**] and Corollary 1 of [**13**]: $\|W_{2^k}\|_\infty = 1$ and $\|W_{2^k}\|_p \geq C(n,p)$. Since

$$\|P_{n_k}\|_\infty = 2^{k/p}\|W_{2^k}\|_\infty = (2^k)^{1+(1/p)-1}$$

for all $k \geq 1$, thus $f_p \in \mathcal{B}^{1+(1/p)}$ by Proposition 2. On the other hand, for each $0 < p < \infty$, by Corollary 1 of [**13**], we have

$$\begin{aligned}\sum_{k=0}^{\infty} 2^{-k} \sum_{n_j \in I_k} \|P_{n_j}\|_p^p &= \sum_{k=1}^{\infty} 2^{-k} \cdot (2^{k/p})^p \|W_{2^k}\|_p^p \\ &\geq C(n,p) \sum_{k=1}^{\infty} 1 = \infty.\end{aligned}$$

By Proposition 3, $f_p \notin L_a^p$. This shows that $\mathcal{B}^{\alpha(<1+(1/p))} \subset L_a^p$ is best possible.

The proof of the Theorem is finished. □

Corollary 3. *For the unit ball B of $\mathbf{C}^n$, we have*

(i)

$$\mathcal{B}^1 \subset \bigcap_{0<p<\infty} L_a^p \subset \mathcal{B}^{\alpha(>1)};$$

$$\mathcal{B}^{\alpha(<1)} \subset \bigcap_{0<p<\infty} H^p \subset \mathcal{B}^{\alpha(>1)}.$$

(ii) *For $\mathcal{B}^\alpha$ and $\cap_{0<p<\infty} L_a^p$, $\cap_{0<p<\infty} H^p$, all of the inclusions in* (i) *are strict and best possible in the sense that the index α of $\mathcal{B}^\alpha$ cannot be increased* (*reduced*) *further.*

Proof. It is easy to see that the inclusions in (i) hold from Theorem (a).

Next we prove the conclusion (ii). For $1 < \alpha < \infty$, let

$$f_\alpha(z) = \sum_{k=1}^{\infty} P_{n_k}(z) = \sum_{k=1}^{\infty} 2^{k(\alpha-1)} W_{2^k}(z)$$

where $\{W_{2^k}(z)\}$ is a sequence of Ryll-Wojtaszczyk polynomials with Hadamard gaps as mentioned above. Similar to the argument about the best possible of "$\mathcal{B}^{\alpha(<1+(1/p))} \subset L_a^p$" in the Theorem, we get

$$f_\alpha \in \mathcal{B}^{\alpha(>1)} \quad \text{but } f_\alpha \notin L_a^p$$

provided that $p = 1/(\alpha - 1)$, and thus $f_\alpha \notin \cap_{0<p<\infty} L_a^p$. Of course, we know $f_\alpha \notin \cap_{0<p<\infty} H^p$. Therefore the strictness of both inclusions at the right sides in (i) is proved.

In [**1**] it was shown that $g \in \cap_{0<p<\infty} H^p(D)$ exists but $g \notin \mathcal{B}^1(D)$. Let

$$f(z_1, \dots, z_n) = g(z_1).$$

It follows from 1.4.5(2) of [**9**] that $f \in \cap_{0<p<\infty} H^p(B)$. On the other hand, we see $f \notin \mathcal{B}^1(B)$, since for a function f depending only on z_1,

$$f \in \mathcal{B}^1(B) \quad \text{if and only if } g \in \mathcal{B}^1(D),$$

see 3.7(1) of [**12**]. This proves the strictness of both inclusions at the left sides in (i). At the same time, it shows that $\cap_{0<p<\infty} H^p \subset \mathcal{B}^{\alpha(>1)}$ is best possible in the sense that the index α of $\mathcal{B}^\alpha$ cannot be reduced further.

From Corollary 1.9 of [**10**], $\mathcal{B}^{\alpha(<1)} \subset \cap_{0<p<\infty} H^p$ is best possible in the sense that the index α of $\mathcal{B}^\alpha$ cannot be increased further.

It follows that the inclusions $\mathcal{B}^1 \subset \cap_{0<p<\infty} L^p_a \subset \mathcal{B}^{\alpha(>1)}$ are best possible immediately from their strictness.

5. Concluding remarks. Based on [**7**], in [**8**], we introduced a class of function spaces $Q_p(B)$ and $Q_{p,0}(B)$, associated with the Green's function for the unit ball of $\mathbf{C}^n$ and proved that $Q_p = \text{Bloch}$, $Q_{p,0} = \text{little Bloch}$ when $1 < p < n/(n-1)$, $Q_1 = \text{BMOA}$ and $Q_{1,0} = \text{VMOA}$. This fact makes it possible for us to deal with Bloch (little Bloch) and BMOA (VMOA) spaces in a unified expression.

By Corollary 1 and the definition of VMOA, it is easy to see that $\mathcal{B}^{\alpha(<1)} \subset \text{VMOA}$.

Summarizing the results in this article, for $0 < p < \infty$, we can get the diagram as follows

$$\begin{array}{ccccccccc}
\text{little Bloch} = \underset{1<p<\frac{n}{n-1}}{Q_{p,0}} & \supset & Q_{1,0} & \subset & Q_1 & \subset & Q_{p,1<p<\frac{n}{n-1}} = \text{Bloch} & & \\
 & & \| & & \| & & \cap & & \\
 & & \text{VMOA} & \subset & \text{BMOA} & \subset & \mathcal{B}^\alpha_{1\le\alpha<1+\frac{1}{p}} & & \\
 & & \cup & & \cap & & \cap & & \\
 & & \mathcal{B}^\alpha_{0<\alpha<1} & \subset & H^p & \subset & L^p_a & \subset & \mathcal{B}^{1+\frac{n+1}{p}} \\
 & & & & \cap & & & & \\
 & & & & \mathcal{B}^{1+\frac{n}{p}}. & & & &
\end{array}$$

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REFERENCES

1. D.M. Campbell, *Functions in all H^p spaces, $p < \infty$*, Canad. Math. Bull. **25** (1982), 110–113.

2. P.L. Durem, *Theory of H^p spaces*, Academic Press, New York, 1970.

3. G.H. Hardy and J.E. Littlewood, *Some properties of fractional integrals* II, Math. Z. **34** (1932), 403–439.

4. M. Mateljević and M. Pavlović, *L^p-behavior of power series with positive coefficients and Hardy spaces*, Proc. Amer. Math. Soc. **87** (1983), 309–316.

5. J. Miao, *A property of analytic functions with Hadamard gaps*, Bull. Austral. Math. Soc. **45** (1992), 105–112.

6. C.H. Ouyang, *Proper inclusions of function spaces and Ryll-Wojtaszczyk polynomials on the unit ball of* $\mathbf{C}^n$, to appear.

7. C.H. Ouyang, W.S. Yang and R.H. Zhao, *Characterizations of Bergman spaces and Bloch space in the unit ball of* $\mathbf{C}^n$, Trans. Amer. Math. Soc. **347** (1995), 4301–4313.

8. ———, *Möbius invariant Q_p spaces associated with the Green's function on the unit ball of* $\mathbf{C}^n$, Pacific J. Math. **182** (1998), 69–99.

9. W. Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$, Springer-Verlag, New York, 1980.

10. J. Ryll and P. Wojtaszczyk, *On homogeneous polynomials on a complex ball*, Trans. Amer. Math. Soc. **276** (1983), 107–116.

11. J.H. Shi, *Inequalities for the integral means of holomorphic functions and their derivatives in the unit ball of* $\mathbf{C}^n$, Trans. Amer. Math. Soc. **328** (1991), 619–637.

12. R.M. Timoney, *Bloch functions in several complex variables*, I, Bull. London Math. Soc. **12** (1980), 241–267.

13. D. Ullrich, *Radial divergence in* BMOA, Proc. London Math. Soc. **68** (1994), 145–160.

14. S. Yamashita, *Gap series and α-Bloch functions*, Yokohama Math. J. **28** (1980), 31–36.

15. W.S. Yang, *Characterizations of α-Bloch spaces*, Acta Math. Sci. **17** (1997), 471–477.

16. Kehe Zhu, *Operator theory in function spaces*, Marcel Dekker, Inc., New York, 1990.

17. W.X. Zhuo, *Derivatives of Bloch functions and α-Carleson measure on the unit ball*, Acta Math. Sci. (Chinese edition) **14** (1994), 351–360.

18. A. Zygmund, *Trigonometric series*, Cambridge Univ. Press, London, 1959.

Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, P.O. Box 71010, Wuhan 430071, P.R. China
Current E-mail address: wy7639@csc.albany.edu

Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, P.O. Box 71010, Wuhan 430071, P.R. China
E-mail address: ouyang@wipm.whcnc.ac.cn