ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 30, Number 3, Fall 2000

A THEOREM ON TRANSCENDENCE OF INFINITE SERIES

M.A. NYBLOM

1. Introduction. There are a number of sufficient conditions known within the literature for an infinite series, $\sum_{n=1}^{\infty} 1/a_n$, of positive rationals to converge to an irrational number (see [**3**], [**4**], [**11**], [**10**] and the references cited therein). These conditions, which are quite varied in form, share one common feature, namely, they all require rapid growth of the sequence $\{a_n\}$ to deduce irrationality of the series. As an illustration consider the following results of Sándor which have been taken from [**11**] and [**12**].

Theorem 1.1. *Let $\{a_m\}$, $m \geq 1$, be a sequence of positive integers such that*

$$\limsup_{m\to\infty} \frac{a_{m+1}}{a_1 a_2 \cdots a_m} = \infty \quad \text{and} \quad \liminf_{m\to\infty} \frac{a_{m+1}}{a_m} > 1.$$

Then the sum of the series $\sum_{m=1}^{\infty} 1/a_m$ is an irrational number. Alternatively, if $\{a_m\}$ and $\{b_m\}$ are a sequence of positive integers with $b_m | b_{m+1}$, $b_m \to \infty$ and $\lambda > 2$ exists such that $b_N^{\lambda} \sum_{m>N} a_m/b_m < 1$, for infinitely many N, then the sum of the series $\sum_{m=1}^{\infty} a_m/b_m$, when convergent, is a transcendental number.

In view of the fact that all algebraic numbers cannot be approximated by infinitely many rationals m/n to within $1/n^r$ for any $r \in N\backslash\{0\}$, one possible approach to demonstrating the transcendence of a given series having sum s would be to produce a sequence of rapidly converging rational approximations to s, for example, using the partial sums of the series. Such an approximation, in the absence of methods for accelerating the convergence of a series, may still be achieved if the sequence $\{a_n\}$ has sufficiently strong growth as in Theorem 1.1. In this paper we do precisely this by showing that, under the following growth

Received by the editors on March 21, 1999, and in revised form on July 29, 1999.

condition,

$$\liminf_{m\to\infty}\frac{a_{m+1}}{a_m^m} > 1, \tag{1}$$

the series $\sum_{m=1}^{\infty} 1/a_m$ will converge to a transcendental number. Despite the severity of the above assumption, one can still find specific examples within the family of series determined by (1). In particular, we prove as a direct corollary of the main result in Section 2 (see Theorem 2.1) the transcendence of $\sum_{m=1}^{\infty} 1/U_{m!}$ and $\sum_{m=1}^{\infty} 1/V_{m!}$, where $\{U_m\}$ and $\{V_m\}$ are the sequence of generalized Fibonacci and Lucas numbers, respectively. In addition, an unexpected consequence of Theorem 2.1 is deduced in which every convergent infinite series of positive rationals, $\sum_{m=1}^{\infty} 1/a_m$, with $a_n \in N\backslash\{0\}$, has infinitely many disjoint subseries (to be defined) having transcendental sums.

2. Main result. To prove the main result of this paper, we will first require a preliminary lemma, which incidentally shows that the first group of conditions in Theorem 1.1 is implied by (1).

Lemma 2.1. *Suppose that* $\{a_m\}_{m=1}^{\infty}$ *is a sequence of reals greater than unity and such that*

$$\liminf_{m\to\infty}\frac{a_{m+1}}{a_m^m} > 1.$$

If n *is any fixed positive integer and* $b_m = (a_1 a_2 \cdots a_m)^n/a_{m+1}$*, then* $b_m = o(1)$ *as* $m \to \infty$.

Proof. By the above assumption, $\delta > 0$ must exist such that for all $m' > N(\delta)$, say, $a_{m'+1}/a_{m'}^{m'} \geq (1+\delta)$. Choose a fixed integer $p > \max\{N(\delta), n\}$ and for $m > p$ consider $b'_m = (a_p \cdots a_m)^n/a_{m+1}$. The result will follow upon showing that $b'_m \to 0$. To this end consider

$\log(1/b'_m)$. Now

$$\begin{aligned}\log(1/b'_m) &= \sum_{r=p}^{m}(\log a_{r+1} - \log a_r) + \log a_p - n\sum_{r=p}^{m}\log a_r \\ &= \sum_{r=p}^{m}\log\left(\frac{a_{r+1}}{a_r^{n+1}}\right) + \log a_p \\ &\geq \sum_{r=p}^{m}\log\left(\frac{a_{r+1}}{a_r^{n+1}}\right).\end{aligned}$$

However, since $p > \max\{N(\delta), n\}$, one has for each $r = p, \dots, m$,

$$\log\left(\frac{a_{r+1}}{a_r^{n+1}}\right) \geq \log\left(\frac{a_{r+1}}{a_r^{r}}\right) \geq \log(1+\delta).$$

Consequently, $\log(1/b'_m) \geq (m-p+1)\log(1+\delta) \to \infty$ as $m \to \infty$. □

Using this lemma one can deduce the following result.

Theorem 2.1. *Suppose* $\{a_m\}_{m=1}^{\infty}$ *is a sequence of integers greater than unity and such that*

$$\liminf_{m\to\infty}\frac{a_{m+1}}{a_m^m} > 1.$$

Then the series $\sum_{m=1}^{\infty} 1/a_m$ *converges to a transcendental number.*

Proof. From the assumption it is clear that the series under consideration is convergent. Choosing k sufficiently large so that $\xi_1 = \sum_{r=k}^{\infty} 1/a_r < 1$, it will suffice to demonstrate the transcendence of ξ_1. Thus, assume the contrary and suppose that ξ_1 is an algebraic number of degree n. Let

$$f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

be a polynomial over the integers with $c_n \neq 0$ such that $f(\xi_1) = 0$, and set $L = \sum_{r=1}^{n} r|c_r|$. Via the assumption, an integer $N > 0$ must exist such that $a_{m+1}/a_m^m > 1$ for all $m > N$; moreover, as $a_m \uparrow \infty$ and so $a_k \cdots a_m < a_1 \cdots a_m$, we may by Lemma 2.1 choose a fixed $m > \max\{k, N\}$ such that $a_m > 2$ with

$$0 < 2L\frac{(a_k \cdots a_m)^n}{a_{m+1}} < 1. \tag{2}$$

In addition, define $\xi_2 = \sum_{r=k}^{m} 1/a_r$, which can be expressed in the form $\xi_2 = A/(a_k \cdots a_m)$ for some positive integer A. We shall examine the following nonnegative real number

$$M = |f(\xi_1) - f(\xi_2)|(a_k \cdots a_m)^n.$$

Firstly, $M > 0$ since if $M = 0$, that is, if $f(\xi_2) = 0$, then $f(x) = (x - \xi_2)q(x)$ where $q(x) \in Q[x]$ has degree $n - 1$ and as $\xi_1 \neq \xi_2$ one would have to conclude that $q(\xi_1) = 0$, which contradicts the degree of the assumed algebraic number ξ_1. Furthermore, from the definition of M it is clear that

$$\begin{aligned} M &= |f(\xi_2)|(a_k \cdots a_m)^n \\ &= \left| c_n \frac{A^n}{(a_k \cdots a_m)^n} + c_{n-1}\frac{A^{n-1}}{(a_k \cdots a_m)^{n-1}} + \cdots + c_0 \right| (a_k \cdots a_m)^n \end{aligned}$$

and so M is a positive integer. We now obtain an upper bound for M. Writing $f(\xi_1)-f(\xi_2)$ in terms of a difference of two polynomials, observe after an application of the triangle inequality and noting $\xi_2 < \xi_1 < 1$ that

$$\begin{aligned} |f(\xi_1) - f(\xi_2)| &= \left| \sum_{r=1}^{n} c_r(\xi_1^r - \xi_2^r) \right| \\ &= \left| \sum_{r=1}^{n} c_r(\xi_1 - \xi_2) \sum_{l=1}^{r} \xi_1^{r-l}\xi_2^{l-1} \right| \\ &\le (\xi_1 - \xi_2) \sum_{r=1}^{n} |c_r| \left(\sum_{l=1}^{r} \xi_1^{r-l}\xi_2^{l-1} \right) \\ &\le (\xi_1 - \xi_2) \sum_{r=1}^{n} |c_r| \sum_{l=1}^{r} 1 \\ &= (\xi_1 - \xi_2)L. \end{aligned}$$

From the above construction, one has $a_{r-1}/a_r < 1/a_{r-1}^{r-2}$ for $r = m+2, m+3, \dots$. Consequently, as $a_{m+1} > a_m > 2$, observe that

$$\begin{aligned}\xi_1 - \xi_2 &= \frac{1}{a_{m+1}}\left(1 + \sum_{r=m+2}^{\infty} \frac{a_{m+1}}{a_r}\right)\\ &< \frac{1}{a_{m+1}}\left(1 + \sum_{r=m+2}^{\infty} \frac{a_{r-1}}{a_r}\right)\\ &< \frac{1}{a_{m+1}}\left(1 + \sum_{r=m+2}^{\infty} \frac{1}{a_{r-1}^{r-2}}\right)\\ &< \frac{1}{a_{m+1}}\left(1 + \sum_{r=m+2}^{\infty} \frac{1}{a_{m+1}^{r-2}}\right)\\ &< \frac{1}{a_{m+1}}\left(1 + \sum_{r=m+2}^{\infty} \frac{1}{2^{r-2}}\right)\\ &< \frac{2}{a_{m+1}}.\end{aligned}$$

Thus, as a result of our choice of m in (2.1) we have

$$|f(\xi_1) - f(\xi_2)|(a_k \cdots a_m)^n < 2L\frac{(a_k \cdots a_m)^n}{a_{m+1}} < 1.$$

Hence, we have produced an integer M such that $0 < M < 1$; a contradiction. Therefore, as n was arbitrarily chosen, ξ_1 cannot be an algebraic number to any degree. □

To conclude this section we quickly illustrate how the above result can be used to provide a partial answer to a question of John Brillhart (see [**2**, Problem 5]) in which sufficient conditions were sought on a sequence $n_k \in N$ such that the series of the form $\sum_{k=1}^{\infty} 1/n_1 n_2 \cdots n_k$ would sum to a transcendental number. Clearly, if we define $a_k = n_1 n_2 \cdots n_k$, then by Theorem 2.1, one such sufficient condition is that the $\liminf_{k\to\infty} n_{k+1}/(n_1 n_2 \cdots n_k)^{k-1} > 1$.

3. Some comparisons. We examine now some immediate consequences of Theorem 2.1 which show the similarity of our results with those of Sandor and Erdös who provided sufficient conditions for irrational and transcendental valued series. The following result is an easy consequence of the main theorem and mirrors Theorem 1.1 of Sandor.

Corollary 3.1. *Let $\{a_m\}_{m=1}^{\infty}$ be a sequence of positive integers greater than unity and such that*

$$c_m = \frac{a_{m+1}}{(a_1 \cdots a_m)^m}$$

is strictly monotone increasing for all but finitely many m. Then the series $\sum_{m=1}^{\infty} 1/a_m$ converges to a transcendental number.

Proof. As the sequence $\{c_m\}_{m=1}^{\infty}$ is eventually strictly monotone increasing we have that $c_{m-1} < c_m$ for m sufficiently large. Upon simplifying this inequality, we find

$$(a_1 \cdots a_m) < \frac{a_{m+1}}{a_m^m},$$

from which it is immediately apparent that $a_{m+1}/a_m^m \to \infty$ as $m \to \infty$. Hence the condition of Theorem 2.1 is satisfied. □

Consider now the following results of Erdös taken from [**3**] and [**5**].

Theorem 3.1. *Let $a_1 < a_2 < \cdots < a_m < \cdots$ be a sequence of positive integers such that*

$$\limsup_{m\to\infty} a_m^{1/2^m} = \infty \quad \text{and} \quad a_m > m^{1+\varepsilon}$$

for a number $\varepsilon > 0$ and for every $m > m_0(\varepsilon)$. Then the sum of the series $\sum_{m=1}^{\infty} 1/a_m$ is a transcendental number.

Theorem 3.2. *If $n_1 < n_2 < \cdots < n_k < \cdots$ is a sequence of positive integers such that*

$$\limsup_{k\to\infty} \frac{\log(n_k)}{\log(k)} = \infty,$$

then for any integer $t \geq 2$, the sum of the series $\sum_{k=1}^{\infty} t^{-n_k}$ is a transcendental number.

In connection with these results of Erdös we now state two further corollaries.

Corollary 3.2. *Let* $\{a_m\}_{m=1}^{\infty}$ *be a sequence of integers greater than unity and such that*

$$c_m = a_m^{1/m^m}$$

is strictly monotone increasing for all but finitely many m*. Then the series* $\sum_{m=1}^{\infty} 1/a_m$ *converges to a transcendental number.*

Proof. By the above assumption one has that $c_m < c_{m+1}$ for m sufficiently large. Raising both sides of this inequality to the power $(m+1)^{m+1}$ we find upon rearrangement that

$$a_m^{b_m} < \frac{a_{m+1}}{a_m^m},$$

where $b_m = m(((m+1)/m)^{m+1} - 1) \sim m(e-1)$ as $m \to \infty$. Thus, as $a_m > 1$ one has $a_m^{b_m} \to \infty$ as $m \to \infty$ and so the condition of Theorem 2.1 is satisfied. □

Corollary 3.3. *Suppose* $\{n_k\}$ *is a strictly monotone increasing sequence of positive integers such that*

$$\liminf_{k\to\infty}(n_{k+1} - kn_k) > 0.$$

Then for any integer $t \geq 2$ *the series* $\sum_{k=1}^{\infty} t^{-n_k}$ *converges to a transcendental number.*

Proof. For the integer t and sequence $\{n_k\}$ define $a_k = t^{n_k}$. Via the assumption, $\delta > 0$ exists such that $n_{k+1} - kn_k \geq \delta$ for $k > N(\delta)$, say. Hence, for all $k > N(\delta)$, we have that

$$\frac{a_{k+1}}{a_k^k} = t^{n_{k+1} - kn_k} \geq t^{\delta} \geq 2^{\delta} > 1.$$

Consequently, the condition of Theorem 2.1 is again satisfied. □

4. Disjoint subseries with transcendental sums. The following definition of disjoint subsequences and subseries is taken from [**6**].

Definition 4.1. We say that $\sum_{m=1}^{\infty} v_m$ is a subseries of a given series $\sum_{m=1}^{\infty} u_m$ if $\{v_m\}$ is a subsequence of the sequence $\{u_n\}$. Two subseries $\sum_{m=1}^{\infty} v_m$ and $\sum_{m=1}^{\infty} w_m$ of the same series $\sum_{m=1}^{\infty} u_m$ are disjoint if $\{w_m\}$ is a subsequence of the sequence $\{u_n\}$ from which the subsequence $\{v_m\}$ has been taken, that is, $\{w_m\}$ and $\{v_m\}$ are disjoint subsequences of the sequence $\{u_n\}$.

As an application of Theorem 2.1, we shall demonstrate the following unexpected result.

Theorem 4.1. *Every convergent infinite series of positive rationals $\sum_{m=1}^{\infty} 1/a_m$ where $a_m \in N\backslash\{0\}$, has infinitely many disjoint subseries converging to a transcendental number.*

Proof. Let $\sum_{m=1}^{\infty} 1/a_m$ be a convergent series of positive rationals. Then the sequence $\{1/a_m\}$ tends to zero when $m \to \infty$. Thus, as $a_m \to \infty$ as $m \to \infty$, one can find infinitely many disjoint subsequences, $\{a_{m(k)}^{(p)}\}_{k=1}^{\infty}$ of the sequence $\{a_m\}_{m=1}^{\infty}$ where $p = 1, 2, \ldots$, and such that

$$\frac{a_{m(k+1)}^{(p)}}{(a_{m(k)}^{(p)})^{m(k)}} > k.$$

Consequently, the subsequence $\{a_{m(k)}^{(p)}\}_{k=1}^{\infty}$ satisfies the condition of Theorem 2.1 and so $\sum_{k=1}^{\infty} 1/a_{m(k)}^{(p)}$ converges to a transcendental number. □

The above result is similar to a proposition in [**1**] where it was shown that every convergent series of positive rational $\sum_{m=1}^{\infty} b_m/a_m$ had infinitely many disjoint subseries with irrational sums.

Remark 4.1. There is (see [**9**]) a convergent series of positive rationals such that all its subseries have transcendental sums. Consequently, one cannot substitute the word transcendental by the word rational in the previous result. This is in contrast with the known fact, see [**11**], that if $\sum_{m=1}^{\infty} a_m$ is a divergent series with $a_m \to 0$ as $m \to \infty$, then every positive real number may be given as the sum of a convergent subseries of the original series.

5. Application. In this section we shall demonstrate, via Theorem 2.1, the transcendence of a family of series involving the generalized Fibonacci and Lucas sequences, denoted by U_m and V_m, respectively. These sequences can be defined as follows. Let (P, Q) be a relatively prime pair of integers such that the roots α and β of $x^2 - Px + Q = 0$ are distinct. Then U_m, V_m are given by

$$U_m = \frac{\alpha^m - \beta^m}{\alpha - \beta} \quad \text{and} \quad V_m = \alpha^m + \beta^m.$$

It is well known that, when the discriminant $\Delta = P^2 - 4Q > 0$, both $\{U_m\}$ and $\{V_m\}$ are an increasing sequence of positive integers. In particular, for $(P, Q) = (1, -1)$, one has $U_m = F_m$ and $V_m = L_m$ where F_m and L_m are the Fibonacci and Lucas numbers, respectively. The transcendence of the following series

$$\sum_{m=1}^{\infty} \frac{1}{m! F_{2^m}}$$

was proved independently by Mahler [**7**] and Mignotte [**8**]. As a related result, we now establish the transcendence of $\sum_{m=1}^{\infty} 1/U_{m!}$ and $\sum_{m=1}^{\infty} 1/V_{m!}$.

Corollary 5.1. *Let (P, Q) be a relatively prime pair of integers with $P > |Q + 1|$ and $Q \neq 1$ and $\{U_m\}, \{V_m\}$ the associated generalized Fibonacci and Lucas sequences. If $a_m = U_{m!}$ or $a_m = V_{m!}$, then $\sum_{m=1}^{\infty} 1/a_m$ converges to a transcendental number.*

Remark 5.1. We note that the restriction on Q is required as the sequence $\{U_m\}$ will contain infinitely many zero elements when $(P, Q) = (1, 1)$.

Proof. In view of Theorem 2.1, it will suffice to demonstrate in either case that $a_{m+1}/a_m^m \to \infty$ as $m \to \infty$. Clearly, from definition $\alpha = (P + \sqrt{\Delta})/2$ and $\beta = (P - \sqrt{\Delta})/2$ where $\Delta = P^2 - 4Q$. Now from assumption $\sqrt{\Delta} > \sqrt{(Q+1)^2 - 4Q} = |Q - 1| > 1$ and so

$$|\beta| = \left| \frac{P - \sqrt{\Delta}}{2} \right| = \frac{|2Q|}{P + \sqrt{\Delta}} < \frac{|2Q|}{|Q+1| + |Q-1|} = 1,$$

noting here that the righthand equality holds for all $Q \in R$ with $|Q| \geq 1$. Consequently, $|\alpha| = |Q|/|\beta| > |Q| \geq 1$ and $|\beta/\alpha| < 1$. Now, in the case when $a_m = U_{m!}$ observe

$$\frac{a_{m+1}}{a_m^m} > \alpha^{m!}(\sqrt{\Delta})^{m-1}\Big(1-(\beta/\alpha)^{(m+1)!}\Big) \sim \alpha^{m!}(\sqrt{\Delta})^{m-1},$$

as $m \to \infty$. While, in the latter case,

$$\frac{a_{m+1}}{a_m^m} > \alpha^{m!}\frac{(1+(\beta/\alpha)^{(m+1)!})}{(1+|\beta/\alpha|/m)^m} \sim \alpha^{m!}\, e^{-|\beta/\alpha|},$$

as $m \to \infty$. Hence, in both cases, a_{m+1}/a_m^m will grow unboundedly with increasing m. $\square$

REFERENCES

1. C. Badea, *The irrationality of certain infinite series*, Glasgow Math. J. **29** (1987), 221–228.

2. J. Brillhart, *Research problems* (A.A. Gioia and M.V. Subbarao, eds.), Rocky Mountain J. Math. **15** (1985), 385–388.

3. P. Erdös, *Some problems and results on the irrationality of the sum of infinite series*, J. Math. Sci. **10** (1975), 1–7.

4. P. Erdös and R.L. Graham, *Old and new problems and results in combinatorial number theory*, Imprimetrie Kunding, 1980.

5. P. Erdös and E.G. Straus, *Problem* 154, Elem. Math. **9** (1954), 18.

6. B. Gelbaum, *Problems in analysis*, Springer, Berlin, 1982.

7. K. Mahler, *On the transcendence of the solutions of a special class of functional equations*, Bull. Austral. Math. Soc. **13** (1975), 389–410.

8. M. Mignotte, *Quelques Problèmes d'Effectivité en Théorie des Nombres*, Thesis, Univ. Paris XIII, Paris, 1974.

9. M.A. Nyblom, *A note on a family of transcendental valued series*, submitted.

10. A. Oppenheim, *The irrationality or rationality of certain infinite series*, in *Studies in Pure Math.* (presented to Richard Rado) Academic Press, London, 1971, 195–201.

11. J. Sándor, *Some classes of irrational numbers*, Studia Univ. Babeş-Bolyai Math. **29** (1984), 3–12.

12. ———, *On some irrational factorial series*, Studia Univ. Babeş-Bolyai Math. **32** (1987), 13–17.

Department of Mathematics, Royal Melbourne Institute of Technology, GPO Box 2476V, Melbourne, Victoria 3001, Australia
E-mail address: michael.nyblom@rmit.edu.au