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UNIVERSAL BINARY POSITIVE DEFINITE HERMITIAN LATTICES

HIROMU IWABUCHI

ABSTRACT. We will determine all universal integral lattices on binary positive definite Hermitian spaces over arbitrary imaginary quadratic fields, where a positive definite lattice is said to be 'universal' if it represents all positive rational integers. A.G. Earnest and A. Khosravani determined universal binary Hermitian lattices when the imaginary quadratic fields have class number 1. In this paper we will extend the result to the case of fields with arbitrary class numbers and obtain nine new universal binary Hermitian lattices up to equivalence, including nonfree lattices.

1. Introduction. Lagrange [**5**] proved that any positive rational integer is a sum of four squares. In other words, the quaternary quadratic form $f(x,y,z,w) = x^2 + y^2 + z^2 + w^2$ represents all positive rational integers. We call such a positive definite form a *universal* form. Ramanujan [**7**] showed that there are 55 universal diagonal quaternary quadratic forms in all. More generally, we know that there are only a finite number of universal integral forms with cross products but have not yet determined them completely, when they have an odd cross product. Recently, Earnest and Khosravani [**2**] investigated the similar problem for binary classic integral Hermitian forms over the imaginary quadratic fields of class number one and determined the classes of universal forms in the case. We say a lattice or a form is *classic integral* if its scale ideal is integral. As for the representation of numbers, binary Hermitian forms over quadratic fields can be regarded as quaternary quadratic forms over the rational number field. Accordingly, universal binary Hermitian forms provide universal quaternary quadratic forms, which may not be classic integral whereas Hermitian forms are classic. In this paper we will determine all universal binary Hermitian lattices for arbitrary imaginary quadratic fields.

In Section 2 we will give a correspondence of Hermitian lattices to quadratic lattices. In Section 3 we will obtain universal and potentially

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universal lattices by checking the representability of a few certain integers and state the main theorem which lists all universal Hermitian lattices for imaginary quadratic fields with class number greater than or equal to 2. Then we will show a proposition which is needed to prove the theorem. In the last section we will show the universality of some nonclassic quadratic forms to complete the proof of the theorem.

2. Hermitian lattices and quadratic lattices. We establish the correspondence of a quadratic form over the rational number field $\mathbf{Q}$ to a Hermitian form over an imaginary quadratic field. Notation for lattices and spaces is due to O'Meara's book [**6**]. Let F denote an imaginary quadratic field over $\mathbf{Q}$ and $m < 0$ be a square-free integer for which $F = \mathbf{Q}(\sqrt{m})$. We write a mapping: $x \to \bar{x}$ for the nontrivial involutive $\mathbf{Q}$-automorphism of F. The ring $\mathfrak{o}$ of integers in F has a basis $[1, \omega]$ as a $\mathbf{Z}$-module of rank 2, where $\omega := \omega_{-m} = (1+\sqrt{m})/2$ if $m \equiv 1 \pmod 4$ or $\omega_{-m} = \sqrt{m}$ if $m \equiv 2, 3 \pmod 4$. Let V be an n-dimensional Hermitian space over F with nondegenerate Hermitian form H and L an $\mathfrak{o}$-lattice on V. Consider the corresponding $2n$-dimensional quadratic space $\tilde{V}$ over $\mathbf{Q}$ to V as defined in Jacobson [**4**]. Taking an F-basis $[u_1, u_2, \dots, u_n]$ of V, we can regard V as a $2n$-dimensional vector space $\tilde{V}$ over $\mathbf{Q}$ with a basis $[u_1, \omega u_1, \dots, u_n, \omega u_n]$. We associate a symmetric bilinear form $B(x,y) = \{H(x,y)+H(y,x)\}/2$ to $\tilde{V}$ as to be a quadratic space over $\mathbf{Q}$. The associated quadratic form Q is also defined by $Q(u) := B(u,u)$ for $u \in \tilde{V}$. A Hermitian lattice L on V is a finitely generated $\mathfrak{o}$-module in V, which is represented as $L = \mathfrak{p}_1 u_1 + \mathfrak{p}_2 u_2 + \cdots + \mathfrak{p}_n u_n$ for fractional ideals $\mathfrak{p}_i$ of F and an F-basis $[u_1, \dots, u_n]$. Each $\mathfrak{p}_i$ has a $\mathbf{Z}$-basis $[\alpha_i, \beta_i]$. Using the above correspondence of $\tilde{V}$ to V, we obtain a quadratic $\mathbf{Z}$-lattice $\tilde{L}$ on $\tilde{V}$:

$$\tilde{L} = \mathbf{Z}\alpha_1 u_1 + \mathbf{Z}\beta_1 u_1 + \mathbf{Z}\alpha_2 u_2 + \mathbf{Z}\beta_2 u_2 + \cdots + \mathbf{Z}\alpha_n u_n + \mathbf{Z}\beta_n u_n.$$

We let $\mathfrak{v}L$ and $d\tilde{L}$ denote the volume ideal of L and the discriminant of the corresponding $\tilde{L}$, respectively. Then we have

$$\mathfrak{v}L = \det\left(H(u_i, u_j)\right) \cdot \mathfrak{p}_1\bar{\mathfrak{p}}_1 \cdot \mathfrak{p}_2\bar{\mathfrak{p}}_2 \cdots \mathfrak{p}_n\bar{\mathfrak{p}}_n,$$

and

$$d\tilde{L} = \{\text{disc }(F/\mathbf{Q})/4\}^2 N(\mathfrak{v}L),$$

where disc $(F/\mathbf{Q})$ denotes the discriminant of F over $\mathbf{Q}$, and $N(\mathfrak{a})$ the absolute norm of a fractional ideal $\mathfrak{a}$. Note that $\tilde{L}$ satisfies $Q(\tilde{L}) =$

$H(L)$, and L is a universal binary Hermitian lattice if and only if $\tilde{L}$ is a universal quaternary quadratic lattice. In [**3**], Earnest and Khosravani showed that the discriminant of a universal positive definite integral quaternary quadratic form over $\mathbf{Z}$ cannot exceed 1073/4, which implies that $|\mathrm{disc}\,(F/\mathbf{Q})|$ is less than 65. We look for imaginary quadratic fields with a class number greater than or equal to 2 and a discriminant greater than -65. There are 14 such fields: $\mathbf{Q}(\sqrt{m})$ for $m = -5$, -6, -10, -13, -14, -15, -23, -31, -35, -39, -47, -51, -55, -59.

3. Universal Hermitian lattices. We suppose that L is a universal classic integral Hermitian lattice on a positive definite binary Hermitian space V, where an $\mathfrak{o}$-lattice L is said to be classic integral if $H(u, v) \in \mathfrak{o}$ for any u, v in L. From the universality of L, we see that a vector u exists in L such that $H(u) = 1$ and $\mathfrak{o}u \subset L$ is an orthogonal component of L. We can choose an appropriate vector v in V such that $[u, v]$ is a basis of V and $L = \mathfrak{o}u \perp \mathfrak{p}v$ where $\mathfrak{p}$ is an element of fixed representatives of the ideal class group of F. Then elements $\alpha \in \mathfrak{o}$ and $\beta \in \mathfrak{p}$ exist such that $H(\alpha u + \beta v) = |\alpha|^2 + |\beta|^2 H(v) = 2$, because L represents 2. Put $\mathfrak{N}(\mathfrak{a}) := \{|\alpha|^2 : \alpha \neq 0 \text{ and } \alpha \in \mathfrak{a}\}$ and $m(\mathfrak{a}) = \min \mathfrak{N}(\mathfrak{a})$ for a fractional ideal $\mathfrak{a}$. In any field listed at the end of the last section, the set $\mathfrak{N}(\mathfrak{o})$ does not contain 2, so we have an inequality $H(v) \leqq 2/m(\mathfrak{p})$. Because L is classic integral, we see that $H(\beta_1 v, \beta_2 v) = \beta_1 \bar{\beta}_2 H(v) \in \mathfrak{o}$ for any β_1, β_2 in $\mathfrak{p}$, which provides that $H(v)$ is in $N(\mathfrak{p})^{-1}\mathbf{Z}$. It follows that there are only a few lattices which we have to check on the universality. We will treat a quaternary quadratic form $f(x, y, z, w)$ associated with a quadratic lattice $\tilde{L}$, and check whether it represents integers 2, 3, 5, 13 or not.

First we consider the case where L is a free $\mathfrak{o}$-lattice. We can assume that $L = \mathfrak{o}u + \mathfrak{o}v$, $H(u) = 1, H(u, v) = 0, H(v) = 1$ or 2. If $\mathrm{disc}\,(F/\mathbf{Q})$ is not equal to -20 then the set $\mathfrak{N}(\mathfrak{o})$ contains neither 3 nor 5. If $H(v) = 1$ then L does not represent 3, and if $H(v) = 2$ then L does not represent 5. Thus, no universal lattice exists except for the case of $\mathrm{disc}\,(F/\mathbf{Q}) = -20$ ($m = -5$). In this case, if $H(v) = 1$ then L does not represent 3, and if $H(v) = 2$, then the associated quadratic form with $\tilde{L}$ equals $f(x, y, z, w) = x^2 + 2y^2 + 5z^2 + 10w^2$, which is listed in [**7**].

Next we consider the nonfree case, in which $L = \mathfrak{o}u + \mathfrak{p}v$. An ideal $\mathfrak{p}$ is nonprincipal, and we write it $[\alpha, \beta]$ which is a basis of $\mathfrak{p}$ as a $\mathbf{Z}$-module. We can assume that $\mathfrak{p}$ is integral and that $u \in L$ and $v \in V$ satisfy

$H(u) = 1$, $H(u, v) = 0$, $H(v) \in N(\mathfrak{p})^{-1}\mathbf{Z}$ in what follows.

In the case of $m = -5$, the class number of F is equal to 2, so we set $\mathfrak{p} = [2, 1 + \omega]$, $m(\mathfrak{p}) = 4$. Since $H(v) \in (1/2)\mathbf{Z}$ and $H(v) \leqq 1/2$, $H(v)$ is equal to $1/2$. In this case, the quadratic form associated with $\tilde{L}$ is $f(x, y, z, w) = x^2 + 2y^2 + 2yz + 3z^2 + 5w^2$. It is in one class genus, therefore its universality is proven from local conditions.

In the case of $m = -6$, there is only one universal lattice: $L = \mathfrak{o}u + \mathfrak{p}v$ ($\mathfrak{p} = [2, \omega]$, $H(v) = 1/2$). The quadratic form associated with $\tilde{L}$ is $f(x, y, z, w) = x^2 + 2y^2 + 3z^2 + 6w^2$, which is listed in [**7**].

In the case of $m = -10$, there is only one universal lattice: $L = \mathfrak{o}u + \mathfrak{p}v$ ($\mathfrak{p} = [2, \omega]$, $H(v) = 1/2$). The associated form $f(x, y, z, w) = x^2 + 2y^2 + 5z^2 + 10w^2$ is universal as is listed in [**7**].

In the case of $m = -13$ or -14, no lattices represent all of 2, 3, and 5.

In the case of $m = -15$, $L = \mathfrak{o}u + \mathfrak{p}v$ ($\mathfrak{p} = [2, \omega]$, $H(v) = 1/2$) is potentially universal. The associated form is $f(x, y, z, w) = x^2 + 2y^2 + y^2 + yz + 2z^2 + xw + 4w^2$, which has not been shown to be universal yet. The other lattices are not universal.

In the case of $m = -23$, both $L = \mathfrak{o}u + \mathfrak{p}v$ ($\mathfrak{p} = [2, \omega]$, $H(v) = 1/2$) and $L' = \mathfrak{o}u + \mathfrak{p}v$ ($\mathfrak{p} = [2, \bar{\omega}]$, $H(v) = 1/2$) are potentially universal. They are not equivalent, but their corresponding quadratic lattices are equivalent. The associated quadratic form is $f(x, y, z, w) = x^2 + 2y^2 + yz + 3z^2 + xw + 6w^2$. The other lattices are not universal.

In the case of $m = -31$, both $L = \mathfrak{o}u + \mathfrak{p}v$ ($\mathfrak{p} = [2, 1 + \omega]$, $H(v) = 1/2$) and $L' = \mathfrak{o}u + \mathfrak{p}v$ ($\mathfrak{p} = [2, -2 + \omega]$, $H(v) = 1/2$) are potentially universal. Their associated forms coincide. It is $f(x, y, z, w) = x^2 + 2y^2 + yz + 4z^2 + xw + 8w^2$. The other lattices are not universal.

In the case of $m < -35$, any lattice does not represent at least one of four integers, 2, 3, 5, 13. Thus any more universal lattice does not exist.

Consequently, we obtained four universal and five potentially universal lattices. Here we state our theorem.

Theorem. *Let L be a classic integral Hermitian lattice on a binary positive definite Hermitian space over an arbitrary imaginary quadratic field with class number greater than or equal to 2. If L represents all positive integers, L is equivalent to one of the following lattices*:

$$\begin{array}{lll}
L_1 = \mathfrak{o}u + \mathfrak{o}v & (H(v) = 2) & \text{in } \mathbf{Q}(\sqrt{-5}), \\
L_2 = \mathfrak{o}u + \mathfrak{p}v & (H(v) = 1/2,\ \mathfrak{p} = [2, 1 + \omega_5]) & \text{in } \mathbf{Q}(\sqrt{-5}), \\
L_3 = \mathfrak{o}u + \mathfrak{p}v & (H(v) = 1/2,\ \mathfrak{p} = [2, \omega_6]) & \text{in } \mathbf{Q}(\sqrt{-6}), \\
L_4 = \mathfrak{o}u + \mathfrak{p}v & (H(v) = 1/2,\ \mathfrak{p} = [2, \omega_{10}]) & \text{in } \mathbf{Q}(\sqrt{-10}), \\
L_5 = \mathfrak{o}u + \mathfrak{p}v & (H(v) = 1/2,\ \mathfrak{p} = [2, \omega_{15}]) & \text{in } \mathbf{Q}(\sqrt{-15}), \\
L_6 = \mathfrak{o}u + \mathfrak{p}v & (H(v) = 1/2,\ \mathfrak{p} = [2, \omega_{23}]) & \text{in } \mathbf{Q}(\sqrt{-23}), \\
L_7 = \mathfrak{o}u + \mathfrak{p}v & (H(v) = 1/2,\ \mathfrak{p} = [2, \bar{\omega}_{23}]) & \text{in } \mathbf{Q}(\sqrt{-23}), \\
L_8 = \mathfrak{o}u + \mathfrak{p}v & (H(v) = 1/2,\ \mathfrak{p} = [2, 1 + \omega_{31}]) & \text{in } \mathbf{Q}(\sqrt{-31}), \\
L_9 = \mathfrak{o}u + \mathfrak{p}v & (H(v) = 1/2,\ \mathfrak{p} = [2, -2 + \omega_{31}]) & \text{in } \mathbf{Q}(\sqrt{-31}),
\end{array}$$

where $H(u) = 1,\ H(u, v) = 0$ in any case.

Conversely, these nine lattices represent all positive integers.

The former part of the Theorem has already been verified. To prove the latter part, we have to show that the last five lattices are universal. These are $L = \mathfrak{o}u + \mathfrak{p}v$ $(H(u) = 1,\ H(u, v) = 0,\ H(v) = N(\mathfrak{p})^{-1})$ with which associated Hermitian forms are $|\alpha|^2 + N(\mathfrak{p})^{-1}|\beta|^2$ $(\alpha \in \mathfrak{o}, \beta \in \mathfrak{p})$. The numbers represented by such a form are multiplicatively closed, as to be shown in the following proposition. It plays an important role in the proof of the theorem.

Proposition 1. *If $H(\alpha, \beta) = |\alpha|^2 + N(\mathfrak{p})^{-1}|\beta|^2$ $(\alpha \in \mathfrak{o}, \beta \in \mathfrak{p})$ then*

$$n_1 n_2 \in H(L) \quad \text{for any } n_1, n_2 \in H(L).$$

Proof. Put $p = N(\mathfrak{p})$ for brevity. We call a pair (α, β), $(\alpha \in \mathfrak{o}, \beta \in \mathfrak{p})$ a representation of a number k if $H(\alpha, \beta) = k$. We let (α_i, β_i) be a representation of any elements n_i in $H(L)$, $i = 1, 2$. We obtain that

$$\begin{aligned} n_1 n_2 &= \left(|\alpha_1|^2 + \frac{1}{p}|\beta_1|^2\right)\left(|\alpha_2|^2 + \frac{1}{p}|\beta_2|^2\right) \\ &= |\alpha_1|^2|\alpha_2|^2 + \frac{1}{p^2}|\beta_1|^2|\beta_2|^2 + \frac{1}{p}\{|\alpha_1|^2|\beta_2|^2 + |\alpha_2|^2|\beta_1|^2\} \\ &= |\alpha_1\bar{\alpha}_2 + \frac{1}{p}\beta_1\bar{\beta}_2|^2 + \frac{1}{p}|\alpha_1\beta_2 - \alpha_2\beta_1|^2, \end{aligned}$$

and $\alpha_1\bar{\alpha}_2 + p^{-1}\beta_1\bar{\beta}_2 \in \mathfrak{o}$, $\alpha_1\beta_2 - \alpha_2\beta_1 \in \mathfrak{p}$ because of $\alpha_i \in \mathfrak{o}$, $\beta_i \in \mathfrak{p}$. Hence, $(\alpha_1\bar{\alpha}_2 + p^{-1}\beta_1\bar{\beta}_2, \alpha_1\beta_2 - \alpha_2\beta_1)$ is a representation of $n_1 n_2$. □

Then we only have to show that each form represents all prime integers. This will be done in the following section.

4. Universality of quadratic forms. In the previous section, we obtained potentially universal Hermitian lattices, whose associated quadratic forms are:

$$\begin{aligned} f_{15}(x,y,z,w) &= x^2 + 2y^2 + yz + 2z^2 + xw + 4w^2, \\ f_{23}(x,y,z,w) &= x^2 + 2y^2 + yz + 3z^2 + xw + 6w^2, \\ f_{31}(x,y,z,w) &= x^2 + 2y^2 + yz + 4z^2 + xw + 8w^2. \end{aligned}$$

We can show the universality of the form f_{15} immediately using some of the included ternary forms in f_{15}. We consider the following two ternary forms:

$$\begin{aligned} g_1(x,y,z) &:= f_{15}(x,y,z,0) = x^2 + 2y^2 + yz + 2z^2, \\ g_2(x,y,z) &:= f_{15}(x,z,z,y) = x^2 + xy + 4y^2 + 5z^2. \end{aligned}$$

From the results of Watson [**9**] and Schulze-Pillot [**8**], the form g_1 can represent all positive integers but ones in the form $5^{2\mu+1}\nu$ ($\mu \geqq 0$, $\gcd(5,\nu) = 1$, $(\nu/5) = -1$) and g_2 can represent all but in the form $5^{2\mu}\nu$ ($\mu \geqq 0$, $\gcd(5,\nu) = 1$, $(\nu/5) = -1$) where (q/p) denotes the Legendre symbol. There are no integers which cannot be represented by g_1 nor g_2. Hence f_{15} represents all positive integers.

Our problem for the two remainder forms is more complicated. By means of Proposition 1 it is enough to show that they represent every prime integer. First we note the well-known fact: *The binary quadratic form x^2+4y^2 represents any prime integer p such that $p \equiv 1 \pmod 4$.* Because both of the two forms represent this binary form as

$$f_{23}(x,y,-y,0) = f_{31}(x,0,y,0) = x^2+4y^2,$$

they can represent all prime integers congruent to 1 modulo 4. To show the representations of prime integers congruent to 3 modulo 4, we define two ternary forms represented by f_{23} and f_{31} respectively:

$$\begin{aligned} h_{23}(x,y,z) &:= x^2+2y^2+yz+3z^2, \\ h_{31}(x,y,z) &:= x^2+2y^2+yz+4z^2. \end{aligned}$$

We have the following lemma about these two ternary forms.

Lemma. *If a positive integer k is coprime to 23, then the form h_{23} represents $4k$. If k is coprime to 31, then h_{31} represents $4k$.*

Proof. The genus of h_{23} consists of three classes and their representatives are h_{23} and

$$\begin{aligned} \phi_1(x,y,z) &:= x^2+y^2+xz+6z^2, \\ \phi_2(x,y,z) &:= x^2+xy+y^2+xz+8z^2, \end{aligned}$$

which are given in Brandt and Intrau [**1**]. In view of localization at each prime, if k is coprime to 23, then the genus represents k, that is, at least one of these three forms represents k. If h_{23} represents k, then h_{23} represents $4k$, too. If ϕ_1 represents k, then

$$\begin{aligned} k &= x^2+y^2+xz+6z^2, \\ 4k &= 4x^2+4y^2+4xz+24z^2, \\ &= (2y)^2+2(-x+2z)^2+(-x+2z)(x+2z)+3(x+2z)^2 \\ &= h_{23}(2y,-x+2z,x+2z). \end{aligned}$$

If ϕ_2 represents k then

$$\begin{aligned} k &= x^2+xy+y^2+xz+8z^2, \\ 4k &= 4x^2+4xy+4y^2+4xz+32z^2 \\ &= (x+2y)^2+2(4z)^2+(4z)x+3x^2 \\ &= h_{23}(x+2y,4z,x). \end{aligned}$$

Thus h_{23} represents $4k$ for any integer k coprime to 23. As for h_{31}, we can prove in a similar way. The genus of h_{31} also consists of three classes, whose representatives are h_{31}, $x^2+y^2+xz+8z^2$ and $x^2+xy+2y^2+2yz+5z^2$. If k is coprime to 31, then h_{31} represents k or

$$\begin{aligned} k &= x^2+y^2+xz+8z^2, \\ 4k &= 4x^2+4y^2+4xz+32z^2 \\ &= (2y)^2+2(4z)^2+(4z)x+4x^2 \\ &= h_{31}(2y,4z,x), \end{aligned}$$

or

$$\begin{aligned} k &= x^2+xy+2y^2+2yz+5z^2, \\ 4k &= 4x^2+4xy+8y^2+8yz+20z^2, \\ &= (2x+y)^2+2(y-2z)^2+(y-2z)(y+2z)+4(y+2z)^2, \\ &= h_{31}(2x+y,y-2z,y+2z). \qquad \square \end{aligned}$$

If a prime integer $p \equiv 3 \pmod 4$ and $p > 23$, then $p = 4k+23$ for a positive integer k coprime to 23. As a result of the Lemma the form h_{23} represents $4k$, and

$$\begin{aligned} f_{23}(x-1,y,z,2) &= (x-1)^2+2y^2+yz+3z^2+2(x-1)+6\cdot 2^2 \\ &= x^2+2y^2+yz+3z^2+23 \\ &= h_{23}(x,y,z)+23. \end{aligned}$$

Hence f_{23} represents p. Similarly, if $p \equiv 3 \pmod 4$ and $p > 31$, then $p = 4k+31$ for a positive integer k coprime to 31. The form h_{31} represents $4k$ and

$$\begin{aligned} f_{31}(x-1,y,z,2) &= (x-1)^2+2y^2+yz+4z^2+2(x-1)+8\cdot 2^2 \\ &= x^2+2y^2+yz+4z^2+31 \\ &= h_{31}(x,y,z)+31. \end{aligned}$$

Hence f_{31} represents p. As we can verify that f_{23} and f_{31} represent all integers less than 31, we obtain the following proposition.

Proposition 2. *The quaternary quadratic forms*

$$f_{15}(x,y,z,w) = x^2 + 2y^2 + yz + 2z^2 + xw + 4w^2,$$
$$f_{23}(x,y,z,w) = x^2 + 2y^2 + yz + 3z^2 + xw + 6w^2,$$
$$f_{31}(x,y,z,w) = x^2 + 2y^2 + yz + 4z^2 + xw + 8w^2,$$

are all universal.

This completes the proof of the Theorem. □

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Department of Mathematics, Kobe University, 657-0013 Nada, Kobe, Japan
E-mail address: iwabuchi@math.kobe-u.ac.jp