ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 30, Number 3, Fall 2000

QUASINORMAL OPERATORS SIMILAR TO IRREDUCIBLE ONES

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ABSTRACT. We show that a quasinormal operator T on an infinite-dimensional Hilbert space is similar to an irreducible operator if and only if T is not quadratic and $T - \lambda I$ is not finite-rank for any $\lambda \in \mathbf{C}$.

1. Introduction. Throughout this paper, all operators are bounded and linear on complex Hilbert spaces. An operator T is said to be quasinormal if T commutes with T^*T. An operator is said to be irreducible if it commutes with no projection other than 0 and I, and is said to be reducible otherwise. The aim of this paper is to obtain necessary and sufficient conditions for a quasinormal operator to be similar to an irreducible operator.

Every operator T on a nonseparable Hilbert space is reducible. However, there are some operators on a separable Hilbert space which are reducible but are similar to irreducible ones. From now on, we only have to consider separable Hilbert space operators. Gilfeather [**5**] proved that every normal operator without eigenvalue is similar to an irreducible operator. Later on, Fong and Jiang [**4**] improved Gilfeather's work by allowing the presence of eigenvalues. In this paper we extend Fong and Jiang's result to quasinormal operators as follows.

Main Theorem. *A quasinormal operator T on an infinite-dimensional Hilbert space is similar to an irreducible operator if and only if T is not quadratic and $T - \lambda I$ is not finite-rank for any $\lambda \in \mathbf{C}$.*

We provide a similar theorem on finite-dimensional Hilbert spaces in [**7**].

Gilfeather [**5**] used binormal operators (defined in [**2**]) to prove that every quadratic operator T is always reducible. Here we give a much

Received by the editors on December 10, 1998, and in revised form on June 14, 1999.

easier proof. Let $\mathcal{B}(\mathcal{H},\mathcal{K})$ denote the set of all operators from $\mathcal{H}$ to $\mathcal{K}$. In particular, $\mathcal{B}(\mathcal{H})$ denotes $\mathcal{B}(\mathcal{H},\mathcal{H})$.

Proposition 1.1. *Let $T \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is an infinite-dimensional Hilbert space. If T is similar to an irreducible operator, then T is not quadratic and $T-\lambda I$ is not finite-rank for any $\lambda \in \mathbf{C}$.*

Proof. We first show that if T is quadratic, then it is reducible. Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ exist such that T is unitarily equivalent to

$$\alpha I \oplus \beta I \oplus \begin{bmatrix} \alpha I & T_1 \\ 0 & \beta I \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_3),$$

for some $\alpha, \beta \in \mathbf{C}$, and a one-to-one positive operator $T_1 \in \mathcal{B}(\mathcal{H}_3)$ [**8**]. Therefore, it suffices to consider the case when

$$T = \begin{bmatrix} \alpha I & T_1 \\ 0 & \beta I \end{bmatrix}.$$

Since T_1 is positive, a function $\phi \in L^\infty(X,\Omega,\mu)$ exists such that T_1 is unitarily equivalent to the multiplication operator M_ϕ on $L^2(X,\Omega,\mu)$ ($M_\phi f = \phi f$ for $f \in L^2(X,\Omega,\mu)$) where $X \subset \mathbf{R}$ is compact, Ω is a σ-algebra of subsets of X and μ is a σ-finite positive measure on Ω. Thus T is unitarily equivalent to

$$T' = \begin{bmatrix} \alpha I & M_\phi \\ 0 & \beta I \end{bmatrix}$$

on $H' = L^2(X,\Omega,\mu) \oplus L^2(X,\Omega,\mu)$. Choose a μ-measurable subset $\Delta \subset X$ such that $0 < \mu(\Delta) < \mu(X)$. Let χ_Δ be the characteristic function of Δ, and let P be the direct sum of multiplication operators $M_{\chi\Delta} \oplus M_{\chi\Delta}$. Then P is a nontrivial projection on $\mathcal{H}'$ and commutes with T'. Therefore, T' and hence T is reducible.

To complete the proof, we suppose that $n \in \mathbf{N}$ exists such that $0 < \operatorname{rank}(T-\lambda I) < n$ for some $\lambda \in \mathbf{C}$. For any invertible operator S on $\mathcal{H}$, let $\mathcal{M}$ be the linear span of the ranges of $S^{-1}(T-\lambda I)S$ and $(S^{-1}(T-\lambda I)S)^*$. Thus $1 \le \dim \mathcal{M} \le 2n-2$. Let P be the projection from $\mathcal{H}$ onto the subspace $\mathcal{M}$. Then P is a nontrivial projection and commutes with $S^{-1}TS$. This proves the proposition. $\square$

Proposition 1.1 establishes necessity in the Main Theorem. In addition, by Fong and Jiang's work for normal operators [**4**], we only have to show that every nonnormal quasinormal operator is similar to an irreducible operator to obtain the Main Theorem. In Section 2 we consider completely nonnormal quasinormal operators, defined therein. In Section 3 we consider a class of operators more general than nonnormal quasinormal operators, defined therein.

2. Completely nonnormal quasinormal operators. A quasinormal operator $T \in \mathcal{B}(\mathcal{H})$ is said to be completely nonnormal if for any nonzero reducing subspace $\mathcal{K}$ for T, the restriction of T to $\mathcal{K}$ is not normal. In this section we will prove that a completely nonnormal quasinormal operator is always similar to an irreducible operator. We start with the following lemma.

Lemma 2.1. *Let $M \in \mathcal{B}(\mathcal{K})$ be a one-to-one and positive operator, and let*

$$A = \begin{bmatrix} 0 & & & & \\ M & 0 & & & \\ X_1 & M & 0 & & \\ X_2 & & M & 0 & \\ \vdots & & & \ddots & \ddots \end{bmatrix} \in \mathcal{B}\Big(\sum_{i=1}^{\infty} \oplus \mathcal{K}\Big).$$

If P is a projection which commutes with A, then $P = \sum_{i=1}^{\infty} \oplus P_1 \in \mathcal{B}(\sum_{i=1}^{\infty} \oplus \mathcal{K})$, where P_1 commutes with M.

Proof. Suppose that $P = [P_{ij}]_{i,j=1}^{\infty} \in \mathcal{B}(\sum_{i=1}^{\infty} \oplus \mathcal{K})$. By comparing the $(1, j)$ entries of $PA = AP$, we get $P_{1,j+1}M = 0$ for $j \geq 2$. Since M has dense range, we have $P_{1,j} = 0$ for $j \geq 3$. It follows that $P_{1,2} = 0$ by comparing the (1,1) entries of $PA = AP$. By induction on i and comparing the (i, j) entries, where $j \geq i$, we get $P_{i,j+1}M = 0$, and hence $P_{i,j+1} = 0$. Since P is self-adjoint, we may assume that $P = \sum_{i=1}^{\infty} \oplus P_i$.

By $PA = AP$ again, we obtain, for each $i \in \mathbf{N}$,

$$P_{i+1}M = MP_i. \tag{1}$$

Since P_i and M are self-adjoint,

$$MP_{i+1} = P_i M. \tag{2}$$

By (1) and (2), $M^2P_i = MP_{i+1}M = P_iM^2$ and so $MP_i = P_iM$ since M is positive. By (2) again, we have $MP_{i+1} = P_iM = MP_i$ and so all the P_i are equal since M is one-to-one. This completes the proof. $\square$

Before we prove that every completely nonnormal quasinormal operator is similar to an irreducible operator, we need the following representation. Let $T \in \mathcal{B}(\mathcal{H})$ be a quasinormal operator. By [**1**], Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$, a normal operator $N \in \mathcal{B}(\mathcal{K}_1)$ and a one-to-one positive operator $M \in \mathcal{B}(\mathcal{K}_2)$ exist such that T is unitarily equivalent to T', where

$$T' = N \oplus \begin{bmatrix} 0 & & & \\ M & 0 & & \\ & M & 0 & \\ & & \ddots & \ddots \end{bmatrix} \in \mathcal{B}\Big(\mathcal{K}_1 \oplus \sum_{i=1}^{\infty} \oplus \mathcal{K}_2\Big). \tag{3}$$

For convenience, we use J_n to denote the $n \times n$ nilpotent Jordan block

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}, \tag{4}$$

and V to denote the Volterra operator on $L^2(0,1)$ defined by

$$(Vf)(x) = \int_0^x f(t)\,dt. \tag{5}$$

Lemma 2.2. *If $T \in \mathcal{B}(\mathcal{H})$ is completely nonnormal quasinormal, then T is similar to an irreducible operator.*

Proof. Since T is completely nonnormal quasinormal, by (3), a Hilbert space $\mathcal{K}$ and a one-to-one positive operator $M \in \mathcal{B}(\mathcal{K})$ exist such that T is unitarily equivalent to T', where

$$T' = \begin{bmatrix} 0 & & & \\ M & 0 & & \\ & M & 0 & \\ & & \ddots & \ddots \end{bmatrix} \in \mathcal{B}\Big(\sum_{i=1}^{\infty} \oplus \mathcal{K}\Big).$$

If $\dim \mathcal{K} = n$ for some $n \in \mathbf{N}$, we choose $X_1 = J_n$, where J_n is defined as in (4). Otherwise, if $\mathcal{K}$ is infinite-dimensional, we choose $X_1 = U^*VU$, where V is defined as in (5) and U is a unitary operator from $L^2(0,1)$ onto $\mathcal{K}$. Thus X_1 is irreducible in either case. Let

$$X = \left[\begin{array}{c|ccc} I & & & \\ \hline -X_1 & I & & \\ 0 & & I & \\ \vdots & & & \ddots \end{array}\right] \in \mathcal{B}\Big(\sum_{i=1}^{\infty} \oplus \mathcal{K}\Big).$$

Then T is similar to A, where

$$A = XT'X^{-1} = \left[\begin{array}{c|cccc} 0 & & & & \\ \hline M & 0 & & & \\ MX_1 & M & 0 & & \\ 0 & & M & 0 & \\ \vdots & & & \ddots & \ddots \end{array}\right] \in \mathcal{B}\Big(\sum_{i=1}^{\infty} \oplus \mathcal{K}\Big).$$

It suffices to show that A is irreducible. Let P be a projection which commutes with A. By Lemma 2.1, we may assume that $P = \sum_{i=1}^{\infty} \oplus P_1 \in \mathcal{B}(\sum_{i=1}^{\infty} \oplus \mathcal{K})$, where P_1 commutes with M. In addition, by the fact that $PA = AP$, we have $MP_1X_1 = MX_1P_1$. Since M is one-to-one, $P_1X_1 = X_1P_1$ and so $P_1 = 0$ or I on $\mathcal{K}$ by the irreducibility of X_1. Hence, $P = 0$ or I on $\sum_{j=1}^{\infty} \oplus \mathcal{K}$. $\square$

3. Nonnormal quasinormal operators. Let $N \in \mathcal{B}(\mathcal{H})$ with $\ker N^* \subset \ker N$, and let $M \in \mathcal{B}(\mathcal{K})$ be completely nonnormal quasinormal. In this section we prove that $N \oplus M$ is always similar to an irreducible operator. Recall that every nonnormal quasinormal operator is unitarily equivalent to $N \oplus M$, where N is normal and M is completely nonnormal quasinormal. It is well known that if N is a normal operator, then $\ker N^* = \ker N$. Therefore, the operator $N \oplus M$ considered here is in fact more general than nonnormal quasinormal operators. We recall the following definition.

Definition 3.1. Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Operators A and B are said to be disjoint if $R = 0$ is the only solution to the operator equation $RA = BR$.

Lemma 3.2. *Let $B \in \mathcal{B}(\mathcal{K})$ be similar to an irreducible operator and disjoint from $A \in \mathcal{B}(\mathcal{H})$. If $Y \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ exists such that $YB - AY$ has dense range, then $A \oplus B$ is similar to an irreducible operator.*

Proof. An invertible $Z \in \mathcal{B}(\mathcal{K})$ exists such that ZBZ^{-1} is irreducible. Then $A \oplus B$ is similar to

$$T = \begin{bmatrix} I & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & Z \end{bmatrix}^{-1} = \begin{bmatrix} A & (YB - AY)Z^{-1} \\ 0 & ZBZ^{-1} \end{bmatrix}.$$

Suppose that

$$P = \begin{bmatrix} P_1 & P_3^* \\ P_3 & P_2 \end{bmatrix}$$

is a projection commuting with T. Since A and B are disjoint, A and ZBZ^{-1} are also disjoint. Hence $P_3 A = (ZBZ^{-1})P_3$ implies that $P_3 = 0$. Therefore, both P_1 and P_2 are projections. Irreducibility of ZBZ^{-1} means $P_2 = 0$ or I. Since $P_1(YB - AY) = (YB - AY)P_2$, and since $YB - AY$ has dense range $P_1 = 0$, or I, when $P_2 = 0$, or I. Therefore $P = 0$ or I, and so $A \oplus B$ is similar to the irreducible operator T. $\square$

By Lemma 3.2 we get the following corollary. As usual, $\sigma(A)$ denotes the spectrum of an operator $A \in \mathcal{B}(\mathcal{H})$.

Corollary 3.3. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ where $\dim \mathcal{H} \leq \dim \mathcal{K}$. Suppose that $\sigma(A)$ and $\sigma(B)$ are disjoint. If B is similar to an irreducible operator, then so is $A \oplus B$.*

Proof. Define

$$\pi : \mathcal{B}(\mathcal{K}, \mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{K}, \mathcal{H}), \pi(X) = XB - AX$$

for all $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Then

$$\sigma(\pi) \subset \sigma(B) - \sigma(A) = \{\beta - \alpha : \beta \in \sigma(B), \alpha \in \sigma(A)\}$$

([**6**, Corollary 3.2]), and so $0 \notin \sigma(\pi)$ because of disjointness of $\sigma(A)$ and $\sigma(B)$. In particular, injectivity of π implies A and B are disjoint. Also surjectivity of π yields an operator $Y \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $YB - AY$ has dense range. By Lemma 3.2, $A \oplus B$ is similar to an irreducible operator. $\square$

We are now ready to prove that $N \oplus M$ is similar to an irreducible operator, where $\ker N^* \subset \ker N$ and M is completely nonnormal quasinormal.

By (3) a Hilbert space $\mathcal{K}_1$ and a one-to-one positive operator $M_1 \in \mathcal{B}(\mathcal{K}_1)$ exist such that M is unitarily equivalent to

$$M' = \begin{bmatrix} 0 & & & \\ M_1 & 0 & & \\ & M_1 & 0 & \\ & & \ddots & \ddots \end{bmatrix} \in \mathcal{B}\Big(\sum_{i=1}^{\infty} \oplus \mathcal{K}_1\Big). \tag{6}$$

Lemma 3.4. *Let $N \in \mathcal{B}(\mathcal{H})$ and $M' \in \mathcal{B}(\sum_{i=1}^{\infty} \oplus \mathcal{K}_1)$ be defined as in* (6). *Then*

(i) *$Y \in \mathcal{B}(\sum_{i=1}^{\infty} \oplus \mathcal{K}_1, \mathcal{H})$ exists such that $YM' - NY$ has dense range*;

(ii) *If* $\ker N^* \subset \ker N$, *then M' and N are disjoint.*

Proof. (i) Fix a nonzero vector $y \in \mathcal{K}_1$ and a sequence $\{x_n\}_{n=1}^{\infty}$ which is dense in the unit ball of $\mathcal{H}$. For each $n \in \mathbf{N}$, define $Y_n : \mathcal{K}_1 \to \mathcal{H}$ by $Y_n(x) = (1/n)\langle x, y\rangle x_n$ for every $x \in \mathcal{K}_1$. Then $Y_n \in \mathcal{B}(\mathcal{K}_1, \mathcal{H})$ is a rank one operator. Define $Y = [\,0\ Y_1\ 0\ Y_2\ \cdots] \in \mathcal{B}(\sum_{i=1}^{\infty} \oplus \mathcal{K}_1, \mathcal{H})$. Then the range of $YM' - NY$ contains each x_n and so is dense in $\mathcal{H}$.

(ii) We want to show that

$$RN = M'R \Longrightarrow R = 0. \tag{7}$$

We decompose $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where $\mathcal{H}_1 = \ker N^*$ and $\mathcal{H}_2 = (\ker N^*)^{\perp}$. Relative to this decomposition, we have

$$R = [\,R_1 \quad R_2\,], \qquad N = \begin{bmatrix} 0 & 0 \\ 0 & N_2 \end{bmatrix}, \tag{8}$$

where $R_j \in \mathcal{B}(\mathcal{H}_j, \mathcal{K})$ and $N_2^* \in \mathcal{B}(\mathcal{H}_2)$ is one-to-one. Now apply (8) to the matrix equation $RN = M'R$ and get two equations

$$0 = M'R_1 \tag{9}$$

and

$$R_2N_2 = M'R_2. \tag{10}$$

By (9) we have $R_1 = 0$ since M' is one-to-one. By (10) since $N_2^{*n}R_2 = R_2M'^{*n}$ for $n \in \mathbf{N}$ and N_2^* is one-to-one, we have $R_2 = 0$. This proves (7), and the proposition follows. □

By Lemma 3.4, we obtain the following proposition and thus fulfill the goal of this section.

Proposition 3.5. *Let $N \in \mathcal{B}(\mathcal{H})$ with $\ker N^* \subset \ker N$, and let $M \in \mathcal{B}(\mathcal{K})$ be completely nonnormal quasinormal. Then $N \oplus M$ is similar to an irreducible operator.*

Proof. As before, M is unitarily equivalent to M', where M' is defined as in (6). Applying Lemmas 2.2, 3.4 and 3.2 to $N \oplus M'$, we can make sure that $N \oplus M'$ is similar to an irreducible operator. Therefore, $N \oplus M$ is also similar to an irreducible operator. □

We now prove the Main Theorem as follows.

Proof of Main Theorem. Let T be a nonnormal quasinormal operator. It suffices to show that T is similar to an irreducible operator. Here T may or may not be completely nonnormal quasinormal. The former is taken care of by Lemma 2.2, and the latter by Proposition 3.5. This proves the Main Theorem. □

We end this article with the following conjecture, kindly suggested by the referee.

Conjecture 3.6. *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, where $\mathcal{K}$ is an infinite-dimensional Hilbert space. If B is similar to an irreducible operator, then so is $A \oplus B$.*

When B is similar to an irreducible operator, it is easy to see that $A \oplus B$ is not quadratic and $(A \oplus B) - \lambda I$ is not finite rank for any $\lambda \in \mathbf{C}$. So in this case the conjecture is consistent with our Main Theorem. Proposition 3.5 deals with a special case of this conjecture. Also Corollary 3.3 confirms it in the situation where $\sigma(A)$ and $\sigma(B)$ are disjoint.

Acknowledgments. This work was supported in part by the National Science Council of Taiwan. The author would like to thank P.Y. Wu for helpful suggestions. The referee generously provided many improvements in the arguments in Section 3. Consequently, the present article is more concise and readable than its earlier version.

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