

**GENERATION OF ANALYTIC SEMI-GROUPS
BY SECOND-ORDER DIFFERENTIAL OPERATORS
WITH NONSEPARATED BOUNDARY CONDITIONS**

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ABSTRACT. We investigate under which boundary conditions a second order differential operator of the form $Lu = u'' + q_1(x)u' + q_0(x)u$ generates an analytic semi-group in $L^p(a, b)$, $1 \leq p \leq \infty$. The boundary conditions are not supposed to be separated:

$$B_i(u) \equiv a_i u(a) + b_i u'(a) + c_i u(b) + d_i u'(b) = 0, \quad i = 1, 2,$$

and this allows us to obtain quite general results. The generation of analytic semi-groups is proved by showing the estimate

$$\|R(\lambda : L)\| \leq M|\lambda|^{-1}$$

for the resolvent operator in a suitable sector of the complex plane.

1. Introduction. We consider the formal second-order differential operator l given by

$$l(u) = u'' + q_1(x)u' + q_0(x)u, \quad x \in (a, b),$$

where each q_i is a regular function. We associate to the operator l two linearly independent nonseparated boundary conditions:

$$B_i(u) \equiv a_i u(a) + b_i u'(a) + c_i u(b) + d_i u'(b) = 0, \quad i = 1, 2$$

with complex coefficients. The formal operator l , together with the boundary forms $\{B_1, B_2\}$, defines a linear operator L_p in the Banach space $L^p(a, b)$, $1 \leq p \leq \infty$, as $L_p u = l(u)$ with domain

$$D(L_p) = \{u \in W^{2,p}(a, b) : B_i(u) = 0, i = 1, 2\}.$$

Here $W^{2,p}(a, b)$ stands for the Sobolev space of order $(2, p)$.

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Our purpose is to determine the cases, depending on the boundary conditions, for which the operator L_p is the generator of an analytic semi-group of bounded linear operators in $L^p(a, b)$. As is well known (see [5, 7]), sufficient conditions for assuring that an operator L on a Banach space X is the generator of an analytic semigroup are:

- The resolvent set $\rho(L)$ contains a sector of the form

$$\Sigma_{\delta, r} = \{\lambda \in \mathbf{C} : |\arg(\lambda - r)| < \delta, \lambda \neq r\}$$

for some $\pi/2 < \delta < \pi$ and $r \in \mathbf{R}$.

- A positive constant M exists such that

$$(1.1) \quad \|R(\lambda : L)\| \leq \frac{M}{|\lambda - r|}, \quad \forall \lambda \in \Sigma_{\delta, r},$$

where $R(\lambda : L) \equiv (\lambda I - L)^{-1}$ is the resolvent operator associated to λ .

It is convenient to note that we are using the definition of analytic semi-group given in [5], where we do not assume that the domain of L is dense in X . When this happens, the semi-group generated by L is a C_0 -semi-group (strongly continuous).

In order to invert the operator $\lambda I - L_p$ we introduce the Green's function $G(x, s; \lambda)$ of the differential system

$$(1.2) \quad \begin{cases} l(u) - \lambda u \equiv u'' + q_1(x)u' + q_0(x)u - \lambda u = f & \text{in } (a, b), \\ B_1(u) \equiv a_1u(a) + b_1u'(a) + c_1u(b) + d_1u'(b) = 0, \\ B_2(u) \equiv a_2u(a) + b_2u'(a) + c_2u(b) + d_2u'(b) = 0, \end{cases}$$

where f is an arbitrary function in $L^p(a, b)$. In this way we can express each resolvent operator $R(\lambda : L_p)$ as a Hilbert-Schmidt operator:

$$(1.3) \quad R(\lambda : L_p)f = - \int_a^b G(\cdot, s; \lambda)f(s) ds, \quad \forall f \in L^p(a, b).$$

By means of an adequate expression for $G(x, s; \lambda)$, we will be able to obtain bounds of the form (1.1) for a generic class of boundary conditions that are called *regular*. For doing this we will use (1.3) to bound $R(\lambda : L_p)$ both in the spaces $L^1(a, b)$ and $L^\infty(a, b)$, and then, by

interpolation, we will deduce the bound (1.1) in $L^p(a, b)$ for $1 \leq p \leq \infty$. In the case of regular boundary conditions, we can assume that the operator L_p is the generator of an analytic semi-group of bounded linear operators on $L^p(a, b)$.

The first references about differential equations with nonseparated boundary conditions are the papers by Birkhoff [2–4] in which he gave an expression of the Green's function for a general n -order differential equation with nonseparated boundary conditions and studied some spectral properties of the associated operator. While doing this, Birkhoff introduced the so-called *regular* boundary conditions. A modern treatment of the work by Birkhoff can be found in the book of Naimark [6].

To conclude this introduction, we briefly outline the structure of the paper. We begin with some algebraic considerations on the boundary conditions. Later, in Section 3, we give a suitable expression for the Green's function of system (1.2). We also define a function, the *characteristic determinant*, that allows us to determine the spectrum of the operator L_p . In Section 4 we introduce some useful simplifications in order to avoid complicated calculations. The principal part is Section 5, where we obtain the bound (1.1) in the case of regular boundary conditions. In the last section we establish the results about generation of analytic semi-groups on $L^p(a, b)$, $1 \leq p \leq \infty$.

2. Algebraic preliminaries. Consider the differential system

$$(2.1) \quad \begin{cases} l(u) = u'' + q_1 u' + q_0 u & \text{in } (a, b), \\ B_1(u) = a_1 u(a) + b_1 u'(a) + c_1 u(b) + d_1 u'(b) = 0, \\ B_2(u) = a_2 u(a) + b_2 u'(a) + c_2 u(b) + d_2 u'(b) = 0, \end{cases}$$

where $q_1 \in C^1([a, b], \mathbf{C})$, $q_0 \in C([a, b], \mathbf{C})$ and the boundary conditions $\{B_1, B_2\}$ are linearly independent, i.e., the coefficient's matrix

$$A = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}$$

has rank two. We define the unbounded linear operator

$$L_p : D(L_p) \subset L^p(a, b) \longrightarrow L^p(a, b)$$

as $L_p u = l(u)$ for every $u \in D(L_p)$, where

$$D(L_p) = \{u \in W^{2,p}(a, b) : B_i(u) = 0, i = 1, 2\}.$$

That is, L_p is the L^p -realization of problem (2.1).

By making some algebraic considerations on the boundary conditions, it is easy to see that the definition of the operator L_p does not depend on the representation of these boundary conditions. In fact, the boundary conditions $\{B_1, B_2\}$ are uniquely determined (up to a multiplicative constant) by the numbers A_{ij} , $1 \leq i < j \leq 4$, where A_{ij} is the determinant of the 2×2 matrix formed with the i th and j th columns of A . We have the following result, whose proof is elementary:

Proposition 2.1. *For some $1 \leq i < j \leq 4$, $A_{ij} \neq 0$ and*

$$A_{12}A_{34} + A_{14}A_{23} - A_{13}A_{24} = 0.$$

Conversely, let a_{ij} , $1 \leq i < j \leq 4$, be complex numbers that satisfy the equation

$$a_{12}a_{34} + a_{14}a_{23} - a_{13}a_{24} = 0,$$

with some a_{ij} nonzero. Then a 2×4 matrix A of rank two exists and a number $r \neq 0$ such that

$$A_{ij} = ra_{ij}, \quad 1 \leq i < j \leq 4.$$

The matrix A is unique up to products for a nonsingular matrix.

3. Characteristic determinant and Green's function. We shall define two functions which will play a key role henceforth. The first one, the characteristic determinant, is an entire function of one complex variable that allows us to completely determine the spectrum of the operator L_p . The second one, the Green's function, will help us to study the resolvent set of L_p in order to obtain the appropriate bounds for assuring the generation of analytic semi-groups.

Take an arbitrary $\lambda \in \mathbf{C}$. Let $u_1(x) \equiv u_1(x; \lambda)$ and $u_2(x) \equiv u_2(x; \lambda)$ be two solutions of the equation $l(u) = \lambda u$ with boundary conditions given, respectively, by $u_1(a) = 0$, $u_1'(a) = 1$ and $u_2(a) = 1$, $u_2'(a) = 0$. It is clear that the functions $\{u_1, u_2\}$ form a fundamental system of

solutions of the equation $l(u) = \lambda u$ in the sense that any other solution can be written as $u = C_1 u_1 + C_2 u_2$ for certain constants C_1 and C_2 . We define the *characteristic determinant* $\Delta(\lambda)$ as

$$(3.1) \quad \Delta(\lambda) = \begin{vmatrix} B_1(u_1) & B_1(u_2) \\ B_2(u_1) & B_2(u_2) \end{vmatrix}.$$

If $\Delta(\lambda) \neq 0$, the constants C_1, C_2 can be determined in a unique way and vice versa. This means that the homogeneous problem

$$\begin{cases} l(u) - \lambda u = 0 & \text{in } (a, b), \\ B_1(u) = 0, \\ B_2(u) = 0, \end{cases}$$

has a nontrivial solution if and only if $\Delta(\lambda) \neq 0$. We then have the following result.

Proposition 3.1. *A number $\lambda \in \mathbf{C}$ is an eigenvalue of the operator L_p if and only if it is a zero of the characteristic determinant. That is, the point-spectrum of L_p coincides with the set of zeros of Δ .*

By construction, $\Delta(\lambda)$ is an entire function. Therefore, either

- $\Delta(\lambda) \equiv 0$, so the spectrum of L_p is \mathbf{C} , or
- $\Delta(\lambda)$ is not identically zero. Then the set of zeros is at most a determinable set with no accumulation points.

In the following, we will suppose that the second condition holds.

Let λ be such that $\Delta(\lambda) \neq 0$ and consider the function

$$(3.2) \quad N(x, s; \lambda) = \begin{vmatrix} u_1(x) & u_2(x) & g(x, s; \lambda) \\ B_1(u_1) & B_1(u_2) & B_1(g)_x \\ B_2(u_1) & B_2(u_2) & B_2(g)_x \end{vmatrix}$$

(the symbol $B_i(g)_x$ means that the operator B_i is made over the function $g(x, s; \lambda)$ with respect to the variable x). The function $g(x, s; \lambda)$ is defined as follows:

$$(3.3) \quad g(x, s; \lambda) = \pm \frac{1}{2} \frac{\begin{vmatrix} u_1(x) & u_2(x) \\ u_1(s) & u_2(s) \end{vmatrix}}{\begin{vmatrix} u'_1(s) & u'_2(s) \\ u_1(s) & u_2(s) \end{vmatrix}},$$

where it takes the + sign for $x > s$ and the – sign for $x < s$.

Proposition 3.2. *The differential system*

$$(3.4) \quad \begin{cases} l(u) - \lambda u = f & \text{in } (a, b), \\ B_1(u) = 0, \\ B_2(u) = 0, \end{cases}$$

in which $f \in L^p(a, b)$ and $\Delta(\lambda) \neq 0$ has a unique Green's function $G(x, s; \lambda)$, which is given by

$$(3.5) \quad G(x, s; \lambda) = \frac{N(x, s; \lambda)}{\Delta(\lambda)}.$$

The proof of this proposition is quite simple and we refer the reader to [6, Section III-7]. The first reference we have for formulae (3.2)–(3.5) is the paper [3] of Birkhoff.

The existence of a Green's function allows us to express the solution of problem (3.4) in integral form:

$$u(x) = \int_a^b G(x, s; \lambda) f(s) ds, \quad x \in [a, b].$$

As $f \in L^p(a, b)$, the solution u belongs to $W^{2,p}(a, b)$ and it satisfies the boundary conditions $B_i(u) = 0$, $i = 1, 2$.

We can now translate the above facts to the terminology of operator theory. Every $\lambda \in \mathbf{C}$ such that $\Delta(\lambda) \neq 0$ belongs to $\rho(L_p)$, and the associated resolvent operator $R(\lambda : L_p)$ can be expressed as a Hilbert-Schmidt operator:

$$\begin{aligned} (\lambda I - L_p)^{-1} f &\equiv R(\lambda : L_p) f = - \int_a^b G(\cdot, s; \lambda) f(s) ds, \\ &f \in L^p(a, b). \end{aligned}$$

4. Some simplifications on the original problem. We are going to make some modifications of the original problem in order to simplify the calculations to be made in Section 5.

We begin by considering the differential problem (3.4) on the interval $(0, 1)$ instead of (a, b) by the linear transformation $x \mapsto (x - a)/(b - a)$. Then the system becomes:

$$(4.1) \quad \begin{cases} \tilde{l}(v) = v'' + \tilde{q}_1 v' + \tilde{q}_0 v & \text{in } (0, 1), \\ \tilde{B}_1(v) \equiv \tilde{a}_1 v(0) + \tilde{b}_1 v'(0) + \tilde{c}_1 v(1) + \tilde{d}_1 v'(1) = 0, \\ \tilde{B}_2(v) \equiv \tilde{a}_2 v(0) + \tilde{b}_2 v'(0) + \tilde{c}_2 v(1) + \tilde{d}_2 v'(1) = 0, \end{cases}$$

where \tilde{q}_1 is of class C^1 and \tilde{q}_0 is continuous in $[0, 1]$. As in [6, Section II], we are going to eliminate the term $\tilde{q}_1 v'$ in the expression of $\tilde{l}(v)$. Consider the C^2 -diffeomorphism ϕ given by

$$(4.2) \quad \phi(t) = \exp\left(-\frac{1}{2} \int_0^t \tilde{q}_1(s) ds\right), \quad t \in [0, 1],$$

and let $M_\phi : L^p(0, 1) \rightarrow L^p(0, 1)$ be multiplication by ϕ :

$$M_\phi u = \phi u, \quad u \in L^p(0, 1).$$

It is clear that M_ϕ is a bijective bounded linear operator; its inverse is given by $M_\phi^{-1} v = \phi^{-1} v$. M_ϕ also establishes a bijection from $W^{2,p}(0, 1)$ to itself.

Now we multiply system (4.1) by ϕ to get:

$$(4.3) \quad \begin{cases} l(u) = u'' + qu & \text{in } (0, 1), \\ B_1(u) \equiv a_1 u(0) + b_1 u'(0) + c_1 u(1) + d_1 u'(1) = 0, \\ B_2(u) \equiv a_2 u(0) + b_2 u'(0) + c_2 u(1) + d_2 u'(1) = 0, \end{cases}$$

where $u = M_\phi^{-1} v$, $q = \tilde{q}_0 - \tilde{q}_1^2/4 - \tilde{q}_1'/2$ and

$$(4.4) \quad \begin{aligned} a_i &= \tilde{a}_i + \tilde{b}_i \phi'(0), & b_i &= \tilde{b}_i, \\ c_i &= \tilde{c}_i \phi(1) + \tilde{d}_i \phi'(1), & d_i &= \tilde{d}_i \phi(1), \\ & & i &= 1, 2. \end{aligned}$$

We also have that $\tilde{l}(v) = \phi l(u)$.

Associated to problems (4.1) and (4.3), we have, respectively, the operators \tilde{L}_p and L_p defined as

$$\tilde{L}_p v = \tilde{l}(v), \quad D(\tilde{L}_p) = \{v \in W^{2,p}(0, 1) : \tilde{B}_i(v) = 0, i = 1, 2\},$$

and

$$L_p u = l(u), \quad D(L_p) = \{u \in W^{2,p}(0,1) : B_i(u) = 0, i = 1, 2\}.$$

Both operators are related in the following way:

$$\tilde{L}_p = M_\phi L_p M_\phi^{-1},$$

where the equality must be understood as an operator equality: $D(\tilde{L}_p) = D(M_\phi L_p M_\phi^{-1})$ and $\tilde{L}_p u = M_\phi L_p M_\phi^{-1} u$ for each $u \in D(\tilde{L}_p)$. Then we have the following straightforward result.

Proposition 4.1. *The resolvent sets of the operators L_p and \tilde{L}_p are the same:*

$$\rho(L_p) = \rho(\tilde{L}_p).$$

As a consequence, their spectra are equal and, for every $\lambda \in \rho(L)$, we have the following relation between the associated resolvent operators:

$$R(\lambda : \tilde{L}_p) = M_\phi R(\lambda : L_p) M_\phi^{-1}.$$

We can write the operator L_p as follows:

$$L_p = T_p + Q_p,$$

where $T_p u = u''$ and $Q_p u = qu$ for every $u \in D(L_p) = D(T_p) = D(Q_p)$. It is clear that Q_p is a bounded linear operator, and the following holds.

Proposition 4.2. *If $\lambda \in \rho(T_p)$ verifies $\|R(\lambda : T_p)\| \leq \|Q_p\|^{-1}/2$, then $\lambda \in \rho(L_p)$ and we have the following inequality*

$$\|R(\lambda : L_p)\| \leq 2\|R(\lambda : T_p)\|.$$

Proof. As $\|R(\lambda : T_p)Q_p\| \leq \|R(\lambda : T_p)\|\|Q_p\| \leq 1/2$, we have that the operator $I - R(\lambda : T_p)Q_p$ is invertible and

$$[I - R(\lambda : T_p)Q_p]^{-1} = \sum_{n=0}^{\infty} [R(\lambda : T_p)Q_p]^n,$$

so

$$(4.5) \quad \|[I - R(\lambda : T_p)Q_p]^{-1}\| \leq 1/(1 - 1/2) = 2.$$

Let $R_\lambda = [I - R(\lambda : T_p)Q_p]^{-1}R(\lambda : T_p)$. In $D(L_p) = D(T_p)$ we have:

$$\begin{aligned} R_\lambda(\lambda I - L_p) &= [I - R(\lambda : T_p)Q_p]^{-1}R(\lambda : T_p)[(\lambda I - T_p) - Q_p] \\ &= [I - R(\lambda : T_p)Q_p]^{-1}[I - R(\lambda : T_p)Q_p] = I. \end{aligned}$$

This means that $R_\lambda = R(\lambda : L_p)$ so $\lambda \in \rho(L_p)$. Finally,

$$\|R(\lambda : L_p)\| \leq 2\|R(\lambda : T_p)\|,$$

which is an immediate consequence of (4.5). \square

Proposition 4.2 allows us to pass from the problem of obtaining appropriate bounds on the resolvent of the operator L_p to the same problem for the operator T_p . The advantage is that for the operator T_p the calculations to be made are much simpler.

5. Bounds on the resolvent set of the operator T_p . Due to the considerations made in the previous section, we will center our attention on the operator T_p defined as

$$T_p u = u'', \quad D(T_p) = \{u \in W^{2,p}(0, 1) : B_i(u) = 0, i = 1, 2\},$$

where $B_i(u) = a_i u(0) + b_i u'(0) + c_i u(1) + d_i u'(1)$, $i = 1, 2$. We also consider the characteristic determinant $\Delta(\lambda)$ associated to T_p .

Let λ be an arbitrary complex number. If $\Delta(\lambda) \neq 0$, we know that $\lambda \in \rho(T_p)$ and the associated resolvent operator is given by

$$(5.1) \quad R(\lambda : T_p)f = - \int_0^1 G(\cdot, s; \lambda)f(s) ds, \quad f \in L^p(0, 1),$$

where $G(x, s; \lambda)$ is the corresponding Green's function.

The idea now is to use the formula (5.1) to obtain a bound on $R(\lambda : T_p)$ of the form $M/|\lambda|$, where M is a constant that does not depend on λ . These are the kind of bounds that we need in order to

assure the generation of analytic semi-groups. For doing this, we will bound (5.1) both in the spaces $L^\infty(0, 1)$ and $L^1(0, 1)$, and then we will use interpolation results in order to bound $R(\lambda : T_p)$ in all the scale of spaces $L^p(0, 1)$, $1 \leq p \leq \infty$.

We must make a last consideration. Instead of the fundamental system of solutions of the equation $u'' = \lambda u$ constructed in Section 3, we consider this one:

$$u_1(t) = e^{-\rho t}, \quad u_2(t) = e^{\rho t},$$

where $\lambda = \rho^2 \neq 0$, and the associated characteristic determinant, that will also be denoted by $\Delta(\lambda)$. In this way, we have simple expressions for the system $\{u_1, u_2\}$ that can be substituted in the formula for $G(x, s; \lambda)$. Note that the condition $\Delta(\lambda) = 0$ does not imply that λ is an eigenvalue.

For $\lambda = 0$, the above functions $\{u_1, u_2\}$ are not linearly independent. So we choose $u_1(t) = t$ and $u_2(t) = 1$. By substituting them in the formula for $\Delta(\lambda)$, we have that $\lambda = 0$ is an eigenvalue if and only if the following condition holds

$$A_{12} + A_{13} + A_{14} + A_{34} - A_{23} = 0.$$

5.1. Bounds in $L^\infty(0, 1)$. Suppose that $\lambda \neq 0$ is such that $\Delta(\lambda) \neq 0$. Then, from (5.1) we deduce that

$$\begin{aligned} \|R(\lambda : T_\infty)f\|_{L^\infty(0,1)} &= \sup_{0 \leq x \leq 1} \int_0^1 |G(x, s; \lambda)| |f(s)| ds \\ &\leq \left(\sup_{0 \leq x \leq 1} \int_0^1 |G(x, s; \lambda)| ds \right) \|f\|_{L^\infty(0,1)} \\ &= \frac{1}{|\Delta(\lambda)|} \sup_{0 \leq x \leq 1} \int_0^1 |N(x, s; \lambda)| ds. \end{aligned}$$

Hence we must bound $\sup_{0 \leq x \leq 1} \int_0^1 |N(x, s; \lambda)| ds$ in an appropriate way.

Take an arbitrary $\delta \in (\pi/2, \pi)$ and consider the sector Σ_δ defined as

$$\Sigma_\delta = \{\lambda \in \mathbf{C} : |\arg(\lambda)| < \delta, \lambda \neq 0\}.$$

For a given $\lambda \in \Sigma_\delta$, we define ρ as the square root of λ with positive real part

$$\rho = |\lambda|^{1/2} \exp((\arg(\lambda)i/2)), \quad \arg(\lambda) \in (-\delta, \delta).$$

A fundamental system of solutions for the homogeneous equation $u'' - \lambda u = 0$ is given by

$$u_1(x) = e^{-\rho x}, \quad u_2(x) = e^{\rho x}.$$

We can evaluate the boundary forms $\{B_1, B_2\}$ in the functions above to obtain

$$B_i(u_j) = a_i + (-1)^j b_i \rho + c_i e^{(-1)^j \rho} + (-1)^j d_i \rho e^{(-1)^j \rho}, \\ 1 \leq i, j \leq 2.$$

By substituting the above expressions in (3.1), we deduce the following formula for the characteristic determinant, valid for $\lambda = \rho^2 \neq 0$ with $|\arg(\rho)| < \delta/2$,

$$(5.2) \quad \Delta(\lambda) = -(A_{24}\rho^2 + (A_{23} - A_{14})\rho - A_{13})e^\rho \\ + (A_{24}\rho^2 + (A_{14} - A_{23})\rho - A_{13})e^{-\rho} \\ + 2(A_{12} + A_{34})\rho$$

(the numbers A_{ij} were defined in Section 2). From (5.2) we can see that, if $\Delta(\lambda)$ is not identically zero, a number $r > 0$ exists such that $\Delta(\lambda) \neq 0$ for all $\lambda \in \Sigma_\delta$ with $|\lambda| > r$. So we have that the sector $\Sigma_{\delta,r}$ defined as

$$\Sigma_{\delta,r} \equiv r + \Sigma_\delta = \{\lambda \in \mathbf{C} : |\arg(\lambda - r)| < \delta, \lambda \neq r\}$$

is contained in the resolvent set $\rho(T_\infty)$.

Now we must obtain a suitable expression for the numerator $N(x, s; \lambda)$ of the Green's function $G(x, s; \lambda)$. To do so, we will make use of the formulas given in Section 3. First of all, we obtain from (3.3)

$$g(x, s; \lambda) = \begin{cases} \frac{1}{4\rho}(e^{\rho(x-s)} - e^{\rho(s-x)}) & \text{if } x > s, \\ \frac{1}{4\rho}(e^{\rho(s-x)} - e^{\rho(x-s)}) & \text{if } x < s. \end{cases}$$

Evaluating the boundary forms in $g(x, s; \lambda)$, we obtain

$$\begin{aligned} B_i(g)_x &= a_i g(0, s; \lambda) + b_i \frac{\partial g}{\partial x}(0, s; \lambda) \\ &\quad + c_i g(1, s; \lambda) + d_i \frac{\partial g}{\partial x}(1, s; \lambda) \\ &= (a_i - b_i \rho - c_i e^{-\rho} + d_i \rho e^{-\rho}) \frac{e^{\rho s}}{4\rho} \\ &\quad + (-a_i - b_i \rho + c_i e^{\rho} + d_i \rho e^{\rho}) \frac{e^{-\rho s}}{4\rho}, \end{aligned}$$

for $i = 1, 2$. The next step is to substitute the expressions obtained before in the determinant (3.2) that defines $N(x, s; \lambda)$. After a long calculation, we obtain the following formula

$$\begin{aligned} N(x, s; \lambda) &= \frac{e^{\rho(x+s)}}{2\rho} (A_{13} - (A_{14} + A_{23})\rho + A_{24}\rho^2) e^{-\rho} \\ &\quad + \frac{e^{-\rho(x+s)}}{2\rho} (A_{13} + (A_{14} + A_{23})\rho + A_{24}\rho^2) e^{\rho} \\ &\quad + \varphi(x, s; \lambda), \end{aligned}$$

where

$$(5.3) \quad \varphi(x, s; \lambda) = \begin{cases} \varphi_1(x, s; \lambda) & \text{if } x > s, \\ \varphi_2(x, s; \lambda) & \text{if } x < s. \end{cases}$$

The functions $\varphi_i(x, s; \lambda)$ are defined by

$$\begin{aligned} \varphi_1(x, s; \lambda) &= \frac{e^{\rho(s-x)}}{2\rho} [(-A_{13} + (A_{23} - A_{14})\rho + A_{24}\rho^2) e^{\rho} - 2A_{12}\rho] \\ &\quad + \frac{e^{\rho(x-s)}}{2\rho} [(-A_{13} + (A_{14} - A_{23})\rho + A_{24}\rho^2) e^{-\rho} + 2A_{12}\rho], \end{aligned}$$

and

$$\begin{aligned} \varphi_2(x, s; \lambda) &= \frac{e^{\rho(s-x)}}{2\rho} [(-A_{13} + (A_{14} - A_{23})\rho + A_{24}\rho^2) e^{-\rho} + 2A_{34}\rho] \\ &\quad + \frac{e^{\rho(x-s)}}{2\rho} [(-A_{13} + (A_{23} - A_{14})\rho + A_{24}\rho^2) e^{\rho} - 2A_{34}\rho]. \end{aligned}$$

From the above expressions, we can bound $|N(x, s; \lambda)|$ as follows

$$|N(x, s; \lambda)| \leq \frac{\Gamma(\rho)}{2|\rho|} e^{-\operatorname{Re}(\rho)} e^{(x+s)\operatorname{Re}(\rho)} + \frac{\Gamma(\rho)}{2|\rho|} e^{\operatorname{Re}(\rho)} e^{-(x+s)\operatorname{Re}(\rho)} + |\varphi(x, s; \lambda)|,$$

where

$$\Gamma(\rho) = |A_{13}| + (|A_{14}| + |A_{23}|)|\rho| + |A_{24}||\rho|^2.$$

In a similar way, we have that

$$|\varphi_1(x, s; \lambda)| \leq (2|A_{12}||\rho| + \Gamma(\rho)e^{-\operatorname{Re}(\rho)}) \frac{e^{(x-s)\operatorname{Re}(\rho)}}{2|\rho|} + (2|A_{12}||\rho| + \Gamma(\rho)e^{\operatorname{Re}(\rho)}) \frac{e^{(s-x)\operatorname{Re}(\rho)}}{2|\rho|},$$

and

$$|\varphi_2(x, s; \lambda)| \leq (2|A_{34}||\rho| + \Gamma(\rho)e^{\operatorname{Re}(\rho)}) \frac{e^{(x-s)\operatorname{Re}(\rho)}}{2|\rho|} + (2|A_{34}||\rho| + \Gamma(\rho)e^{-\operatorname{Re}(\rho)}) \frac{e^{(s-x)\operatorname{Re}(\rho)}}{2|\rho|}.$$

The next step is to bound $\int_0^1 |N(x, s; \lambda)| ds$:

$$\begin{aligned} \int_0^1 |N(x, s; \lambda)| ds &\leq \frac{\Gamma(\rho)}{2|\rho|} e^{-\operatorname{Re}(\rho)} \int_0^1 e^{(x+s)\operatorname{Re}(\rho)} ds \\ &\quad + \frac{\Gamma(\rho)}{2|\rho|} e^{\operatorname{Re}(\rho)} \int_0^1 e^{-(x+s)\operatorname{Re}(\rho)} ds \\ &\quad + \int_0^1 |\varphi(x, s; \lambda)| ds, \end{aligned}$$

that is to say,

$$\begin{aligned} &\int_0^1 |N(x, s; \lambda)| ds \\ &\leq \frac{\Gamma(\rho)}{2|\rho|\operatorname{Re}(\rho)} (e^{x\operatorname{Re}(\rho)} - e^{-x\operatorname{Re}(\rho)} + e^{(1-x)\operatorname{Re}(\rho)} - e^{(x-1)\operatorname{Re}(\rho)}) \\ &\quad + \int_0^1 |\varphi(x, s; \lambda)| ds. \end{aligned}$$

In order to bound $\int_0^1 |\varphi(x, s; \lambda)| ds$, we can use (5.3) to write:

$$\int_0^1 |\varphi(x, s; \lambda)| ds = \int_0^x |\varphi_1(x, s; \lambda)| ds + \int_x^1 |\varphi_2(x, s; \lambda)| ds.$$

Now we evaluate each summand separately. First we have

$$\begin{aligned} \int_0^x |\varphi_1(x, s; \lambda)| ds &\leq (2|A_{12}||\rho| + \Gamma(\rho)e^{-\operatorname{Re}(\rho)}) \int_0^x \frac{e^{(x-s)\operatorname{Re}(\rho)}}{2|\rho|} ds \\ &\quad + (2|A_{12}||\rho| + \Gamma(\rho)e^{\operatorname{Re}(\rho)}) \int_0^x \frac{e^{(s-x)\operatorname{Re}(\rho)}}{2|\rho|} ds \\ &= \frac{\Gamma(\rho)}{2|\rho|\operatorname{Re}(\rho)} (e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)}) \\ &\quad + e^{(x-1)\operatorname{Re}(\rho)} - e^{(1-x)\operatorname{Re}(\rho)} \\ &\quad + \frac{|A_{12}|}{\operatorname{Re}(\rho)} (e^{x\operatorname{Re}(\rho)} - e^{-x\operatorname{Re}(\rho)}). \end{aligned}$$

In the same way, we have

$$\begin{aligned} \int_x^1 |\varphi_2(x, s; \lambda)| ds &\leq (2|A_{34}||\rho| + \Gamma(\rho)e^{\operatorname{Re}(\rho)}) \int_x^1 \frac{e^{(x-s)\operatorname{Re}(\rho)}}{2|\rho|} ds \\ &\quad + (2|A_{34}||\rho| + \Gamma(\rho)e^{-\operatorname{Re}(\rho)}) \int_x^1 \frac{e^{(s-x)\operatorname{Re}(\rho)}}{2|\rho|} ds \\ &= \frac{\Gamma(\rho)}{2|\rho|\operatorname{Re}(\rho)} (e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)} + e^{-x\operatorname{Re}(\rho)} - e^{x\operatorname{Re}(\rho)}) \\ &\quad + \frac{|A_{34}|}{\operatorname{Re}(\rho)} (e^{(1-x)\operatorname{Re}(\rho)} - e^{(x-1)\operatorname{Re}(\rho)}). \end{aligned}$$

Finally we obtain

$$\begin{aligned} \int_0^1 |N(x, s; \lambda)| ds &\leq \frac{\Gamma(\rho)}{|\rho|\operatorname{Re}(\rho)} (e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)}) \\ &\quad + \frac{|A_{12}|}{\operatorname{Re}(\rho)} (e^{x\operatorname{Re}(\rho)} - e^{-x\operatorname{Re}(\rho)}) \\ &\quad + \frac{|A_{34}|}{\operatorname{Re}(\rho)} (e^{(1-x)\operatorname{Re}(\rho)} - e^{(x-1)\operatorname{Re}(\rho)}), \end{aligned}$$

that can also be written as

$$\int_0^1 |N(x, s; \lambda)| ds \leq \frac{1}{|\rho| \operatorname{Re}(\rho)} [|A_{12}| |\rho| (e^{x \operatorname{Re}(\rho)} - e^{-x \operatorname{Re}(\rho)}) + |A_{34}| |\rho| (e^{(1-x) \operatorname{Re}(\rho)} - e^{(x-1) \operatorname{Re}(\rho)}) + \Gamma(\rho) (e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)})].$$

Now taking the supremum in the preceding inequality, we deduce

$$\sup_{0 \leq x \leq 1} \int_0^1 |N(x, s; \lambda)| ds \leq ((|A_{12}| + |A_{34}|) |\rho| + \Gamma(\rho)) \frac{e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)}}{|\rho| \operatorname{Re}(\rho)}.$$

From the definition of $\Gamma(\rho)$, we finally obtain

$$\sup_{0 \leq x \leq 1} \int_0^1 |N(x, s; \lambda)| ds \leq (|A_{13}| + (|A_{12}| + |A_{14}| + |A_{23}| + |A_{34}|) |\rho| + |A_{24}| |\rho|^2) \frac{e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)}}{|\rho| \operatorname{Re}(\rho)}.$$

Consider now the function $H(\rho)$ defined as

$$H(\rho) = (|A_{13}| + (|A_{12}| + |A_{14}| + |A_{23}| + |A_{34}|) |\rho| + |A_{24}| |\rho|^2) \frac{e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)}}{|\Delta(\lambda)|}.$$

Then we have the following bound

$$\sup_{0 \leq x \leq 1} \int_0^1 |G(x, s; \lambda)| ds = \frac{1}{|\Delta(\lambda)|} \sup_{0 \leq x \leq 1} \int_0^1 |N(x, s; \lambda)| ds \leq \frac{H(\rho)}{|\rho| \operatorname{Re}(\rho)}.$$

Note that $|\arg(\rho)| < \delta/2$, so $\operatorname{Re}(\rho) \geq |\rho| \cos(\delta/2) > 0$. Then the following inequality holds

$$\sup_{0 \leq x \leq 1} \int_0^1 |G(x, s; \lambda)| ds \leq \frac{H(\rho)}{\cos(\delta/2) |\rho|^2},$$

and from this we obtain the final bound

$$(5.4) \quad \|R(\lambda : T_\infty)\| \leq \frac{H(\rho)}{\cos(\delta/2) |\rho|^2},$$

valid when $|\arg(\rho)| < \delta/2$. Note that the function $H(\rho)$ only depends on ρ and the coefficients of the boundary conditions $\{B_1, B_2\}$. In the cases where $H(\rho)$ is a bounded function we deduce from (5.4) the desired bound in the space $L^\infty(0, 1)$:

$$\|R(\lambda : T_\infty)\| \leq \frac{M}{|\rho|^2} = \frac{M}{|\lambda|},$$

where M is a constant independent of λ .

It is clear from the previous comments that we must now determine the cases for which the function $H(\rho)$ remains bounded as $\rho \rightarrow \infty$ in the sector Σ_δ . From the definition of $H(\rho)$, it is natural to consider five different cases, depending only on the boundary conditions. These cases are:

- (i) $A_{24} \neq 0$
- (ii) $A_{24} = 0$ and $A_{14} - A_{23} \neq 0$.
- (iii) $A_{24} = A_{14} - A_{23} = 0$, $A_{13} \neq 0$ and $A_{12} + A_{34} = 0$.
- (iv) $A_{24} = A_{14} - A_{23} = 0$, $A_{13} \neq 0$ and $A_{12} + A_{34} \neq 0$.
- (v) $A_{24} = A_{14} - A_{23} = A_{13} = 0$.

We will now analyze each case separately.

Case (i). We suppose that $A_{24} \neq 0$. It is convenient to write the formula (5.2) for the characteristic determinant in the following form

$$\begin{aligned} \Delta(\lambda) = A_{24}\rho^2 e^\rho & \left[(e^{-2\rho} - 1) + \frac{1}{A_{24}\rho} ((A_{14} - A_{23})e^{-2\rho} \right. \\ & + 2(A_{12} + A_{34})e^{-\rho} + (A_{14} - A_{23})) \\ & \left. + \frac{A_{13}}{A_{24}\rho^2} (1 - e^{-2\rho}) \right]. \end{aligned}$$

Choose an arbitrary constant $r_0 > 0$ and take $\rho \in \mathbf{C}$ such that $|\arg(\rho)| < \delta/2$ and $\operatorname{Re}(\rho) > r_0$. Then we have that $|e^{-\rho}| = e^{-\operatorname{Re}(\rho)} < e^{-r_0}$ and

$$|e^{-2\rho} - 1| = |e^{-\rho} + 1||e^{-\rho} - 1| \geq |1 - e^{-\operatorname{Re}(\rho)}|^2 = (1 - e^{-r_0})^2.$$

Therefore, we deduce

$$|\Delta(\lambda)| \geq |A_{24}||\rho|^2 e^{\operatorname{Re}(\rho)} \left[(1 - e^{-r_0})^2 - \frac{1}{|A_{24}||\rho|} (|A_{14} - A_{23}| e^{-2r_0} + 2|A_{12} + A_{34}| e^{-r_0} + |A_{14} - A_{23}|) - \frac{|A_{13}|}{|A_{24}||\rho|^2} (1 - e^{-2r_0}) \right].$$

Now choose a constant $r_1 > 0$ such that, if $|\rho| > r_1$, then

$$\frac{1}{|A_{24}||\rho|} (|A_{14} - A_{23}| e^{-2r_0} + 2|A_{12} + A_{34}| + |A_{14} - A_{23}|) + \frac{|A_{13}|}{|A_{24}||\rho|^2} (1 - e^{-2r_0}) < \frac{(1 - e^{-r_0})^2}{2}.$$

Thus, for $|\rho| > r_1$, we have

$$|\Delta(\lambda)| \geq \frac{(1 - e^{-r_0})^2}{2} |A_{24}||\rho|^2 e^{\operatorname{Re}(\rho)} = c_1 |A_{24}||\rho|^2 e^{\operatorname{Re}(\rho)},$$

where $c_1 = [(1 - e^{-r_0})^2/2]$. Hence, from the above inequality and the definition of $H(\rho)$, we obtain the following bound:

$$\begin{aligned} H(\rho) &\leq c_1 \left(\frac{|A_{13}|}{|A_{24}||\rho|^2} + \frac{|A_{12}| + |A_{14}| + |A_{23}| + |A_{34}|}{|A_{24}||\rho|} + 1 \right) (1 - e^{-2\operatorname{Re}(\rho)}) \\ &\leq \left(\frac{|A_{13}|}{|A_{24}|r_1^2} + \frac{|A_{12}| + |A_{14}| + |A_{23}| + |A_{34}|}{|A_{24}|r_1} + 1 \right) (1 - e^{-r_0})^2 =: M_1, \end{aligned}$$

taking into account that

$$(1 - e^{-2\operatorname{Re}(\rho)}) = (1 + e^{-\operatorname{Re}(\rho)})(1 - e^{-\operatorname{Re}(\rho)}) < 1 + e^{-r_0}.$$

This shows that the function $H(\rho)$ remains uniformly bounded.

Case (ii). Suppose now that $A_{24} = 0$ and $A_{14} - A_{23} \neq 0$. Then the function $H(\rho)$ has the form

$$H(\rho) = (|A_{13}| + (|A_{12}| + |A_{14}| + |A_{23}| + |A_{34}|)|\rho|) \frac{e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)}}{|\Delta(\lambda)|},$$

and the characteristic determinant is given by

$$\Delta(\lambda) = ((A_{14} - A_{23})\rho + A_{13})e^\rho + ((A_{14} - A_{23})\rho - A_{13})e^{-\rho} + 2(A_{12} + A_{34})\rho.$$

As in the previous case, we choose ρ such that $|\arg(\rho)| < \delta/2$ and $|\rho| > r_0$, where $r_0 > 0$ is a constant to be chosen conveniently. We can write

$$\Delta(\lambda) = (A_{14} - A_{23})\rho e^\rho \left(1 + e^{-2\rho} + 2\frac{A_{12} + A_{34}}{A_{14} - A_{23}}e^{-\rho} + \frac{A_{13}}{A_{14} - A_{23}}\frac{1}{\rho}(1 - e^{-2\rho}) \right),$$

and therefore

$$\begin{aligned} |\Delta(\lambda)| &\geq |A_{14} - A_{23}||\rho|e^{\operatorname{Re}(\rho)} \\ &\quad \cdot \left(1 - e^{-2\operatorname{Re}(\rho)} - 2\frac{|A_{12} + A_{34}|}{|A_{14} - A_{23}|}e^{-\operatorname{Re}(\rho)} - \frac{|A_{13}|}{|A_{14} - A_{23}||\rho|}|1 - e^{-2\rho}| \right) \\ &\leq |A_{14} - A_{23}||\rho|e^{\operatorname{Re}(\rho)} \\ &\quad \cdot \left(1 - e^{-2r_0} - 2\frac{|A_{12} + A_{34}|}{|A_{14} - A_{23}|}e^{-r_0} - \frac{|A_{13}|}{|A_{14} - A_{23}||\rho|}(1 - e^{-2r_0}) \right). \end{aligned}$$

Now, if we choose $r_0 > 0$ such that

$$e^{-2r_0} + 2\frac{|A_{12} + A_{34}|}{|A_{14} - A_{23}|}e^{-r_0} < \frac{1}{2},$$

we deduce the following inequality, valid for $\operatorname{Re}(\rho) > r_1$:

$$|\Delta(\lambda)| \geq |A_{14} - A_{23}||\rho|e^{\operatorname{Re}(\rho)} \left(\frac{1}{2} - \frac{|A_{13}|}{|A_{14} - A_{23}||\rho|}(1 + e^{-2r_0}) \right).$$

Taking now $r_1 > 0$ such that, for all $|\rho| > r_1$, the following inequality holds:

$$\frac{|A_{13}|}{|A_{14} - A_{23}||\rho|}(1 + e^{-2r_0}) < \frac{1}{4}.$$

Then we have that

$$|\Delta(\lambda)| \geq \frac{1}{4}|A_{14} - A_{23}||\rho|e^{\operatorname{Re}(\rho)}.$$

Combining the previous bound with the expression for $H(\rho)$, we obtain

$$\begin{aligned} H(\rho) &\leq 4 \frac{|A_{13}| + (|A_{12}| + |A_{14}| + |A_{23}| + |A_{34}|)|\rho|}{|A_{14} - A_{23}||\rho|} \frac{e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)}}{e^{\operatorname{Re}(\rho)}} \\ &= 4 \left(\frac{|A_{13}|}{|A_{14} - A_{23}||\rho|} + \frac{|A_{12}| + |A_{14}| + |A_{23}| + |A_{34}|}{|A_{14} - A_{23}|} \right) (1 - e^{-2\operatorname{Re}(\rho)}). \end{aligned}$$

Finally, for every $|\rho| > r_1$ with $|\arg(\rho)| < \delta/2$, we have

$$H(\rho) \leq 4 \left(\frac{|A_{13}|}{|A_{14} - A_{23}|r_1} + \frac{|A_{12}| + |A_{14}| + |A_{23}| + |A_{34}|}{|A_{14} - A_{23}|} \right) (1 + e^{-r_0}) =: M_1,$$

which proves that $H(\rho)$ remains bounded.

In what follows, we will suppose that $A_{24} = A_{14} - A_{23} = 0$. Then $H(\rho)$ can be written as

$$(5.5) \quad H(\rho) = (|A_{13}| + (|A_{12}| + 2|A_{14}| + |A_{34}|)|\rho|) \frac{e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)}}{|\Delta(\lambda)|},$$

and the characteristic determinant is given by

$$(5.6) \quad \Delta(\lambda) = A_{13}e^\rho - A_{13}e^{-\rho} + 2(A_{12} + A_{34})\rho.$$

The last three cases arise now in a natural way.

Case (iii). We suppose that $A_{24} = A_{14} - A_{23} = 0$, $A_{13} \neq 0$ and $A_{12} + A_{34} = 0$. We can write (5.5) and (5.6), respectively, as follows:

$$H(\rho) = (|A_{13}| + 2(|A_{12}| + |A_{14}|)|\rho|) \frac{e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)}}{|\Delta(\lambda)|},$$

and

$$\Delta(\lambda) = A_{13}e^{-\rho} - A_{13}e^{\rho} = A_{13}e^\rho(1 - e^{-2\rho}).$$

Then we have

$$H(\rho) = \frac{|A_{13}| + 2(|A_{12}| + |A_{14}|)|\rho|}{|A_{13}|} \cdot \frac{1 - e^{-2\operatorname{Re}(\rho)}}{|1 - e^{-2\rho}|}.$$

It is clear that the term $(1 - e^{-2\operatorname{Re}(\rho)})/|1 - e^{-2\rho}|$ remains bounded for $|\rho|$ large. Thus, we consider two subcases:

- $|A_{12}| + |A_{14}| \neq 0$. In this case, it is easy to see that $H(\rho)$ is not a bounded function.

- $|A_{12}| + |A_{14}| = 0$. Then

$$H(\rho) = \frac{1 - e^{-2\operatorname{Re}(\rho)}}{|1 - e^{-2\rho}|},$$

and it remains bounded for $|\rho|$ large. Note that this case is equivalent to the case

$$A_{12} = A_{14} = A_{23} = A_{24} = A_{34} = 0,$$

which implies $A_{13} \neq 0$.

Case (iv). $A_{24} = 0$, $A_{14} - A_{23} = 0$, $A_{13} \neq 0$ and $A_{12} + A_{34} \neq 0$. From (5.5) we see that the dominant term in the numerator of $H(\rho)$ is $|\rho|e^{\operatorname{Re}(\rho)}$ but in the denominator is $|e^\rho|$. This shows that $H(\rho)$ is not a bounded function.

Case (v). $A_{24} = A_{14} - A_{23} = A_{13} = 0$. We have that

$$H(\rho) = (|A_{12}| + 2|A_{14}| + |A_{34}|)|\rho| \frac{e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)}}{|\Delta(\lambda)|},$$

and

$$\Delta(\lambda) = 2(A_{12} + A_{34})\rho.$$

We must consider two subcases:

- $A_{12} + A_{34} \neq 0$. Write

$$H(\rho) = \frac{|A_{12}| + 2|A_{14}| + |A_{34}|}{2|A_{12} + A_{34}|} (e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)}),$$

and observe that $H(\rho)$ is not bounded for $|\rho|$ large.

- $A_{12} + A_{34} = 0$. Then

$$\Delta(\lambda) \equiv 0,$$

and as $A_{12} + A_{13} + A_{14} + A_{34} - A_{23} = 0$, then $\sigma(T_\infty) = \mathbf{C}$. This case will not be considered, since we cannot speak about generation of semi-groups.

In the cases where $H(\rho)$ is bounded, we have proved the existence of positive constants r_1 and M_1 such that $\|R(\lambda : T_\infty)\| \leq M_1/|\lambda|$ for $\lambda = \rho^2$ with $|\rho| > r_1$ and $|\arg(\rho)| < \delta/2$. Define $r = r_1^2/\sin(\delta)$ and $M = M_1(1 + 1/\sin(\delta))$; then the sector $r + \Sigma_\delta \equiv \Sigma_{\delta,r}$ is contained in $\rho(T_\infty)$ and, for every $\lambda \in \Sigma_{\delta,r}$, we have

$$\|R(\lambda : T_\infty)\| \leq \frac{M_1}{|\lambda - r|} \frac{|\lambda - r|}{|\lambda|} \leq \frac{M_1}{|\lambda - r|} \left(1 + \frac{r}{|\lambda|}\right) \leq \frac{M}{|\lambda - r|},$$

because $|\lambda| > r_1^2$.

Proposition 5.1. *Choose $\delta \in (\pi/2, \pi)$, consider the sector Σ_δ , and suppose that the coefficients of the boundary conditions $\{B_1, B_2\}$ satisfy one of the following conditions:*

- $A_{24} \neq 0$.
- $A_{24} = 0$ and $A_{14} - A_{23} \neq 0$.
- $A_{12} = A_{14} = A_{23} = A_{24} = A_{34} = 0$.

Then positive constants r and M exist such that $\Sigma_{\delta,r} \subset \rho(T_\infty)$, and the following inequality holds

$$\|R(\lambda : T_\infty)\| \leq \frac{M}{|\lambda - r|}, \quad \forall \lambda \in \Sigma_{\delta,r}.$$

5.2. Bounds in $L^1(0, 1)$. The spirit of this section is similar to the previous one. From (5.1) we have

$$\begin{aligned} \|R(\lambda : T_1)f\|_{L^1(0,1)} &= \int_0^1 |[R(\lambda : T_1)](x)| dx \\ &\leq \int_0^1 \int_0^1 |G(x, s; \lambda)| |f(s)| ds dx \\ &= \int_0^1 \left(\int_0^1 |G(x, s; \lambda)| dx \right) |f(s)| ds \\ &\leq \left(\sup_{0 \leq s \leq 1} \int_0^1 |G(x, s; \lambda)| dx \right) \|f\|_{L^1(0,1)}, \end{aligned}$$

when $\Delta(\lambda) \neq 0$. As we have just calculated an expression for $|N(x, s; \lambda)|$, we can use it in order to bound $\int_0^1 |N(x, s; \lambda)| dx$:

$$\begin{aligned} \int_0^1 |N(x, s; \lambda)| dx &\leq \frac{\Gamma(\rho)}{2|\rho|} e^{-\operatorname{Re}(\rho)} \int_0^1 e^{(x+s)\operatorname{Re}(\rho)} dx \\ &\quad + \frac{\Gamma(\rho)}{2|\rho|} e^{\operatorname{Re}(\rho)} \int_0^1 e^{-(x+s)\operatorname{Re}(\rho)} dx \\ &\quad + \int_0^1 |\varphi(x, s; \lambda)| dx. \end{aligned}$$

Evaluating the integrals, the previous inequality can be written as

$$\begin{aligned} &\int_0^1 |N(x, s; \lambda)| dx \\ &\leq \frac{\Gamma(\rho)}{2|\rho|\operatorname{Re}(\rho)} (e^{s\operatorname{Re}(\rho)} - e^{-s\operatorname{Re}(\rho)} + e^{(1-s)\operatorname{Re}(\rho)} - e^{(s-1)\operatorname{Re}(\rho)}) \\ &\quad + \int_0^1 |\varphi(x, s; \lambda)| dx. \end{aligned}$$

Now from (5.3) we have

$$\int_0^1 |\varphi(x, s; \lambda)| dx = \int_0^s |\varphi_2(x, s; \lambda)| dx + \int_s^1 |\varphi_1(x, s; \lambda)| dx,$$

and we bound each summand separately. On one hand,

$$\begin{aligned} \int_0^s |\varphi_2(x, s; \lambda)| dx &\leq (2|A_{34}||\rho| + \Gamma(\rho)e^{\operatorname{Re}(\rho)}) \int_0^s \frac{e^{(x-s)\operatorname{Re}(\rho)}}{2|\rho|} ds \\ &\quad + (2|A_{34}||\rho| + \Gamma(\rho)e^{-\operatorname{Re}(\rho)}) \int_0^s \frac{e^{(s-x)\operatorname{Re}(\rho)}}{2|\rho|} ds \\ &= \frac{\Gamma(\rho)}{2|\rho|\operatorname{Re}(\rho)} (e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)}) \\ &\quad + e^{(s-1)\operatorname{Re}(\rho)} - e^{(1-s)\operatorname{Re}(\rho)} \\ &\quad + \frac{|A_{34}|}{\operatorname{Re}(\rho)} (e^{s\operatorname{Re}(\rho)} - e^{-s\operatorname{Re}(\rho)}), \end{aligned}$$

and on the other hand,

$$\begin{aligned} \int_s^1 |\varphi_1(x, s; \lambda)| dx &\leq (2|A_{12}||\rho| + \Gamma(\rho)e^{-\operatorname{Re}(\rho)}) \int_s^1 \frac{e^{(x-s)\operatorname{Re}(\rho)}}{2|\rho|} dx \\ &\quad + (2|A_{12}||\rho| + \Gamma(\rho)e^{\operatorname{Re}(\rho)}) \int_s^1 \frac{e^{(s-x)\operatorname{Re}(\rho)}}{2|\rho|} dx \\ &= \frac{\Gamma(\rho)}{2|\rho|\operatorname{Re}(\rho)} (e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)} + e^{-s\operatorname{Re}(\rho)} - e^{s\operatorname{Re}(\rho)}) \\ &\quad + \frac{|A_{12}|}{\operatorname{Re}(\rho)} (e^{(1-s)\operatorname{Re}(\rho)} - e^{(s-1)\operatorname{Re}(\rho)}). \end{aligned}$$

Adding, we obtain

$$\begin{aligned} \int_0^1 |N(x, s; \lambda)| dx &\leq \frac{|A_{12}|}{\operatorname{Re}(\rho)} (e^{(1-s)\operatorname{Re}(\rho)} - e^{(s-1)\operatorname{Re}(\rho)}) \\ &\quad + \frac{|A_{34}|}{\operatorname{Re}(\rho)} (e^{s\operatorname{Re}(\rho)} - e^{-s\operatorname{Re}(\rho)}) \\ &\quad + \frac{\Gamma(\rho)}{|\rho|\operatorname{Re}(\rho)} (e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)}) \\ &= \frac{1}{|\rho|\operatorname{Re}(\rho)} [|A_{12}||\rho| (e^{(1-s)\operatorname{Re}(\rho)} - e^{(s-1)\operatorname{Re}(\rho)}) \\ &\quad + |A_{34}||\rho| (e^{s\operatorname{Re}(\rho)} - e^{-s\operatorname{Re}(\rho)}) \\ &\quad + \Gamma(\rho) (e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)})], \end{aligned}$$

from which we deduce

$$\sup_{0 \leq s \leq 1} \int_0^1 |N(x, s; \lambda)| dx \leq ((|A_{12}| + |A_{34}|)|\rho| + \Gamma(\rho)) \frac{e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)}}{|\rho| \operatorname{Re}(\rho)}.$$

The above expression can also be written as

$$\sup_{0 \leq s \leq 1} \int_0^1 |N(x, s; \lambda)| dx \leq \frac{H(\rho)}{|\rho| \operatorname{Re}(\rho)} (e^{\operatorname{Re}(\rho)} - e^{-\operatorname{Re}(\rho)}),$$

where $H(\rho)$ is the function defined in the previous subsection. Hence we have obtained that

$$\sup_{0 \leq s \leq 1} \int_0^1 |G(x, s; \lambda)| dx = \frac{1}{|\Delta(\lambda)|} \sup_{0 \leq s \leq 1} \int_0^1 |N(x, s; \lambda)| dx \leq \frac{H(\rho)}{|\rho| \operatorname{Re}(\rho)},$$

which implies

$$\|R(\lambda : T_1)\| \leq \frac{H(\rho)}{\cos(\delta/2)|\rho|^2}.$$

Note that the second member of this inequality is the same as in (5.4), so the previous study of $H(\rho)$ is also valid in the case treated here. Therefore we have the following result, analogous to Proposition 5.1.

Proposition 5.2. *Under the hypotheses of Proposition 5.1, positive constants r and M exist such that $\Sigma_{\delta, r} \subset \rho(T_1)$, and the following inequality holds*

$$\|R(\lambda : T_1)\| \leq \frac{M}{|\lambda - r|}, \quad \forall \lambda \in \Sigma_{\delta, r}.$$

Remark 5.1. It is clear by construction that the constants M and r in Propositions 5.1 and 5.2 are the same.

In his early works [2–4] about nonseparated boundary conditions, Birkhoff distinguished a class of boundary conditions that he called *regular*. For regular boundary conditions, he was able to obtain good spectral properties. The regular conditions of Birkhoff are exactly the

same that we consider in Proposition 5.1. This fact motivates the next definition.

Definition 5.1. The nonseparated boundary conditions

$$\begin{cases} B_1(u) \equiv a_1u(0) + b_1u'(0) + c_1u(1) + d_1u'(1) = 0, \\ B_2(u) \equiv a_2u(0) + b_2u'(0) + c_2u(1) + d_2u'(1) = 0, \end{cases}$$

are called *regular* if their coefficients satisfy one of the following three conditions:

- $A_{24} \neq 0$,
- $A_{24} = 0$ and $A_{14} - A_{23} \neq 0$,
- $A_{12} = A_{14} = A_{23} = A_{24} = A_{34} = 0$.

Remark 5.2. From Proposition 2.1 it is straightforward to verify that this definition does not depend on algebraic manipulations on the boundary conditions.

Suppose that the boundary conditions $\{B_1, B_2\}$ are regular. Then we have a bound for the operator $R(\lambda : T_p)$ of the form

$$\|R(\lambda : T_p)\| \leq \frac{M}{|\lambda - r|}, \quad \forall \lambda \in \Sigma_{\delta,r}, \quad p = 1, \infty.$$

From the Riesz-Thorin interpolation theorem [1], we deduce a similar bound in every $L^p(0, 1)$, $1 \leq p \leq \infty$,

$$\|R(\lambda : T_p)\| \leq \frac{M_p}{|\lambda - r|}, \quad \forall \lambda \in \Sigma_{\delta,r},$$

where M_p and r are positive constants depending only on δ , p and on the coefficients of the boundary conditions. We now have all the elements to establish the main result of this section.

Theorem 5.3. For $1 \leq p \leq \infty$, consider the differential operator $T_p u = u''$ with $D(T_p) = \{u \in W^{2,p}(0, 1) : B_1(u) = B_2(u) = 0\}$, where the boundary conditions $\{B_1, B_2\}$ are regular. Fix an arbitrary $\delta \in (\pi/2, \pi)$ and consider the sector $\Sigma_\delta = \{\lambda \in \mathbf{C} : |\arg(\lambda)| < \delta, \lambda \neq 0\}$.

Then positive constants r and M_p exist such that $\Sigma_{\delta,r} \equiv r + \Sigma_{\delta,r} \subset \rho(T_p)$, and the resolvent operator associated to $\lambda \in \Sigma_{\delta,r}$ satisfies the following bound in the space $L^p(0, 1)$:

$$\|R(\lambda : T_p)\| \leq \frac{M_p}{|\lambda - r|}.$$

To conclude this section, we give some examples of nonseparated boundary conditions, both regular and irregular, in $L^2(0, 1)$.

Example 5.1. Consider the classical case of separated boundary conditions:

$$\begin{cases} B_1(u) \equiv au(0) + bu'(0) = 0, \\ B_2(u) \equiv cu(1) + du'(1) = 0. \end{cases}$$

The coefficient's matrix is

$$A = \begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & c & d \end{pmatrix},$$

and it is easy to prove that the conditions are always regular. The operator T_2 is always self-adjoint.

Example 5.2. Consider the generalized periodic boundary conditions

$$\begin{cases} B_1(u) \equiv u(0) - ru(1) = 0, \\ B_2(u) \equiv u'(0) - ru'(1) = 0, \end{cases}$$

where $r \neq 0$. The coefficient's matrix is

$$A = \begin{pmatrix} 1 & 0 & -r & 0 \\ 0 & 1 & 0 & -r \end{pmatrix}.$$

We have that $A_{24} = 0$ and $A_{14} - A_{23} = -2r \neq 0$, which means that the conditions are regular. The operator T_2 is self-adjoint only in the cases $r = 1$ (periodic conditions) or $r = -1$ (anti-periodic conditions).

Example 5.3. Consider a nonclassical case of nonseparated boundary conditions, as follows

$$\begin{cases} B_1(u) \equiv u(0) - u(1) - u'(0) = 0, \\ B_2(u) \equiv u'(0) + 2u(1) + u'(1) = 0. \end{cases}$$

The coefficient's matrix is, in this case,

$$A = \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}.$$

As $A_{24} = -1$, the conditions are regular, but T_2 is not self-adjoint.

Example 5.4. As a case of nonregular boundary conditions, consider the following initial value conditions

$$\begin{cases} B_1(u) \equiv u(0) = 0, \\ B_2(u) \equiv u'(0) = 0. \end{cases}$$

In this case, the characteristic determinant is given by

$$\Delta(\lambda) = -2\rho.$$

As $A_{12} + A_{13} + A_{14} + A_{34} - A_{23} \neq 0$, then $\lambda = 0$ is not an eigenvalue. We then have that $\sigma(T) = \emptyset$. The Green's function is given by

$$G(x, s; \lambda) = \begin{cases} \frac{1}{-2\rho}(e^{\rho(x-s)} - e^{\rho(s-x)}) & \text{if } x > s, \\ 0 & \text{if } x < s. \end{cases}$$

If we take the function $f_0 \equiv 1$, we have that

$$[R(\lambda : T_2)f_0](x) = \int_0^1 G(x, s; \lambda) ds = \frac{1}{\rho^2}(e^{\rho x} - e^{-\rho x} + 1),$$

from which we deduce

$$\begin{aligned} \|R(\lambda : T_2)\| &\geq \int_0^1 |[R(\lambda : T_2)f_0](x)|^2 dx \\ &= \frac{1}{\rho^4} \int_0^1 |e^{\rho x} - e^{-\rho x} + 1|^2 dx, \end{aligned}$$

and the last term goes to infinity as $|\rho| \rightarrow \infty$. So it is not possible to obtain the adequate kind of bounds on $R(\lambda : T_2)$.

6. Generation of analytic semi-groups. We recall a well-known result that gives sufficient conditions for an operator to be the generator of an analytic semi-group [5].

Theorem 6.1. *Let X be a complex Banach space, and consider a linear operator $L : D(L) \subset X \rightarrow X$. Suppose that constants $\delta \in (\pi/2, \pi)$, $M > 0$ and $r \in \mathbf{R}$ exist such that*

- $\rho(L) \supset \Sigma_{\delta,r} = \{\lambda \in \mathbf{C} : |\arg(\lambda - r)| < \delta, \lambda \neq r\}$,
- $\|R(\lambda : L)\| \leq M/|\lambda - r|$, for all $\lambda \in \Sigma_{\delta,r}$.

Then L is the generator of an analytic semi-group of bounded linear operators on X , $\{e^{tL}\}_{t \geq 0}$. When the domain $D(L)$ is dense in X , the semi-group generated by L is strongly continuous.

Now fix $1 \leq p \leq \infty$, consider the operator T_p and suppose that the boundary conditions $\{B_1, B_2\}$ are regular. Then Theorem 5.3 guarantees the existence of positive constants r and M_p such that the sector $\Sigma_{\delta,r}$ is contained in the resolvent set $\rho(T_p)$ and

$$\|R(\lambda : T_p)\| \leq \frac{M_p}{|\lambda - r|}, \quad \forall \lambda \in \Sigma_{\delta,r}.$$

This shows that the hypotheses of Theorem 6.1 are satisfied, so we deduce the following result.

Theorem 6.2. *Take $1 \leq p \leq \infty$ and consider the linear operator*

$$T_p : D(T_p) \subset L^p(0, 1) \longrightarrow L^p(0, 1)$$

defined as $T_p u = u''$ with $D(T_p) = \{u \in W^{2,p}(0, 1) : B_i(u) = 0, i = 1, 2\}$. Suppose that the boundary conditions $\{B_1, B_2\}$ are regular. Then T_p is the infinitesimal generator of an analytic semi-group of bounded linear operators on the space $L^p(0, 1)$. If $p \neq \infty$, this semi-group is strongly continuous.

Now a natural question arises: how do the changes made in Section 4 affect the results obtained here? We are going to see that an analogue of Theorem 6.2 can be stated.

Consider the operator $L_p u = u'' + qu$, for $u \in D(L_p) = D(T_p)$. We can take r sufficiently large for

$$\|R(\lambda : T_p)\| \leq \frac{1}{2} \|Q_p\|^{-1}, \quad \forall \lambda \in \Sigma_{\delta,r},$$

where Q_p is the operator $Q_p u = qu$ with domain $D(Q_p) = D(T_p)$. Proposition 4.2 assures that $\lambda \in \rho(L_p)$ and

$$\|R(\lambda : L_p)\| \leq 2\|R(\lambda : T_p)\| \leq \frac{2M}{|\lambda - r|}, \quad \forall \lambda \in \Sigma_{\delta,r}.$$

As a consequence, we have the following result:

Corollary 6.3. *Let $1 \leq p \leq \infty$, and consider the linear operator*

$$L_p : D(L_p) \subset L^p(0, 1) \longrightarrow L^p(0, 1)$$

defined as $L_p u = u'' + qu$ with $D(L_p) = \{u \in W^{2,p}(0, 1) : B_i(u) = 0, i = 1, 2\}$. Suppose that the boundary conditions $\{B_1, B_2\}$ are regular. Then L_p is the infinitesimal generator of an analytic semi-group of bounded linear operators on $L^p(0, 1)$. If $p \neq \infty$, this semi-group is strongly continuous.

In Section 4 we had obtained the operator L_p from the more general operator $\tilde{L}_p v = v'' + \tilde{q}_1 v' + \tilde{q}_0 v$ with domain $D(\tilde{L}_p) = \{v \in W^{2,p}(0, 1) : \tilde{B}_i(v) = 0, i = 1, 2\}$, where the boundary conditions were given by

$$\begin{cases} \tilde{B}_1(u) \equiv \tilde{a}_1 v(0) + \tilde{b}_1 v'(0) + \tilde{c}_1 v(1) + \tilde{d}_1 v'(1) = 0, \\ \tilde{B}_2(u) \equiv \tilde{a}_2 v(0) + \tilde{b}_2 v'(0) + \tilde{c}_2 v(1) + \tilde{d}_2 v'(1) = 0. \end{cases}$$

From the relations (4.4), it is easy to see that the boundary conditions $\{B_1, B_2\}$ are of regular type if and only if $\{\tilde{B}_1, \tilde{B}_2\}$ are.

The operators L_p and \tilde{L}_p were related as follows:

$$\tilde{L}_p = M_\phi L_p M_\phi^{-1},$$

where M_ϕ is the multiplication operator by the function ϕ defined in (4.2). From Proposition 4.1, we have that $\rho(L_p) = \rho(\tilde{L}_p)$ and, as a consequence of Corollary 6.3, $\Sigma_{\delta,r} \subset \rho(\tilde{L}_p)$. We also have the following equality

$$R(\lambda : \tilde{L}_p) = M_\phi R(\lambda : L_p) M_\phi^{-1}, \quad \forall \lambda \in \rho(L_p) = \rho(\tilde{L}_p).$$

So for every $\lambda \in \Sigma_{\delta,r}$, we have that

$$\|R(\lambda : \tilde{L}_p)\| = \|M_\phi R(\lambda : L_p) M_\phi^{-1}\| \leq \|R(\lambda : L_p)\| \leq \frac{M}{|\lambda - r|}.$$

This shows that Theorem 6.2 is also valid for the operator \tilde{L}_p .

Finally, note that the linear change made to pass from the interval $[0, 1]$ to $[a, b]$ does not affect the results obtained. In particular, the regularity of the boundary conditions is maintained through this change of variables. This observation allows us to give the main result of this work:

Theorem 6.4. *We consider the second-order differential system with nonseparated boundary conditions given by*

$$\begin{cases} l(u) = u'' + q_1(x)u' + q_0(x)u & \text{in } (a, b), \\ B_1(u) \equiv a_1u(a) + b_1u'(a) + c_1u(b) + d_1u'(b) = 0, \\ B_2(u) \equiv a_2u(a) + b_2u'(a) + c_2u(b) + d_2u'(b) = 0, \end{cases}$$

where $q_1 \in C^1([a, b], \mathbf{C})$ and $q_0 \in C([a, b], \mathbf{C})$. Suppose that the boundary conditions are regular, i.e., their coefficients verify one of the following conditions:

- $A_{24} \neq 0$.
- $A_{24} = 0$ and $A_{14} - A_{23} \neq 0$.
- $A_{12} = A_{14} = A_{23} = A_{24} = A_{34} = 0$.

Fix $1 \leq p \leq \infty$ and consider the L^p -realization of the differential system, that is, the linear operator $L_p : D(L_p) \subset L^p(a, b) \rightarrow L^p(a, b)$ defined as

$$L_p u = u'' + q_1(x)u' + q_0(x)u$$

with domain $D(L_p) = \{u \in W^{2,p}(a, b) : B_i(u) = 0, i = 1, 2\}$. Then L_p is the generator of an analytic semi-group $\{e^{tL_p}\}_{t \geq 0}$ of bounded linear operators on the Banach space $L^p(a, b)$. If $p \neq \infty$, the semi-group is strongly continuous.

When $p = \infty$, we can obtain a strongly continuous semi-group on a certain subspace of $L^\infty(a, b)$. Define X_0 as the closure of $D(L_\infty)$ in $L^\infty(a, b)$. Let L_0 be the part of L in X_0 , that is, $D(L_0) = \{u \in D(L_\infty) : L_\infty u \in X_0\}$ and $L_0 u = L_\infty u$ for $u \in D(L_0)$. Then the operator L_0 verifies the hypotheses of Theorem 6.4 and has domain $D(L_0)$ dense in X_0 so it generates an analytic semi-group $\{e^{tL_0}\}_{t \geq 0}$ on X_0 that is strongly continuous. We also have the following relation

$$e^{tL_0} u = e^{tL_\infty} u, \quad \forall u \in X_0, t \geq 0.$$

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