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SINGULAR POINTS FOR TILINGS OF NORMED SPACES

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ABSTRACT. A point x in a normed space X is said to be singular for a given tiling of X whenever each neighborhood of x intersects infinitely many tiles. We show that, when X is infinite-dimensional and all tiles are convex, special points in the boundary of tiles (like extreme points or PC points, if any) must be singular. Under the further assumptions that X is separable and doesn't contain c_0, singular points abound among the smooth points of any bounded tile. Finally, in any normed space a tiling is constructed which is free of singular points and whose members are both bounded and star-shaped; this disproves the conjecture that Corson's theorem might apply to star-shaped bounded coverings.

Introduction. Throughout this paper, X denotes a normed space over the reals.

A collection τ of subsets of X is a *covering* of X whenever each element of X belongs to some member of τ. If n is a cardinal number, a point x of X is said to be *n-singular* for τ if each neighborhood of x meets at least n different members of τ. For simplicity, $\aleph_0$-singular points will be called *singular points*. We say that τ is *locally finite* at x provided x is not a singular point for τ, and that τ is *locally finite* when it is locally finite at each point of X. A subset of X is a *body* if it is different from X itself and is the closure of its nonempty interior. A covering of X by bodies is called a *tiling* of X whenever any two different members of it have disjoint interiors. The elements of such a covering are called *tiles*. When adjectives (like "bounded," "convex," "star-shaped," etc.) are applied to a collection τ of subsets of X, it means that they apply to each member of τ.

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Bounded convex tilings are available in any normed space, as we showed in [**5**]. The situation is completely different when looking for bounded convex tilings that are also locally finite: in fact, the availability of such tilings is severely restricted by the classical theorem of Corson [**1**]. It asserts that, for any bounded convex covering τ of a normed space that has an infinite-dimensional reflexive subspace, there is a finite-dimensional (hence compact) parallelotope that intersects infinitely many members of τ; thus, τ cannot be locally finite. However, locally finite bounded convex tilings can be produced in some (infinite-dimensional) Banach spaces. For instance, it is an easy exercise to verify that, in the space c_0, the family of balls of radius $1/2$ centered at the points with integer coordinates actually provides such a tiling. (Note that the tiling constructed in the same way in the space l_∞ is not even point-finite, in the sense that there are points, namely all the vertices of the balls, that belong to infinitely (even uncountably) many tiles.) More generally, in [**3**] it is proved that, for a separable Banach space, admitting locally finite bounded convex tilings is equivalent to being isomorphic to a polyhedral space. (A normed space is said to be *polyhedral* if the unit ball of each finite-dimensional subspace is a polyhedron.)

Basic references for studying tilings in general situations, and in particular local finiteness and related concepts, are [**7**] and [**8**]. In [**6**] the surprising (and to the best of our knowledge, the only known) example is described of a tiling with pairwise disjoint tiles: such a construction requires X to be a nonseparable space because of Sierpinski's classical theorem [**12**] on continua. The interesting notion of *index of a singular point* x for a given convex tiling τ of a topological linear space was introduced in [**11**] by Nielsen: roughly speaking, the index of x turns out to be the "number of dimensions" required to detect that τ is not locally finite at x.

The present paper deals with tilings of normed spaces: these are assumed to be infinite-dimensional unless otherwise stated. The aims of the paper are to show that:

(i) if a tiling is convex, then some boundary points of any tile that are in a special position (like extreme points or PC points, if any) are necessarily singular for it (Propositions 1 and 6 and Theorems 2 and 5);

(ii) if a tiling of a separable Banach space not containing c_0 is convex and bounded, then in any tile "many" singular points for it can be found among the smooth points (Theorem 7);

(iii) Corson's theorem mentioned earlier does not apply to star-shaped bounded tilings (Construction 8).

Throughout the paper the cardinality of a set Γ will be denoted by $|\Gamma|$. For a subset A of a normed space, the symbols $\mathrm{bdy}\,(A)$, $\mathrm{cl}\,(A)$, $\mathrm{co}\,(A)$ and $\mathrm{int}\,(A)$ denote respectively the boundary, closure, convex hull and interior of A. For a normed space X, $U(X)$ and $S(X)$ denote respectively the closed unit ball and the unit sphere centered at the origin; $\widehat{X}$ denotes the completion of X.

Singular points. For a normed space X, let $\mathrm{tot}\,(X)$ denote the total character of X, that is, the smallest cardinal c such that a set $W \subset X^*$ exists with $|W| = c$, which is total on X. Moreover, let $\mathrm{norm}\,(X)$ denote the norming character of X, that is, the smallest cardinal c such that a norming set W exists for X with $|W| = c$. (Recall that a set $W \subset S(X^*)$ is called a *norming set for* X when, for some $\alpha \in (0, 1]$ it happens that $\sup\{|f(x)| : f \in W\} \geq \alpha\|x\|$ for every $x \in X$.) Finally, let $\mathrm{dens}\,(X)$ denote the density character of X, that is, the smallest cardinal c such that a set $W \subset X$ exists with $|W| = c$, which is dense in X. Trivially,

$$\mathrm{tot}\,(X) \leq \mathrm{tot}\,(\widehat{X}) \leq \mathrm{norm}\,(X) \leq \mathrm{dens}\,(X)$$

for any normed space X. Whenever X is a WCG (infinite-dimensional) normed space, then $\mathrm{tot}\,(X) = \mathrm{dens}\,(X)$ (see [**9**, Proposition 2.2]). The very simple case $X = l_\infty(\Gamma)$, Γ any nonempty set, shows that $\mathrm{norm}\,(X)$ can be strictly smaller than $\mathrm{dens}\,(X)$, even when Γ is finite, that is, X is finite-dimensional; moreover, it can happen that $\mathrm{tot}\,(X) < \mathrm{tot}\,(\widehat{X})$, and so $\mathrm{tot}\,(X) < \mathrm{norm}\,(X)$ (see Example 1.1 in [**5**]), but it seems to be an open question whether $\mathrm{tot}\,(X) = \mathrm{norm}\,(X)$ for every Banach space X.

From now on, τ always denotes a convex tiling of a normed space X and C a member of τ. However, it will be obvious that all the state results, except Theorem 7, remain valid in the following more general context: τ is a convex covering of X and C is a member of τ which is a body whose interior doesn't meet any other member of τ.

Our first result is of a purely "linear nature", so it is meaningful even for finite-dimensional X where it implies that any bounded tile must contain $\dim(X)$-singular points.

Proposition 1. *Any extreme point of C, if any, is a* $\operatorname{tot}(X)$*-singular point for τ.*

Proposition 1 and Theorem 2 below can be proved by the same argument, so we'll prove them at the same time.

Now we pass to consider points in the boundary of C that are in a special position with respect to some topology on X which is strictly weaker than the norm-topology. The relevant case of the weak topology will be settled in Proposition 6.

Let λ be a Hausdorff topology consistent with the linear structure of X and, for $A \subseteq X$, let $(A, \|\cdot\|)$ and (A, λ) denote the set A equipped with the norm-topology and λ, respectively. Suppose that the identity map $I : (X, \|\cdot\|) \to (X, \lambda)$ is continuous while I^{-1} is not (that is, λ is strictly weaker than the norm-topology). Given any set $A \subset X$, a point $x \in A$ is called a *point of λ-continuity* (*in A*) provided that the map

$$I^{-1}_{|A} : (A, \lambda) \longrightarrow (A, \|\cdot\|)$$

is continuous at x.

Under our notation the following holds.

Theorem 2. *Any point in C of λ-continuity (in C) is a singular point for τ when X is a Banach space. Even with X an incomplete space, it is a* $\operatorname{tot}(X)$*-singular point for τ provided the map I is a strictly singular operator (that is, λ is strictly weaker than the norm-topology when restricted to each infinite-dimensional subspace of X).*

To prove Theorem 2, we need the following, possibly known, result. We sketch its proof for the sake of completeness.

Lemma 3. *Let $(X, \|\cdot\|)$ and (X, λ) be as above with $(X, \|\cdot\|)$ a Banach space. Let L be a $\|\cdot\|$-closed subspace of X with* $\operatorname{codim}(L) < \infty$*. Then λ is strictly weaker than the $\|\cdot\|$-topology even when restricted to L.*

In case $(X, \|\cdot\|)$ *is not complete, the above conclusion may fail, even when* λ *is generated by some (weaker) norm.*

Proof. Suppose $(X, \|\cdot\|)$ is a Banach space. Assume, for contradiction, that λ agrees on L with the $\|\cdot\|$-topology. First we claim that, in this case, L must also be λ-closed. In fact, let $\{x_\alpha\}$ be a net in L that λ-converges to some point $x \in X$. Since λ is consistent with the linear structure of X, $\{x_\alpha\}$ is a λ-Cauchy net in L and actually it turns out to be a $\|\cdot\|$-Cauchy net. Then $\{x_\alpha\}$ must $\|\cdot\|$-converge to some point $y \in L$. But $\{x_\alpha\}$ is λ-convergent to y too, and that implies $y = x \in L$.

Now, being a closed subspace of finite codimension, L is topologically complemented both in $(X, \|\cdot\||)$ and in (X, λ) and, of course, the complements are isomorphic having the same finite dimension. So we get a contradiction, because both $\|\cdot\|$ and λ would agree with the same product topology.

Finally, let $(X, |||\cdot|||)$ be any infinite-dimensional normed space and let f be a $|||\cdot|||$-noncontinuous linear functional on X. Consider the normed space $(X, \|\cdot\|)$ where $\|\cdot\|$ is the different norm on X given by

$$\|x\| = |||x||| + |f(x)|, \quad x \in X. \tag{1}$$

Then $\|\cdot\|$ is strictly stronger than $|||\cdot|||$, because f actually turns out to be $\|\cdot\|$-continuous on X; therefore, $L = \ker(f)$ is a $\|\cdot\|$-closed one-codimensional subspace of X. Clearly, $|||\cdot|||$ and $\|\cdot\|$ agree on L, which completes the proof. (Note that norms, which are constructed by starting from an initial norm as in (1), cannot give complete spaces anymore, whether or not the space $(X, |||\cdot|||)$ is complete). □

Proof of Proposition 1 *and Theorem* 2. Just to simplify notation, let us denote $\operatorname{int}(U(X))$ by $\overset{\circ}{U}$. Without any loss of generality, we may assume that the origin is an interior point of C. Fix a point x in $\operatorname{bdy}(C)$, and let ε be a positive number. Denote by $\{C_\gamma\}_{\gamma\in\Gamma(\varepsilon)}$ the family whose elements are precisely those members of τ that are different from C and meet $x+\varepsilon\overset{\circ}{U}$. For each $\gamma \in \Gamma(\varepsilon)$, choose a functional $f_\gamma \in X^*$ separating C_γ from C in such a way that $C \subseteq f_\gamma^{-1}((-\infty, 1])$

and $C_\gamma \subseteq f_\gamma^{-1}([1,+\infty))$. We claim that

$$C \cap (x + \varepsilon \overset{\circ}{U}) = \bigcap_{\gamma \in \Gamma(\varepsilon)} f_\gamma^{-1}((-\infty, 1]) \cap (x + \varepsilon \overset{\circ}{U}). \tag{2}$$

Indeed, take any point y in $(x+\varepsilon\overset{\circ}{U})\backslash C$. There is a $\delta \in (0,1)$ such that $\delta y \in (x+\varepsilon\overset{\circ}{U})\backslash C$, so $\delta y \in C_\gamma$ for some $\gamma \in \Gamma(\varepsilon)$. Thus $f_\gamma(\delta y) \geq 1$, hence $f_\gamma(y) > 1$, which proves that y doesn't belong to the set in the right side of (2). The second inclusion is trivial.

Now consider the (possibly trivial) subspace of X

$$L_\varepsilon = \bigcap_{\gamma \in \Gamma(\varepsilon)} \ker(f_\gamma)$$

equipped with the induced norm. From (2) we immediately get

$$x + \varepsilon U(L_\varepsilon) \subset C. \tag{3}$$

Now assume that x is not a $\operatorname{tot}(X)$-singular point for τ: for ε small enough we have $|\Gamma(\varepsilon)| < \operatorname{tot}(X)$ which implies $\dim(L_\varepsilon) \geq 1$, with L_ε infinite-dimensional when X is.

By (3), C must contain a nontrivial segment centered at x so x cannot be an extreme point of C and that proves Proposition 1.

By (3) also, when X is infinite-dimensional and I is a strictly singular operator, x cannot be a point of λ-continuity in C because any λ-neighborhood of x, being $\|\cdot\|$-unbounded even along L_ε, must actually intersect $x+\varepsilon S(L_\varepsilon)$. So the second claim in the statement of Theorem 2 is also proved.

To complete the proof of Theorem 2, it remains to settle the case in which we only know that λ is strictly weaker than the $\|\cdot\|$-topology on the whole Banach space X. Assume that x is not a singular point for τ. This means that, for ε small enough, $\Gamma(\varepsilon)$ is finite so $\operatorname{codim}(L_\varepsilon)$ is finite too. Then Lemma 3 applies and we conclude that λ is strictly weaker than the $\|\cdot\|$-topology even when restricted to L_ε. Reasoning as above, we are done. □

Remark 4. The above proof contains the following intuitive result which might be useful to have stated separately.

When $x \in \operatorname{bdy}(C)$ is not a $\operatorname{tot}(X)$-singular point for τ, there is a nontrivial affine subspace $x + L$ of X through x such that $\operatorname{bdy}(C) \cap (x + L)$ has nonempty relative interior in $x + L$. Indeed, x being a boundary point of the convex set C, (3) really means

$$x + \varepsilon U(L_\varepsilon) \subset \operatorname{bdy}(C). \tag{4}$$

As Lemma 3 shows, the proof of the first claim in the statement of Theorem 2 doesn't work in incomplete spaces. This gap can be partially filled by the following

Theorem 5. *Suppose that for each $B \in \tau$, $B \neq C$, a λ-continuous linear functional exists separating C from B. Then any point in C of λ-continuity* (*in C*) *is a singular point for τ.*

Proof. Let us use the same notation and agreements as in the proof of Theorem 2. Suppose, for contradiction, that x is a point of λ-continuity that is not singular for τ. Let ε be small enough such that (2) holds with $\Gamma(\varepsilon)$ a finite set. Of course, we can actually assume that the linear functionals f_γ, $\gamma \in \Gamma(\varepsilon)$, are also λ-continuous. Let W be a λ-neighborhood of x such that

$$C \cap W \subseteq x + \frac{\varepsilon}{2} U(X). \tag{5}$$

We are done provided we show that $W \cap (x + L_\varepsilon)$ is a norm-unbounded set: in fact, in this case it would contain some point in $x + \varepsilon S(X)$, contradicting (5). Suppose that $W \cap (x + L_\varepsilon)$ is norm-bounded. Since its λ-interior relative to $x + L_\varepsilon$ is clearly nonempty, we get that topology λ agrees on L_ε with the norm-topology. Because $\Gamma(\varepsilon)$ is finite, $\operatorname{codim}(L_\varepsilon)$ is finite too so L_ε, which is closed because of λ-continuity of the functionals f_γ, has a finite-dimensional topological complement in X. Reasoning as in the first part of the proof of Lemma 3, we get that λ must agree on the whole of X with the norm-topology, a contradiction. $\square$

When λ is the weak topology, we get of course a relevant setting for Theorems 2 and 5; actually in this special case we can be more precise with the following proposition. Briefly call *PC point* (*in* C) any point in C of weak-to-norm continuity (in C).

Proposition 6. *Any PC point in C is a* norm (X)*-singular point for τ.*

Proof. Suppose that the origin is the point under investigation. Fix any positive number δ; let $f_i \in X^*$, $i = 1, \dots, n$ for some $n \in \mathbf{N}$, be such that for the weak neighborhood W of the origin defined by

$$W = \bigcap_{i=1}^{n} f_i^{-1}([-1,1])$$

it is true that $W \cap C \subset \operatorname{int}(\delta U(X))$.

Consider the convex covering τ' of X given by

$$\tau' = (\tau \backslash \{C\}) \cup \{W \cap C\} \cup \{f_i^{-1}((-\infty, 1])\}_{i=1}^{n} \\ \cup \{f_i^{-1}([1, +\infty))\}_{i=1}^{n}.$$

Then τ' and the bounded body $W \cap C$ satisfy the assumptions of Theorem 1.2 in [**5**] (in place of τ and C respectively). Now take into account that what was really proved there is that (according to our new symbols) any ball containing $W \cap C$ in its interior actually meets at least norm(X) different members of τ'. So our theorem follows, since τ' and our initial tiling τ differ by only finitely many members. $\square$

At the present we have proved that, among the points in bdy(C) that are singular for τ, we find all the extreme points (Proposition 1) and all the *PC* points (Proposition 6); in particular, we find all the denting points, because a point is denting if and only if it is both a *PC* and an extreme point (see [**10**] also for definitions).

Recall that a Banach space X is said to have the *point of continuity property* (*PC property*) provided each weakly closed bounded subset of it contains a *PC* point. Any Banach space having the *RN* property (in particular any reflexive space) has the *PC* property. If C is a weakly closed bounded subset of a Banach space with the *PC* property, then

C is a Baire space even with respect to the weak topology and the identity map

$$I_{|C}^{-1} : (C, w) \longrightarrow (C, \|\cdot\|)$$

is continuous at each point of some w-dense w-G_δ subset of C (see [**2**, Proposition 3.9 and Theorem 3.13]). As a consequence, C has uncountably many norm (X)-singular points.

Let us now investigate the separable case, showing that in many "good" Banach spaces (including reflexive spaces) the subset of $\operatorname{bdy}(C)$ consisting of the smooth points that are singular for τ must be uncountable.

Recall that a point $x \in C$ is called a *smooth point* (of C) if exactly one linear continuous functional f exists on X such that $1 = f(x) = \max f(C)$. The set of all the smooth points of C will be denoted by $\operatorname{sm}(C)$. Consider the (possibly empty) set

$$\Phi_C = \{x \in C : x \in B \text{ for some } B \in \tau\backslash\{C\}\}.$$

Each point x (if any) in $\operatorname{bdy}(C)\backslash\Phi_C$ is a singular point for τ: in fact, if not so, $\inf\{\operatorname{dist}(x, B) : B \in \tau\backslash\{C\}\}$ would be strictly positive and τ would not be a covering. The following theorem provides a sufficient condition for the set $\operatorname{bdy}(C)\backslash\Phi_C$ to be "big."

Theorem 7. *Let X be a separable Banach space that doesn't contain* (*isomorphically*) c_0. *If C is bounded, then the set* $\operatorname{sm}(C)\backslash\Phi_C$

(i) *is w-dense in C and*

(ii) *cannot be covered by the union of countably many w-closed subsets of* $\operatorname{bdy}(C)$.

Proof. Without any loss of generality, we may assume that the origin is an interior point of C. For each $B \in \tau\backslash\{C\}$, let f_B denote a linear functional separating C from B in such a way that

$$\sup f_B(C) \leq \inf f_B(B).$$

(i) Suppose, on the contrary, that for some point $x \in C$ and for some w-neighborhood W of x it happens that

$$W \cap \operatorname{sm}(C) \subseteq \Phi_C. \tag{6}$$

Of course, we can confine ourselves to the case of x being an interior point of C so that we may assume that x is the origin and that, for some $n \in \mathbf{N}$ and $\{f_i\}_{i=1}^n \subset X^*$,

$$W = \bigcap_{i=1}^{n} f_i^{-1}([-1,1]).$$

Clearly

$$\mathrm{bdy}\,(W \cap C) \subset \bigcup_{i=1}^{n} f_i^{-1}(\{\pm 1\}) \cup (\mathrm{bdy}\,(C) \cap W)$$

so

$$\mathrm{sm}\,(W \cap C) \subseteq \bigcup_{i=1}^{n} (f_i^{-1}(\{\pm 1\}) \cap C) \cup (W \cap \mathrm{sm}\,(C)).$$

Put

$$\Omega = \{f \in X^* : \max f(W \cap C) = 1 \quad \text{and} \\ f^{-1}(\{1\}) \cap \mathrm{sm}\,(W \cap C) \neq \varnothing\}.$$

Then (6) implies

$$\Omega \subseteq \{\pm f_i\}_{i=1}^n \cup \left\{ \frac{f_B}{\sup f_B(C)} : B \in \tau \backslash \{C\} \right\}.$$

τ being countable (since X is separable), Ω is countable too.

Now it is well known that

$$\text{(7)} \quad \mathrm{sm}\,(W \cap C) = \{x \in W \cap C \cap f^{-1}(\{1\}) : f \in w^*\text{-}\exp((W \cap C)^0)\}$$

where $w^*\text{-}\exp((W \cap C)^0)$ denotes the set of all the w^*-exposed points of the polar set of $W \cap C$ (that is the set of those elements $g \in X^*$ such that $g|_{W \cap C} \leq 1$ and there is $x \in X$ such that $g(x) > h(x)$ for each $h \in X^*$ with $h|_{W \cap C} \leq 1$, $h \neq g$).

Then $w^*\text{-}\exp((W \cap C)^0)$ actually coincides with Ω and turns out to be countable. Thus, Theorem 3 in [**4**] applies and we get the contradiction that X would contain c_0 (to apply it, note that any countable subset of X^* is "thin" in the sense of Section 1 in [**4**]).

(ii) For each $B \in \tau \setminus \{C\}$, consider the w-closed (possibly empty) subset Γ_B of $\mathrm{bdy}\,(C)$ defined by

$$\Gamma_B = \{x \in C : f_B(x) = \sup f_B(C)\}.$$

Clearly

$$\Phi_C \subseteq \bigcup_{B \in \tau \setminus \{C\}} \Gamma_B$$

so our claim immediately follows from Theorem 2 in [**4**]. □

We conclude by constructing a locally finite bounded tiling that is available in any normed space and is, in some sense, special. In fact, each member of it is star-shaped, thus disproving the conjecture that Corson's theorem also applies to star-shaped bounded coverings.

Construction 8. The construction is really simple. Let $(X, \|\cdot\|)$ be any normed space and H any closed half-space of it such that $\mathrm{bdy}\,(H)$ is a (closed) hyperplane through the origin. Let z be any norm-one interior point of H, and let π denote the continuous linear projection of X onto $\mathrm{bdy}\,(H)$ through the line $\mathbf{R}z$. For fixed H and z, consider the renorming $|||\cdot|||$ of X given by

$$|||x||| = \|\pi(x)\| + \|x - \pi(x)\|, \quad x \in X.$$

Let B denote the closed unit ball with respect to the new norm centered at the origin. Let us set

$$T_1 = B \cap H, \qquad T_n = \mathrm{cl}\,(nB \setminus (n-1)B) \cap H, \quad n = 2, 3, 4, \dots .$$

Clearly the family $\{\pm T_n\}_{n=1}^{\infty}$ provides a bounded locally finite tiling of X. Tile T_1 is convex, while, for each fixed $n \geq 2$, tile T_n is star-shaped from any point in the segment $[(n-1)z, nz]$. In fact, for any real $\sigma \in [n-1, n]$, any real $\lambda \in [0, 1]$ and any $y \in T_n$ trivially we have

$$n \geq |||(1-\lambda)y + \lambda\sigma z||| = (1-\lambda)|||y||| + \lambda\sigma \geq n - 1.$$

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