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A NOTE ON SCHUR-CONVEX FUNCTIONS

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ABSTRACT. In this note it is proved that the integral arithmetic mean of a convex function is a Schur-convex function. Applications to Schur-convexity of logarithmic mean and gamma functions are given.

For the convenience of the reader, we recall shortly the main definitions. Function F of n arguments defined on I^n, where I is an interval with nonempty interior, is Schur-convex on I^n if

$$F(x_1, \dots, x_n) \le F(y_1, \dots, y_n) \tag{1}$$

for each two n-tuples $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ in I^n, such that $x \prec y$ holds, i.e.,

$$\begin{aligned} \sum_{i=1}^{k} x_{[i]} &\le \sum_{i=1}^{k} y_{[i]}, \quad k = 1, \dots, n-1, \\ \sum_{i=1}^{n} x_{[i]} &= \sum_{i=1}^{n} y_{[i]}, \end{aligned} \tag{2}$$

where $x_{[i]}$ denotes the ith largest component in x. F is strictly Schur-convex on I^n if a strict inequality holds in (1) whenever $x \prec y$ and x is not a permutation of y.

For $n = 2$, a continuously differentiable function F on I^2 (I being an open interval) is Schur-convex if and only if it is symmetric and the following holds

$$\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right)(y - x) > 0 \quad \text{for all } x, y \in I,\ x \ne y. \tag{3}$$

Of course, F is Schur-concave if and only if $-F$ is Schur-convex.

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In [**3**] some inequalities concerning gamma and digamma functions are proved. One of the main results is the following

Theorem A. *The function* $(x,y)\mapsto F(x,y)$ *defined by*

$$F(x,y)=\frac{\log\Gamma(x)-\log\Gamma(y)}{x-y},\quad x\neq y,\qquad F(x,x)=\Psi(x) \tag{4}$$

is strictly Schur-concave on $x>0$, $y>0$.

We shall generalize this result.

Theorem 1. *Let* f *be a continuous function on* I. *Then*

$$F(x,y)=\frac{1}{y-x}\int_x^y f(t)\,dt,\quad x,y\in I,\ y\neq x,\qquad F(x,x)=f(x) \tag{5}$$

is Schur-convex (*Schur-concave*) *on* I^2 *if and only if* f *is convex* (*concave*), *on* I.

Proof. F is evidently symmetric. The following holds, for all $x,y\in I$

$$\begin{aligned}\left(\frac{\partial F}{\partial y}-\frac{\partial F}{\partial x}\right)(y-x)&=\left[-\frac{1}{(y-x)^2}\int_x^y f(t)\,dt+\frac{f(y)}{y-x}\right.\\&\qquad\left.-\frac{1}{(y-x)^2}\int_x^y f(t)\,dt+\frac{f(x)}{y-x}\right](y-x)\\&=f(x)+f(y)-\frac{2}{y-x}\int_x^y f(t)\,dt.\end{aligned}$$

By extension of the Hermite-Hadamard inequality [**4**, p. 15], the inequality

$$\frac{1}{y-x}\int_x^y f(t)\,dt\leq\frac{f(x)+f(y)}{2}$$

holds for all $x,y\in I$ if and only if f is a convex function. This proves the theorem.

In fact, using Schur-convexity we immediately also obtain the left side of the Hermite-Hadamard inequality: if f is convex on I, then $F(x,y)$ defined by (5) is Schur-convex; therefore the following holds

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(t)\,dt \tag{6}$$

since $[(x+y)/2,(x+y)/2] \prec (x,y)$.

To prove Theorem A, it is sufficient to note that the function $\Psi = \Gamma'/\Gamma$ is concave on $(0,\infty)$:

Corollary 1. *If $f > 0$ is a function defined on I such that f'/f is convex (concave) on I, then*

$$F(x,y) = \frac{\log f(x) - \log f(y)}{x-y}, \quad x \ne y,$$
$$F(x,x) = f(x)$$

is Schur-convex (Schur-concave) on I^2.

Taking $f(x) = x$, it follows that $K(x,y) = (\log x - \log y)/(x-y)$ is Schur-convex on $\mathbf{R}_+^2$, and therefore $L(x,y) = 1/K(x,y)$ is Schur-concave on this set [**2**, Section 3.I.4]. More generally, the following holds:

Corollary 2. *Generalized logarithmic mean*

$$L_r(x,y) = \left(\frac{y^r - x^r}{r(y-x)}\right)^{1/(r-1)}, \quad x,y > 0$$
$$L(x,x) = x$$

is Schur-convex for $r > 2$ and Schur-concave for $r < 2$. (For $r = 0$, we have $L_0 = L$, and for $r = 1$ we have $L_1(x,y) = [(x^x/y^y)^{1/(x-y)}/e]$).

Proof. Function $t \mapsto t^{r-1}$ is convex on $\mathbf{R}_+$ for $r < 1$ or $r > 2$ and concave for $1 < r < 2$. Therefore, by Theorem 1,

$$\frac{y^r - x^r}{r(y-x)}$$

is Schur-convex on $\mathbf{R}_+^2$ for $r < 1$ and $r > 2$, and Schur-concave for $1 < r < 2$. Since $t \mapsto t^{1/(r-1)}$ is increasing for $r > 1$, $L_r(x, y)$ remains Schur-convex for $r > 2$ and Schur-concave for $1 < r < 2$ [**3**, p. 61]. But $t \mapsto t^{1/(r-1)}$ is decreasing for $r < 1$, therefore L_r becomes Schur-concave for $r < 1$. Taking a limit $r \to 1$, the corollary also holds for $r = 1$.

Remark. Using Schur-concavity of the function (4), the following version of Gautschi's inequality is proved in [**3**]:

$$\exp\left(\beta\frac{\Psi(x+\beta)+\Psi(x)}{2}\right) < Q(x,\beta) < \exp(\beta\Psi(x+\beta/2)) \tag{6}$$

where $Q(x, \beta) = \Gamma(x + \beta)/\Gamma(x)$, $x > 0$, $\beta > 0$. The author claims that this inequality is better than the known one, given by Kershaw:

$$\frac{\Psi(x+\beta)+\Psi(x)}{2} > \Psi(x+\beta-1+\sqrt{1-\beta}). \tag{7}$$

But both the proof and this inequality are not correct. It is sufficient to take $x = 0.5$ and $\beta = 0.75$. The following holds ([**1**], up to five decimals): $\Psi(0.5) = -1.96351$, $\Psi(1.25) = -0.22745$, $\Psi(0.75) = \Psi(1.75) - 1/0.75 = -1.08586$, which disproves (7). Therefore, direct application of Schur-concavity does not lead to better bounds.

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