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# EQUAL SUMS OF SEVENTH POWERS

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ABSTRACT. Until now only three numerical solutions of the diophantine equation $x_1^7 + x_2^7 + x_3^7 + x_4^7 = y_1^7 + y_2^7 + y_3^7 + y_4^7$ are known. This paper provides three numerical solutions in positive integers of the hitherto unsolved system of simultaneous diophantine equations $x_1^k + x_2^k + x_3^k + x_4^k = y_1^k + y_2^k + y_3^k + y_4^k$, $k = 1, 3$ and 7.

Parametric solutions of the diophantine equation

$$\sum_{i=1}^{n} x_i^7 = \sum_{i=1}^{n} y_i^7 \tag{1}$$

have been given by Sastri and Rai [**5**] when $n = 6$ and by Gloden [**3**], [**4**] when $n = 5$. When $n = 4$, only three numerical solutions of (1) are known. These were discovered by Ekl [**1**], [**2**] via computer search.

In this paper we obtain three numerical solutions in positive integers of the hitherto unsolved system of diophantine equations

$$\sum_{i=1}^{4} x_i^k = \sum_{i=1}^{4} y_i^k, \quad k = 1, 3, 7. \tag{2}$$

To solve the system of equations (2), we write

$$\begin{aligned} x_1 &= X_1 - X_2 - X_3, & y_1 &= Y_1 - Y_2 - Y_3, \\ x_2 &= -X_1 + X_2 - X_3, & y_2 &= -Y_1 + Y_2 - Y_3, \\ x_3 &= -X_1 - X_2 + X_3, & y_3 &= -Y_1 - Y_2 + Y_3, \\ x_4 &= X_1 + X_2 + X_3, & y_4 &= Y_1 + Y_2 + Y_3. \end{aligned} \tag{3}$$

Then we have the identities

$$\sum_{i=1}^{4} x_i = 0, \qquad \sum_{i=1}^{4} x_i^3 = 24X_1X_2X_3,$$

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and

$$\sum_{i=1}^{4} x_i^7 = 56X_1X_2X_3\{3(X_1^4+X_2^4+X_3^4) + 10(X_1^2X_2^2+X_2^2X_3^2+X_3^2X_1^2)\}.$$

Hence we will get a solution of the system of equations (2) if we can find $X_i$, $Y_i$, $i = 1, 2, 3$, such that

$$X_1X_2X_3 = Y_1Y_2Y_3 \tag{4}$$

and

$$\begin{aligned} 3(X_1^4 + X_2^4 + X_3^4) &+ 10(X_1^2X_2^2 + X_2^2X_3^2 + X_3^2X_1^2) \\ &= 3(Y_1^4 + Y_2^4 + Y_3^4) + 10(Y_1^2Y_2^2 + Y_2^2Y_3^2 + Y_3^2Y_1^2). \end{aligned} \tag{5}$$

To solve the simultaneous equations (4) and (5), we write

$$\begin{aligned} X_1 &= p(x-q), \qquad X_2 = q(x-s), \qquad X_3 = rs, \\ Y_1 &= r(x-s), \qquad Y_2 = s(x-q), \qquad Y_3 = pq. \end{aligned} \tag{6}$$

With these values of $X_i$, $Y_i$, $i = 1, 2, 3$, equation (4) is identically satisfied while equation (5) reduces to the following cubic equation in $x$:

$$\begin{aligned} &(3p^4 + 10p^2q^2 + 3q^4 - 3r^4 - 10r^2s^2 - 3s^4)x^3 \\ &- 4(3p^4q + 5p^2q^3 + 5p^2q^2s + 3q^4s - 5qr^2s^2 \\ &\qquad - 3qs^4 - 3r^4s - 5r^2s^3)x^2 \\ &+ 2(9p^4q^2 + 5p^2q^4 + 20p^2q^3s - 5p^2q^2r^2 + 5p^2r^2s^2 \\ &\qquad + 9q^4s^2 - 9q^2s^4 - 20qr^2s^3 - 9r^4s^2 - 5r^2s^4)x \\ &- 4(3p^4q^3 + 5p^2q^4s - 5p^2q^2r^2s + 5p^2qr^2s^2 \\ &\qquad + 3q^4s^3 - 3q^3s^4 - 5qr^2s^4 - 3r^4s^3) = 0. \end{aligned} \tag{7}$$

If $p, q, r$ and $s$ are real, the cubic equation (7) will have a real root. By trial we will choose $p, q, r$ and $s$ to be integers such that equation (7) has a rational root. This rational root of (7) will lead to a rational solution of the simultaneous equations (4) and (5). Since both the equations (4) and (5) are homogeneous, we may multiply the rational solution of these equations by a suitable constant to obtain a solution of equations (4) and (5) in integers. Using the relations (3), we finally obtain a solution in integers of the system of equations (2).

In order to find suitable integers $p, q, r$ and $s$ by trial such that equation (7) has a rational root, we note that $p$ and $r$ occur in equation (7) only in even degrees, and so we may take them to be positive integers. In fact, a little reflection shows that, without loss of generality, we may take $p, q$ and $r$ to be positive integers while $s$ may be either positive or negative. We further note that when $p, q, r$ and $s$ are all positive, equation (7) is unaltered if we interchange $p$ and $r$ and simultaneously also interchange $q$ and $s$. Thus, while carrying out the trials with $p, q, r$ and $s$ all positive, we may impose the condition $p < r$.

A computer search was carried out for sets of values of $p, q, r$ and $s$ such that equation (7) has a rational root, with $p, q, r$ and $s$ being positive integers in the range $4 \leq (p + q + r + s) \leq 400$. This yielded the following three numerical solutions of the system of equations (2):

(i) when $p = 4$, $q = 49$, $r = 47$ and $s = 19$, equation (7) has the rational root $x = 130$ which leads to the following solution of equations (4) and (5):

$$\begin{aligned} X_1 &= 324, \qquad & X_2 &= 5439, \qquad & X_3 &= 893, \\ Y_1 &= 5217, \qquad & Y_2 &= 1539, \qquad & Y_3 &= 196. \end{aligned}$$

Using the relations (3) we get, after removal of common factors and suitable transposition, the following solution:

$$1741^k + 2435^k + 3004^k + 3476^k = 1937^k + 2111^k + 3280^k + 3328^k,$$

where the equality holds for $k = 1, 3$ and 7.

(ii) when $p = 35$, $q = 24$, $r = 90$ and $s = 189$, equation (7) is satisfied by $x = 10878/107$, and this leads to the following solution of (2):

$$1523^k + 4175^k + 4492^k + 5956^k = 1951^k + 3107^k + 5528^k + 5560^k,$$
$$k = 1, 3, 7.$$

(iii) Finally, when $p = 21$, $q = 156$, $r = 52$ and $s = 133$, equation (7) is satisfied by $x = 1820/47$, and we eventually get the solution:

$$344^k + 902^k + 1112^k + 1555^k = 479^k + 662^k + 1237^k + 1535^k,$$
$$k = 1, 3, 7.$$

A computer search was also carried out for rational solutions of equation (7) with $p, q, r$ being positive integers and $s$ being a negative integer in the range $4 \leq (p+q+r+|s|) \leq 200$. This did not yield any additional solutions of (2).

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